Bull. Math. Soc. Sci. Math. Roumanie
Tome 51(99) No. 4, 2008, 339–353

Einstein-Weyl structures on complex manifolds and conformal version of Monge-Ampère equation

by
L. Ornea* and M. Verbitsky†

Abstract

A Hermitian Einstein-Weyl manifold is a complex manifold admitting a Ricci-flat Kähler covering $\tilde{M}$, with the deck group acting on $\tilde{M}$ by homotheties. If compact, it admits a canonical Vaisman metric, due to Gauduchon. We show that a Hermitian Einstein-Weyl structure on a compact complex manifold is determined by its volume form. This result is a conformal analogue of Calabi's theorem stating the uniqueness of Kähler metrics with a given volume form in a given Kähler class. We prove that the solution of the conformal version of complex Monge-Ampère equation is unique. We conjecture that a Hermitian Einstein-Weyl structure on a compact complex manifold is unique, up to a holomorphic automorphism, and compare this conjecture to Bando-Mabuchi theorem.

Key Words: Einstein-Weyl structure, Vaisman manifold, potential.
2000 Mathematics Subject Classification: Primary 53C55.

Contents

1 Introduction **340**
1.1 Calabi-Yau theorem and Monge-Ampère equations 340
1.2 Monge-Ampère equations on LCK-manifolds 340
1.3 Sasaki-Einstein manifolds and Einstein-Weyl geometry 342

2 Vaisman manifolds and LCK-geometry **342**

3 Monge-Ampère equation on LCK-manifolds **346**

4 Einstein-Weyl LCK manifolds **348**

*Partially supported by CEEX grant 2-CEx 06-11-22/25.07.2006
†Misha Verbitsky is an EPSRC advanced fellow supported by EPSRC grant GR/R77773/01.

5 Bando-Mabuchi theorem in conformal setting 350

1 Introduction

1.1 Calabi-Yau theorem and Monge-Ampère equations

S.-T. Yau [Y] has shown that a compact manifold of Kähler type with vanishing first Chern class admits a unique Kähler-Einstein metric in a given Kähler class. This result is known as Calabi-Yau theorem; see [B] for details and implications of this extremely important work. Such a metric is called now **Calabi-Yau metric**. This theorem was conjectured by E. Calabi ([C]), who also proved that the Calabi-Yau metric is unique, in a given Kähler class.

The idea of the proof was suggested by Calabi, who has shown that existence of Calabi-Yau metric is implied by the following theorem, which is true for all compact Kähler manifolds.

Theorem 1.1. ([Y]) *Let M be a compact Kähler manifold, $[\omega] \in H^{1,1}(M)$ a Kähler class, and $V \in \Lambda^{n,n}(M)$ a nowhere degenerate volume form. Then there exists a unique Kähler form $\omega_1 \in [\omega]$, such that*

$$\omega_1^n = \lambda V, \tag{1.1}$$

where λ is a constant.

Given two Kähler forms ω, ω_1, in the same Kähler class, we can always have

$$\omega_1 = \omega + dd^c\phi$$

for some function ϕ (this statement is a consequence of the famous dd^c*-lemma*; see e.g. [GH]). Then (1.1) becomes

$$(\omega + dd^c\phi)^n = \lambda e^f \omega^n. \tag{1.2}$$

Here the function ϕ is an unknown, the Kähler form ω and the function f are given, and the constant λ is expressed through f and ω as

$$\int_M \omega^n = \lambda \int_M e^f \omega^n.$$

The equation (1.2) is called **the complex Monge-Ampère equation**. Solutions of (1.1) are unique on a compact manifold, as shown by Calabi ([C]).

1.2 Monge-Ampère equations on LCK-manifolds

In this note, we generalize the Monge-Ampère equation to locally conformally Kähler geometry, and show that its solutions are unique.

Recall that a locally conformally Kähler (LCK) manifold is a complex manifold admitting a Kähler covering $\tilde{M}$, with the deck group acting on $\tilde{M}$ by holomorphic homotheties. If $\tilde{M}$ is, in addition, Ricci-flat, M is called Hermitian Einstein-Weyl, or locally conformally Kähler Einstein-Weyl.[1] The Hermitian Einstein-Weyl geometry is a conformal analogue of Kähler-Einstein geometry.

Since the deck group acts on $\tilde{M}$ conformally, the LCK-structure defines a conformal class of Hermitian metrics on M. A metric in this class is called **an LCK-metric**. In the literature, the distinction between "LCK-metrics" and "LCK-structures" is often ignored. Any compact LCK-manifold M is naturally equipped with a special Hermitian metric ω in its conformal class, called **the Gauduchon metric** (see 2.5). It is defined uniquely, up to a constant multiplier. If, in addition, ω is preserved by a holomorphic flow, which acts on $\tilde{M}$ by homotheties, the LCK-manifold M is called **Vaisman** (this notion was first introduced as **generalized Hopf** manifold, but the name proved to be inappropriate)[2]. P. Gauduchon has proven that all compact Hermitian Einstein-Weyl manifolds are Vaisman ([G2]).

