On the geometrized Skyrme and Faddeev models

Radu Slobodeanu

[a] *Faculty of Physics, Bucharest University, CP Mg-11, RO - 077125.*
E-mail: radualexandru.slobodeanu@g.unibuc.ro

Abstract

The higher-power derivative terms involved in the Faddeev-Hopf and Skyrme energy functionals correspond to σ_2–energy, proposed by Eells and Sampson in [6]. I present here a detailed study of Euler-Lagrange equations associated to this energy and its second variation. Geometrically interesting examples of (stable) critical points are outlined.

1 Introduction

Common tools in quantum field theory, non-linear σ-models are known in differential geometry mainly through the problem of *harmonic maps* between Riemannian manifolds. Namely a (smooth) mapping $\varphi : (M, g) \to (N, h)$ is harmonic if it is critical point for the *energy functional* [6],

$$\mathcal{E}(\varphi) = \frac{1}{2}\int_M \|\mathrm{d}\varphi\|^2 \nu_g,$$

a generalization of the *kinetic energy* of Classical Mechanics.

Less discussed from differential geometric point of view are *Skyrme* and *Faddeev-Hopf models*, which are σ–models with additional fourth-power derivative terms (for an overview including recent progress concerning both models, see [10]).

The first one was proposed in the sixties by Tony Skyrme [16], to model baryons as topological solitons of pion fields (meanwhile it has been shown to be the low energy limit of QCD in the $1/N_c$ expansion). So a baryon is represented by an energy minimising, topologically nontrivial map $\varphi : \mathbb{R}^3 \to \mathbb{S}^3$ with $\{|x| \to \infty\} \mapsto 1$, called *skyrmion.* Their topological invariant called *degree* is identified with the *baryon number.* The static (conveniently renormalized) *Skyrme energy functional* is

$$\mathcal{E}_{\mathtt{Skyrme}}(\varphi) = \int \frac{1}{2}\|\mathrm{d}\varphi\|^2 + \frac{1}{4}\|\mathrm{d}\varphi \wedge \mathrm{d}\varphi\|^2. \tag{1.1}$$

This energy has a *topological lower bound*: $\mathcal{E}_{\mathtt{Skyrme}}(\varphi) \geq 12\pi^2|\mathrm{deg}\varphi|$.

In the second one, stated in 1975 by Ludvig Faddeev and Antti J. Niemi [7], the configuration fields are mappings $\varphi : \mathbb{S}^3 \to \mathbb{S}^2 \subset \mathbb{R}^3$, supposed among other things to model the gluon flux tubes in hadrons. The static energy in this case is given by

$$\mathcal{E}_{\mathtt{Faddeev}}(\varphi) = \int c_2\|\mathrm{d}\varphi\|^2 + c_4\langle \mathrm{d}\varphi \wedge \mathrm{d}\varphi, \varphi\rangle^2 \tag{1.2}$$

Again the field configurations are indexed by an invariant, their *Hopf number*: $Q \in \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ ("topological charge", "linking number") and the energy has a *topological* *bound*: $\mathcal{E}_{\texttt{Faddeev}}(\varphi) \geq c \cdot |Q|^{3/4}$. Although this model can be viewed as a constrained variant of the Skyrme model, it exhibits important specific properties, e.g. it allows *knotted solitons* [1].

Both models rise the same kind of topologically constrained minimization problem: find out static energy minimizers among each topological class (i.e. of prescribed baryon or Hopf number). We can give an unitary treatment for both if we take into account that they are particular cases of the following energy-type functional:

$$\mathcal{E}_{\sigma_{1,2}} : \mathcal{C}^{\infty}(M, N) \to \mathbb{R}_{+}, \qquad \mathcal{E}_{\sigma_{1,2}}(\varphi) = \frac{1}{2}\int_{M} \left[\|\mathrm{d}\varphi\|^2 + \kappa \cdot \sigma_2(\varphi)\right] \nu_g, \tag{1.3}$$

where (M, g), (N, h) are (smooth) Riemannian manifolds, $\kappa \geq 0$ is a coupling constant and $\sigma_2(\varphi)$ is the second elementary symmetric function of the eigenvalues of $\varphi^* h$ with respect to g.