Given an LCK-manifold (M, I, ω), its Hermitian form ω satisfies $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form of** M (see Section 2). Its cohomology class $[\theta]$ is called **the Lee class of** M.

The Monge-Ampère equations on LCK-manifolds are equivalent to finding a Gauduchon metric with a prescribed volume. Uniqueness of such metric (hence, uniqueness of solutions of locally conformally Kähler Monge-Ampère) is due to the following theorem.

Theorem 1.2. *Let (M, J) be a compact complex manifold admitting a Vaisman structure, and $V \in \Lambda^{n,n}(M)$ a nowhere degenerate, positive volume form. Then M admits at most one Vaisman structure with the same Lee class, such that the volume form of the corresponding Gauduchon metric is equal to V.*

We give an introduction to LCK-geometry in Section 2, and explain the properties of Einstein-Weyl structures in Section 4. We prove 1.2 in Section 3. The existence result (not proven in this paper) should follow from the same arguments, added to those used for the proof of Calabi-Yau theorem. In the last section, we explain how other results oh Kähler geometry (the Calabi conjecture and the Bando-Mabuchi theorem) generalize to conformal setting.

Remark 1.3. *For a Kähler manifold, the metric is uniquely determined by the volume form and the Kähler class in cohomology. In a conformal setting, the Vaisman metric is defined uniquely by the volume and the Lee class. This happens because a relevant cohomology group is $H^2(M, L)$, where L is the weight bundle of the conformal structure (see Definition 2.2). It is easy to show that all cohomology of the local system L vanish, cf.* $[O, Remark\ 6.4]$.

[1] Normally, one defines Hermitian Einstein-Weyl differently, and then this definition becomes a theorem; see Claim 4.2.

[2] Traditionally, Vaisman manifolds are defined differently, and this definition becomes a theorem; see Remark 2.9.

1.3 Sasaki-Einstein manifolds and Einstein-Weyl geometry

The compatibility between a complex structure and a Weyl structure naturally leads to the LCK-condition. This was observed by I. Vaisman (see also [PPS]). Moreover, as shown by P. Gauduchon ([G2]), a compact Einstein-Weyl locally conformally Kähler manifold is necessarily Vaisman (see 4.4). Then 1.2 is translated into the uniqueness of an Einstein-Weyl Vaisman metric with a prescribed volume form and the Lee class.

The Vaisman manifolds are intimately related to Sasakian geometry (see *e.g.* [OV1]). Given a Sasakian manifold X, the product $S^1 \times X$ has a natural Vaisman structure. Conversely, any compact Vaisman manifold admits a canonical Riemannian submersion to S^1, with fibers which are isometric and equipped with a natural Sasakian structure.

Under this correspondence, the Einstein-Weyl Vaisman manifolds correspond to Sasaki-Einstein manifolds. The Sasaki-Einstein manifolds recently became a focus of much research, due to a number of new and unexpected examples constructed by string physicists (see [MSY], [CLPP], [GMSW1], [GMSW2], and the references therein). For a physicist, Sasaki-Einstein manifolds are interesting because of AdS/CFT correspondence in string theory. From the mathematical point of view, these examples are as mysterious as the Mirror Symmetry conjecture 15 years ago.

The Sasakian manifolds, being transverse Kähler[3], can be studied by the means of algebraic geometry. One might hope to obtain and study the Sasaki-Einstein metrics by the same kind of procedures as used to study the Kähler-Einstein metrics in algebraic geometry. However, this analogy is not perfect. In particular, it is possible to show that the Sasaki-Einstein structures on CR-manifolds are not unique.

One may hope to approach the classification of Sasaki-Einstein structures using the Einstein-Weyl geometry.

2 Vaisman manifolds and LCK-geometry

We first review the necessary notions of locally conformally Kähler geometry. See [DO], [OV1], [OV2], [OV3], [Ve] for details and examples.

Let (M, J, g) be a complex Hermitian manifold of complex dimension n. Denote by ω its fundamental two-form $\omega(X, Y) = g(X, JY)$.

Definition 2.1. *A Hermitian metric g on (M, J) is* **locally conformally Kähler** *(LCK for short) if*

$$d\omega = \theta \wedge \omega,$$

for a closed 1-form θ.

[3]This viewpoint was systematically developed in the work of C.P. Boyer, K. Galicki and collaborators. See *e.g.* [BG].

Clearly, for any smooth function $f: M \longrightarrow \mathbb{R}^{>0}$, $f\omega$ is also an LCK-metric. A conformal class of LCK-metrics is called **an LCK-structure**.

The form θ is called **the Lee form of the LCK-metric**, and the dual vector field $\theta^\sharp$ is called **the Lee field**.

The one-form $\frac{1}{2}\theta$ can be interpreted as a (flat) connection one-form in the bundle of densities of weight 1, usually denoted L. This is the real line bundle associated to the representation

$$A \mapsto \mid \det(A) \mid^{\frac{1}{2n}}, \quad A \in \mathrm{GL}(2n, \mathbb{R})$$

Definition 2.2. *The bundle L, equipped with a connection $\nabla_0 + \theta$, is called* **the weight bundle of the locally conformally Kähler structure**. *One could consider the form ω as a closed, positive $(1,1)$-form, taking values in L^2.*

Remark 2.3. *Passing to a covering, we may assume that the flat bundle L is trivial. Then ω can be considered as a closed, positive $(1,1)$-form taking values in a trivial vector bundle, that is, a Kähler form. Therefore, any LCK-manifold admits a covering $\tilde{M}$ which is Kähler. The deck group acts on $\tilde{M}$ by homotheties. This property can be used as a definition of LCK-structures (see Section 1).*

Definition 2.4. *A* **Vaisman manifold** *is an LCK-manifold equipped with an LCK-metric g whose Lee form is parallel with respect to the Levi-Civita connection of g. In this case, g is called* **a Vaisman metric**.

A Vaisman metric on a compact manifold is unique, up to a constant multiplier. The proof is due to P. Gauduchon ([G1]).

Definition 2.5. *Let M be an LCK-manifold, g an LCK-metric, and θ the corresponding Lee form. The metric g is called* **Gauduchon metric** *if $d^*\theta = 0$.*

Theorem 2.6. ([*G1*]) *Let M be a compact LCK-manifold. Then M admits a Gauduchon metric, which is unique, up to a constant multiplier.*

Remark 2.7. *A Vaisman metric is obviously Gauduchon. Indeed, the Vaisman condition $\nabla\theta = 0$ implies the Gauduchon condition $d^*\theta = 0$. Therefore, 2.6 implies the uniqueness of Vaisman metrics.*

Definition 2.8. *Let $(\mathcal{C}, g, \omega)$ be a Kähler manifold. Assume that ρ is a free, proper action of $\mathbb{R}^{>0}$ on $\mathcal{C}$, and g and ω are homogeneous of weight 2:*

$$\mathrm{Lie}_v\, \omega = 2\omega, \quad \mathrm{Lie}_v\, g = 2g,$$

where v is the tangent vector field of ρ. The quotient $\mathcal{C}/\rho$ is called **a Sasakian manifold**. *If $N = \mathcal{C}/\rho$ is given, $\mathcal{C}$ is called* **the Kähler cone of** *N. As a Riemannian manifold, $\mathcal{C}$ is identified with the* **Riemannian cone** *of (N, g_N), $\mathcal{C}(N) = (N \times \mathbb{R}^{>0}, t^2 g_N + dt^2)$.*

The Sasakian manifolds are discussed in [BG], in great detail.

The following characterization of *compact* Vaisman manifolds is known (see [OV1]):

Remark 2.9. *A compact complex manifold (M, J) is Vaisman if it admits a Kähler covering $(\tilde{M}, J, h) \to (M, J)$ such that:*

- *The monodromy group $\Gamma \cong \mathbb{Z}$ acts on $\tilde{M}$ by holomorphic homotheties with respect to h (this means that (M, J) is equipped with an LCK-structure).*
- *$(\tilde{M}, J, h)$ is isomorphic to a Kähler cone over a compact Sasakian manifold S. Moreover, there exists a Sasakian automorphism ϕ and a positive number $q > 1$ such that Γ is isomorphic to the cyclic group generated by $(x, t) \mapsto (\phi(x), tq)$.*

Remark 2.10. *In these assumptions, denote by $\theta^\sharp$ the vector field $t\frac{d}{dt}$ on $\tilde{M} = (S \times \mathbb{R}^{>0}, g_S t^2 + dt^2)$. Choose the metric $g = g_S + dt^2$ on $M = \tilde{M}/\Gamma$. Clearly, $\theta^\sharp$ descends to a Lee field on M, denoted by the same letter. Then $J(\theta^\sharp)$ is tangent to the fibers of the natural projection $\tilde{M} \longrightarrow \mathbb{R}^{>0}$, hence belongs to TS. This vector field is called* **the Reeb field** *of the Sasakian manifold S. Clearly, the orbits of $J(\theta^\sharp)$ on $\tilde{M}$ are precompact (contained in a compact set).*

Remark 2.11. *It will be important for us to note that the Kähler metric h on the covering $\tilde{M} = \mathcal{C}(S) = S \times \mathbb{R}^{>0}$ has a global Kähler potential ϕ, which is expressed as $\phi(x, t) = t^2$. The metric $\phi^{-1} \cdot h$ projects on M into the LCK metric g. Therefore, the Vaisman Hermitian form ω of M is related to the Kähler form $\tilde{\omega}$ on $\tilde{M}$ as follows:*

$$\omega = \phi^{-1}\tilde{\omega} \tag{2.3}$$

Moreover, $\phi = \mid \theta \mid^2$, the norm being taken with respect to the lift of g.

On a Vaisman manifold, the Lee field $\theta^\sharp$ is Killing, parallel and holomorphic. One easily proves that $\mathcal{L}_{\theta^\sharp}\omega = 2\omega$.

Recall from [To] the notion of transverse geometry:

Definition 2.12. *Consider a manifold endowed with a foliation $\mathcal{F}$ with tangent bundle F and normal bundle Q. A differential, or Riemannian, form α on X is* **basic** *(or* **transverse***) if $X \rfloor \alpha = 0$ and $\mathrm{Lie}_X \alpha = 0$ for every $X \in F$. The* **transverse geometry** *of $\mathcal{F}$ is the geometry defined locally on the leaf space of $\mathcal{F}$. A* **Kähler transverse structure** *on $(M, \mathcal{F})$ is a complex Hermitian structure on Q defined by a pair $g_{\mathcal{F}}, \omega_{\mathcal{F}}$ of transverse forms, in such a way that the induced almost complex structure defined locally on the leaf space $M/\mathcal{F}$ is integrable and Kähler.*

Example 2.13. *Let (M, J, ω) be a Vaisman manifold, $\theta^\sharp$ its Lee field. Consider the holomorphic foliation $\mathcal{F}$, generated by $\theta^\sharp$ and $J\theta^\sharp$. The form $\omega - \theta \wedge J\theta$ is* *transverse Kähler. Hence the Vaisman manifolds provide examples of transverse Kähler foliations ([Va], [Ts1]). Similarly, a Sasakian manifold has a transverse Kähler geometry associated to the foliation generated by the Reeb field.*

A compact complex manifold of Vaisman type can have many Vaisman structures, still the Lee field is unique up to homothety:

Proposition 2.14. *If g_1, g_2 are Vaisman metrics on the same compact manifold (M, J), then $\theta_1^{\sharp_{g_1}} = c\theta_2^{\sharp_{g_2}}$, for some real constant c.*

Proof: The result was proven by Tsukada in [Ts2]. Here we include an alternative proof. Recall from [Ve] that for a Vaisman structure (g, J), the two-form

$$\eta := \omega - \theta \wedge J\theta$$

is exact and positive, with the null-space generated by $\langle \theta^\sharp, J\theta^\sharp \rangle$. It is the transverse Kähler form of $(M, \mathcal{F})$ (see 2.13). Let g_1, g_2 be Vaisman metrics, ω_1, ω_2 the corresponding Hermitian forms, θ_i and $\theta_i^\sharp$ the corresponding Lee forms and Lee fields. Consider the $(1,1)$-forms η_1, η_2, defined as above,

$$\eta_i := \omega_i - \theta_i \wedge J\theta_i.$$

Unless their null-spaces coincide, the sum $\eta_1 + \eta_2$ is strictly positive. Then

$$\int_M (\eta_1 + \eta_2)^{\dim M} > 0.$$

This is impossible, because η_i are exact. We obtained that the 2-dimensional bundles generated by $\theta_i^\sharp, J\theta_i^\sharp$ are equal:

$$\langle \theta_1^\sharp, J\theta_1^\sharp \rangle = \langle \theta_2^\sharp, J\theta_2^\sharp \rangle$$

This implies that $\theta_1^\sharp$, considered as a vector in $T^{1,0}(M)$, is proportional to $\theta_2^\sharp$ over $\mathbb{C}$.

$$\theta_1^\sharp = a\theta_2^\sharp + bJ\theta_2^\sharp, \quad a, b \in \mathbb{R}. \tag{2.4}$$

Since $\theta_i^\sharp$ is holomorphic, the proportionality coefficient is constant.

To finish the proof of Proposition 2.14, it remains to show that this proportionality coefficient is real. Here we use Remark 2.10: the orbits of $J\theta_1^\sharp$ should be pre-compact. From (2.4) we obtain

$$J\theta_1^\sharp = aJ\theta_2^\sharp - b\theta_2^\sharp.$$

But $aJ\theta_2^\sharp - b\theta_2^\sharp$ acts on the metric by a homothety, with a coefficient which is proportional to e^{-b}. Therefore, an orbit of this vector field is contained in a compact set if and only if $b = 0$. ∎

Remark 2.15. *Let $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the weight bundle of the Vaisman manifold (M, J, g). The Lee form then is the connection form of the standard Hermitian connection in $L_{\mathbb{C}}$, and one can prove (see [Ve]) that its curvature can be identified with the above form $\eta = \omega - \theta \wedge J\theta$, hence it is exact.*

3 Monge-Ampère equation on LCK-manifolds

In this Section, we prove Theorem 1.2. Clearly, Theorem 1.2 is implied by the following proposition.

Proposition 3.1. *Let (M, J) be a compact complex n-dimensional manifold admitting two Vaisman metrics ω_1 and ω_2, such that the corresponding Lee classes and the volume forms are equal:*

$$[\theta_1] = [\theta_2], \quad \omega_1^n = \omega_2^n. \tag{3.5}$$

Then $\omega_1 = \omega_2$.

Proof: We start with the following claim, which is implied by Tsukada's theorem (Proposition 2.14).

Claim 3.2. *In these assumptions, denote the corresponding Lee fields by $\theta_i^\sharp$, $i = 1, 2$. Then*

$$\theta_1^\sharp = \theta_2^\sharp.$$

Proof: Denote by $\tilde{\omega}_i$ the Kähler forms on $\tilde{M}$ corresponding to ω_i. By construction, $\operatorname{Lie}_{\theta_i^\sharp} \tilde{\omega}_i = 2\tilde{\omega}_i$, where Lie denotes the Lie derivative. Therefore,

$$\operatorname{Lie}_{\theta_i^\sharp} \tilde{\omega}_i^n = 2n\tilde{\omega}_i^n. \tag{3.6}$$

Since $[\theta_1] = [\theta_2]$, the forms $\tilde{\omega}_1$, $\tilde{\omega}_2$ have the same automorphy factors under the deck group of $\tilde{M}$, hence the function $\frac{\tilde{\omega}_1^n}{\tilde{\omega}_2^n}$ is invariant under the monodromy of $\tilde{M}$. We shall consider $\Psi := \frac{\tilde{\omega}_1^n}{\tilde{\omega}_2^n}$ as a function on M. By 2.14, $\theta_1^\sharp = c\theta_2^\sharp$. From (3.6), we obtain

$$\operatorname{Lie}_{\theta_1^\sharp} \Psi = 2n(1 - c)\Psi. \tag{3.7}$$

The function Ψ and the vector field $\theta_1^\sharp$ are defined on a compact manifold M; but, unless $c = 1$, (3.7) implies that the maximum of Ψ decreases or increases monotonously under the action of the coresponding flow of diffeomorphisms. Therefore, (3.7) is possible only if $c = 1$. We proved 3.2. $\square$

Return to the proof of Proposition 3.1. Consider the form

$$\eta_i := \omega_i - \theta_i \wedge J\theta_i. \tag{3.8}$$

This is a positive, exact $(1, 1)$-form on M, which can be interpreted as a curvature of the weight bundle (see the proof of Proposition 2.14). First of all, we deduce from $\eta_1 = \eta_2$ the statement of Proposition 3.1.

Lemma 3.3. *In the assumptions of Proposition 3.1, assume that* $\eta_1 = \eta_2$*, where* η_i *are* $(1,1)$*-forms defined in* (3.8)*. Then* $\omega_1 = \omega_2$*.*

Proof: As follows from (3.8), to prove $\omega_1 = \omega_2$ it suffices to show $\theta_1 = \theta_2$. Let $\tilde{M}$ be the Kähler $\mathbb{Z}$-covering of M, which is a cone over a compact Sasakian manifold, and ϕ_1, ϕ_2 the corresponding Kähler potentials, obtained as in Remark 2.11. It is easy to see that $\theta_i = d\log\phi_i$ and $\eta_i = d^c\theta_i$ ([Ve]). Therefore,

$$\eta_1 - \eta_2 = d^c d\log\left(\frac{\phi_1}{\phi_2}\right) \tag{3.9}$$

The functions ϕ_i are automorphic under the deck group action on $\tilde{M}$, with the same factors of monodromy. Therefore, their quotient $\frac{\phi_1}{\phi_2}$ is well defined on M. By (3.9), $0 = \eta_1 - \eta_2 = d^c d\log\left(\frac{\phi_1}{\phi_2}\right)$, hence $\psi := \log\frac{\phi_1}{\phi_2}$ is plurisubharmonic on a compact complex manifold M. Therefore ψ is constant. This gives $\theta_1 - \theta_2 = d\psi = 0$. Lemma 3.3 is proven. ∎

Return to the proof of Proposition 3.1. Note that η_i are transverse Kähler forms. Since

$$\det\eta_i = (\theta^\sharp \wedge J\theta^\sharp)\rfloor\det\omega_i,$$

it follows that

$$\det\eta_1 = \det\eta_2.$$

Let ρ be a transverse form, defined as $\rho = \sum_{k+l=n-2}\eta_1^k \wedge \eta_2^l$. Then

$$(\eta_1 - \eta_2)\wedge\rho = 0. \tag{3.10}$$

As η_i are both positive, ρ is strictly positive, transversal $(n-2, n-2)$-form. It is well known that on a complex manifold X, any positive $(\dim X - 1, \dim X - 1)$-form is an $(\dim X - 1)$-st power of a Hermitian form. Therefore, there exists a transverse form α such that $\rho = \alpha^{n-2}$. Then (3.10) gives

$$(\eta_1 - \eta_2)\wedge\alpha^{n-2} = 0.$$

From (3.9), we obtain

$$\eta_1 - \eta_2 = dd^c\psi,$$

where $\psi := \log\left(\frac{\phi_1}{\phi_2}\right)$ is a smooth, transversal function on M.

We now associate to α a second-order differential operator $\mathcal{D}$ acting on transverse $\mathcal{C}^\infty$ functions, which is defined as follows. For any transverse function f, $dd^c f\wedge\alpha^{n-2}$ is a transverse top $(n-1, n-1)$ form, and hence there exists a unique transverse function g such that $dd^c f\wedge\alpha^{n-2} = g\cdot\alpha^{n-1}$. We define

$$\mathcal{D}(f) = g, \quad \text{where } dd^c f\wedge\alpha^{n-2} = g\cdot\alpha^{n-1}.$$

In other words,

$$\mathcal{D}(f) = \frac{dd^c f \wedge \alpha^{n-2}}{\alpha^{n-1}}.$$

From the definition, we have $D(\psi) = 0$. Obviously $\mathcal{D}$ has positive symbol on the ring of transverse functions, identified locally with functions on a space of leaves of $\mathcal{F}$[4]. This allows us to apply the generalized maximum principle:

Proposition 3.4. ([PW]) *Let $\mathcal{D}$ be a second order differential operator on $\mathbb{R}^n$ with positive symbol, satisfying $\mathcal{D}(const.) = 0$, and let $f \in \ker \mathcal{D}$ be a function in its kernel. Assume that f has a local maximum. Then f is constant.*

Return to the proof of Theorem 1.2. Recall that from (3.9), we have

$$\eta_1 - \eta_2 = d^c d \log(\psi), \quad \psi \in \ker \mathcal{D}.$$

To show that $\eta_1 = \eta_2$ it is enough to prove that the kernel of $\mathcal{D}$ contains only constant functions. As follows from the generalized maximum principle, a function in $\ker \mathcal{D}$ which has a local maximum is necessarily constant. Since M is compact, any continuous function on M must have a maximum. Therefore, $\psi \in \ker \mathcal{D}$ is constant, and $\eta_1 - \eta_2 = dd^c\psi = 0$. The proof of 3.1 is finished. $\square$

Remark 3.5. *In this Section, we proved the uniqueness of a Vaisman metric satisfying $\omega^n = V$, for a given non-degenerate, positive volume form V on a given compact manifold admitting a Vaisman structure. Following the argument used to prove the Calabi-Yau theorem it seems to be possible to prove* existence *of such solutions as well. Indeed, $\omega^n = V$ is equivalent to*

$$\eta^{n-1} = (\theta^\sharp \wedge J\theta^\sharp) \rfloor V,$$

and this is a transversal Calabi-Yau equation, which is implied by the same arguments as used for the usual Calabi-Yau theorem.

Remark 3.6. *Assuming that Calabi-Yau-type result (as in Remark 3.5) is true in LCK geometry, we obtain a very simple description of the moduli of Vaisman metrics. Indeed, the Vaisman metrics in this case are in one-to-one correspondence with non-degenerate, positive volume forms.*

4 Einstein-Weyl LCK manifolds

Einstein-Weyl structures are defined and studied for their own, see *e.g.* [CP]. Here we specialize the definitions to LCK structures.

The Levi-Civita connection ∇^g of g is not the best tool to study the conformal properties of an LCK manifold. Instead, the **Weyl connection** defined by

[4] In fact, the symbol of $\mathcal{D}$ is equal to the Riemannian form associated with α.

$$\nabla = \nabla^g - \frac{1}{2}\{\theta \otimes Id + Id \otimes \theta + g \otimes \theta^\sharp\}$$

is torsion-free and satisfies $\nabla g = \theta \otimes g$.

The Ricci tensor of the Weyl connection is not symmetric. Hence, to obtain the analogue of the Einstein condition one gives:

Definition 4.1. *An LCK-manifold is* **Einstein-Weyl** *if the symmetric part of the Ricci tensor of the Weyl connection is proportional to the metric. An Einstein-Weyl LCK-manifold is also called* **Hermitian Einstein-Weyl**.

Let ∇ be the Weyl connection on an LCK-manifold. One can see that ∇ is the covariant derivative associated to the connection one-form θ in the weight bundle L. Since θ is closed, we can take a covering $\tilde{M}$ of M, with $\theta = df$, for some function f on $\tilde{M}$. The Weyl connection becomes the Levi-Civita connection for the metric $e^{-f}g$ on $\tilde{M}$. Since $\nabla(e^{-f}g) = \nabla(J) = 0$, $e^{-f}g$ is a Kähler metric. This way one obtains a Kähler covering of an LCK-manifold, starting from a Weyl connection. The converse construction is also clear: The Levi-Civita connection on a Kähler covering $\tilde{M}$ of an LCK-manifold M is independent from homotheties, hence descends to M, and satisfies the conditions for Weyl connection.

This gives the following claim.

Claim 4.2. *Let ∇ be a Weyl connection on a complex Hermitian manifold. Then ∇ satisfies the Einstein-Weyl condition if and only if ∇ is Ricci-flat on the Kähler covering of M.*

Remark 4.3. *Claim 4.2 also follows from Proposition 4.5 (below). Indeed, a trivialization of the weight bundle $L_{\mathbb{C}}$ induces a trivialization of canonical class $K = L_{\mathbb{C}}^{-n}$.*

From a deep result of Gauduchon in [G2], it follows that:

Theorem 4.4. *Let (M, J, g) be a compact Einstein-Weyl LCK manifold. Then the Ricci tensor of the Weyl connection vanishes identically and the Lee form is parallel. In particular, (M, J, g) is Vaisman.*

From Theorem 4.4, we obtain that all Kähler coverings of an Einstein-Weyl LCK-manifold are Ricci-flat. This property can be used as a definition of Einstein-Weyl LCK-manifolds.

The locally conformally Kähler Einstein-Weyl structures can be expressed in terms of the complexified weight bundle.

Proposition 4.5. ([*Ve, Proposition* 5.6]) *Let M be an Einstein-Weyl LCK-manifold, K its canonical class, $L_{\mathbb{C}}$ its weight bundle. Consider K, $L_{\mathbb{C}}$ as Hermitian holomorphic bundles, with the metrics induced from M. Then $L_{\mathbb{C}}^n \cong K^{-1}$.*

5 Bando-Mabuchi theorem in conformal setting

The covering of an Einstein-Weyl LCK-manifold is Ricci-flat. Exploiting the analogy with the Calabi-Yau theorem, one could hope to infer the uniqueness, or even existence, of Einstein-Weyl structure on a complex manifold admitting a Vaisman structure. Unfortunately, this analogy does not work that well. The group of holomorphic automorphisms of the simplest Hermitian Einstein-Weyl manifold, a Hopf surface $H = (\mathbb{C}^2\backslash 0)/\mathbb{Z}$, does not act on H by conformal automorphisms. A closer look at the Monge-Ampère equation controlling the Einstein-Weyl condition explains this problem immediately.

Let (M, J, g) be an Einstein-Weyl LCK-manifold equipped with a Vaisman metric, and $\tilde{M}$ its Kähler covering, which trivializes L. From Proposition 4.5, it is clear that $\tilde{M}$ has trivial canonical class. Let Ω be a section of canonical class of $\tilde{M}$ which is equivariant under the monodromy action. Such a section is unique up to a constant. Indeed, if Ω_1, Ω_2 are two equivariant sections of canonical class, the quotient $\frac{\Omega_1}{\Omega_2}$ is a holomorphic function on $\tilde{M}$ which is invariant under monodromy, hence descends to a global holomorphic function on M. Therefore $\frac{\Omega_1}{\Omega_2} = const$. Rescaling Ω such that $|\Omega| = 1$, we obtain

$$\Omega \wedge \bar{\Omega} = \frac{1}{n!\,2^n}\tilde{\omega}^n,$$

where $n = \dim_{\mathbb{C}} M$ and $\tilde{\omega}$ is the Kähler form on $\tilde{M}$. In particular, given two Einstein-Weyl structures with the Kähler forms $\tilde{\omega}_1$ and $\tilde{\omega}_2$, we always have $\tilde{\omega}_1^n = \lambda\tilde{\omega}_2^n$, where λ is a positive constant. After rescaling, we may also assume that

$$\det\tilde{\omega}_1 = \det\tilde{\omega}_2, \tag{5.11}$$

where $\det\tilde{\omega}_i = \tilde{\omega}_i^n$, $n = \dim_{\mathbb{C}} M$. Comparing this equation with (2.3), we find that (5.11) is equivalent to

$$\frac{\det\omega_1}{\det\omega_2} = \frac{\phi_2}{\phi_1}, \tag{5.12}$$

where ϕ_i denotes the Kähler potential of the Kähler metrics ω_i. Writing $\psi_i := \log\phi_i$, $\psi := \psi_1 - \psi_2$, $\eta_i := dd^c\psi_i$, $\eta := \eta_1 - \eta_2$, and using

$$\frac{\det\omega_1}{\det\omega_2} = \frac{\det\eta_1}{\det\eta_2} \tag{5.13}$$

(see the proof of Proposition 3.1), we find that (5.12) is equivalent to the transversal version of Aubin-Calabi-Yau equation:

$$\log\left(\frac{\det(\eta - dd^c\psi)}{\det\eta}\right) = \epsilon\psi + const, \tag{5.14}$$

with $\epsilon = 1$. This equation is well-known in the theory of Kähler-Einstein manifolds. It is easy to see that it has a unique solution $\psi = 0$ for $\epsilon \leq 0$. When

$\epsilon = 1$, this becomes much more difficult. Bando and Mabuchi ([BM]) studied this equation in order to prove that a Kähler-Einstein metric on a Fano manifold is unique, up to a constant multiplier and a holomorphic automorphism. We hope that their argument will carry over in Einstein-Weyl geometry, proving the following conjecture.

Conjecture 5.1. *Let (M, J) be a compact complex manifold. Then it admits at most one Einstein-Weyl structure, up to a holomorphic automorphism.*

This conjecture cannot be very easy, because it implies Bando-Mabuchi theorem. Indeed, assume that $M = \mathrm{Tot}(K\backslash 0)/\langle q\rangle$, where $K\backslash 0$ is the space of non-zero vectors in the canonical line bundle of a Fano manifold X, and $\langle q\rangle$ a holomorphic $\mathbb{Z}$-action generated by $v \longrightarrow qv$, where q is a fixed complex number, $|q| > 1$. The manifold M clearly admits a Vaisman structure; it is Einstein-Weyl if and only if X is Kähler-Einstein, and the Kähler-Einstein metrics on X correspond uniquely to the Einstein-Weyl structures on M. Therefore, 5.1 for such M is equivalent to the Bando-Mabuchi theorem for X.

Acknowledgements: This note originated from discussions at the Max Planck Institute in Bonn whose support is gratefully acknowledged. The second author is grateful to Semyon Alesker, who explained to him the generalized maximum principle.

References

[B] A. Besse, Einstein manifolds, Springer-Verlag, 1987.

[BG] C.P. Boyer, K. Galicki, Sasakian geometry, Oxford mathematical monographs, Oxford Univ. Press, 2008.

[BM] S. Bando and T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions,* Algebraic Geometry, Sendai 1985, 11-40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam-New York, 1987.

[C] E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 78–89. Princeton University Press, Princeton, N. J., 1957.

[CP] D. Calderbank, H. Pedersen, *Einstein-Weyl geometry*, in "Surveys in Differential Geometry: Essays on Einstein manifolds", M. Wang and C. LeBrun eds., International Press 2000, 387–423.

[CLPP] M. Cvetic, H. Lu, D. N. Page and C. N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions,* Phys. Rev. Lett. 95, 071101 (2005) [arXiv:hep-th/0504225].

[DO] S. Dragomir and L. Ornea, Locally conformal Kähler geometry, Progress in Math. **155**, Birkhäuser, Boston, Basel, 1998.

[G1] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann., 267 (1984), 495–518.

[G2] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*, J. reine Angew. Math. **469** (1995), 1–50.

[GMSW1] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$,* Adv. Theor. Math. Phys. 8, 711 (2004) [arXiv:hep-th/0403002].

[GMSW2] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, *A new infinite class of Sasaki-Einstein manifolds,* Adv. Theor. Math. Phys. 8, 987 (2006) [arXiv:hep-th/0403038].

[GH] Griffiths, Ph., Harris, J., *Principles of Algebraic Geometry*, Wiley-Intersience, New York, 1978.

[MSY] D. Martelli, J. Sparks, S.-T. Yau *Sasaki-Einstein Manifolds and Volume Minimisation*, 83 pages, hep-th/0603021

[O] L. Ornea, *Locally conformally Kähler manifolds. A selection of results*, Lecture Notes of *Seminario Interdisciplinare di Matematica*, **4** (2005), 121–152. See also math.DG/0411503.

[OV1] L. Ornea and M. Verbitsky, *Structure theorem for compact Vaisman manifolds*, Math. Res. Lett. **10** (2003), 799–805, see also math.DG/0305259

[OV2] L. Ornea and M. Verbitsky, *An immersion theorem for compact Vaisman manifolds*, Math. Ann. **332** (2005), no. 1, 121–143, see also math.AG/0306077

[OV3] L. Ornea and M. Verbitsky, *Locally conformal manifolds with potential*, Math. Ann. (to appear), math.AG/0407231.

[PPS] H. Pedersen, Y.S. Poon, A. Swann, *The Einstein-Weyl equations in complex and quaternionic geometry*, Diff. Geom. Appl. **3** (1993), 309–321.

[PW] M.H. Protter, H.F. Weinberger, Maximum principles in Differential equations, Springer-Verlag, 1984.

[To] P. Tondeur, Foliations on Riemannian manifolds, Universitext, Springer-Verlag 1988.

[Ts1] K. Tsukada, *Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds*, Compositio Math. 93 (1994), no. 1, 1-22.

[Ts2] K. Tsukada, *The canonical foliation of a compact generalized Hopf manifold*, Differential Geom. Appl. **11** (1999), 13–28.

[Va] VAISMAN, I. *Generalized Hopf manifolds*, Geom. Dedicata 13 (1982), no. 3, 231-255.

[Ve] M. VERBITSKY, *Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds* (Russian). Tr. Mat. Inst. Steklova **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 64–91; translation in Proc. Steklov Inst. Math. 2004, **246**, 54–78. Also available as math.DG/0302219.

[Y] YAU, S. T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I.*, Comm. on Pure and Appl. Math. 31, 339-411 (1978).

Received: 19.07.2008.

University of Bucharest,
Faculty of Mathematics,
14 Academiei str., 70109 Bucharest, Romania.
and
Institute of Mathematics Simion Stoilow
of the Romanian Academy,
21, Calea Grivitei Str.
010702-Bucharest, Romania
E-mail: `Liviu.Ornea@imar.ro`,
`lornea@gta.math.unibuc.ro`

University of Glasgow, Department of Mathematics,
15 University Gardens, Glasgow, Scotland.
Institute of Theoretical and Experimental Physics
B. Cheremushkinskaya, 25, Moscow, 117259, Russia
E-mail: `verbit@maths.gla.ac.uk`
`verbit@mccme.ru`