Even if the variational problem for the σ_p-energy has already been treated in [3, 5, 21], very little is known about its solutions. From our point of view, the particularities of $p = 2$ case are worth to be outlined for their differential geometric interest in its own, if not for providing possible hints in the identification or description of solitons for the original physical models.

The present generalization of (1.1) and (1.2) was proposed in [11, 13]. Other generalizations of the Skyrme and Faddeev energies are discussed in [9, 18, 22].

2 Higher power energies and the Cauchy-Green tensor

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ a smooth mapping between Riemannian manifolds. The so called **first fundamental form** of φ is the symmetric, positive semidefinite 2-covariant tensor field on M, defined as $\varphi^* h$, cf. [5]. Alternatively, using the musical isomorphism, we can see it as the endomorphism $\mathsf{C}_{\varphi} = \mathrm{d}\varphi^t \circ \mathrm{d}\varphi : TM \to TM$, where $\mathrm{d}\varphi^t : TN \to TM$ denote *the adjoint* of $\mathrm{d}\varphi$. When $m = n = 3$, this corresponds to the **(right) Cauchy-Green (strain) tensor** of a deformation in non-linear elasticity (we shall maintain this name for C_{φ} in the general case).

The Cauchy-Green tensor is always diagonalizable; let $\lambda_1^2 \geq \cdots \geq \lambda_r^2 \geq \lambda_{r+1}^2 = ... = \lambda_m^2 = 0$ be its (real, non-negative) eigenvalues (where $r := \mathrm{rank}(\mathrm{d}\varphi)$ everywhere). Recall that λ_i are also called *principal distortion coefficients of* φ.

The *elementary symmetric functions* in the eigenvalues of $\varphi^* h$ represent a measure of the geometrical distortion induced by the map [2]. They are called *principal invariants* [3] of $\mathrm{d}\varphi$:

$$\sigma_1(\varphi) = \sum_{i=1}^{n} \lambda_i^2; \qquad \sigma_2(\varphi) = \sum_{i<j=1}^{n} \lambda_i^2 \lambda_j^2; \qquad ...... ; \qquad \sigma_n(\varphi) = \lambda_1^2 \lambda_2^2 \cdots \lambda_n^2,$$

[1] V. Arnold gives a nice interpretation of this energy: "... the functional on such mappings that is a (weighted) sum of two terms. The first term is the Dirichlet integral (of the squared derivative) of the map φ. The second term is the energy of the corresponding vector field directed along the fibers of the map".

[2] The first characterizes the behaviour of lengths ratio: $\|\mathrm{d}\varphi(X)\|^2 \leq \sigma_1 \|X\|^2$, the second of area elements ratio: $\|\mathrm{d}\varphi(X) \wedge \mathrm{d}\varphi(Y)\|^2 \leq \sigma_2 \|X \wedge Y\|^2$ and so on.

[3] The reason behind this name is that two linear mappings are *orthogonally equivalent* if and only if they have the same principal invariants.

or, with alternative notations:

$$\sigma_1(\varphi) = 2e(\varphi); \qquad \sigma_2(\varphi) = \| \wedge^2 \mathrm{d}\varphi\|^2; \qquad ...; \qquad \sigma_n(\varphi) = [v(\varphi)]^2,$$

where $e(\varphi) = \frac{1}{2}\|\mathrm{d}\varphi\|^2$ is the *energy density of* φ and $v(\varphi) = \sqrt{\det(\varphi^* h)}$ is the *volume (density) of* φ, cf. [6, (3)].

Remark 2.1. At any point of M, there is an orthonormal basis $\{e_i\}$ of corresponding eigenvectors for $\varphi^* h$ at that point. Moreover, according to [14, Lemma 2.3], we have a *local* orthonormal frame of eigenvector fields, around any point of a dense open subset of M. In particular, for such local "eigenfields" we have: $\varphi^* h(e_i, e_j) = \delta_{ij}\lambda_i^2$, so $\{\mathrm{d}\varphi(e_i)\}$ are orthogonal with norm $\|\mathrm{d}\varphi(e_i)\| = \lambda_i$.

Remark 2.2 (Classes of smooth mappings characterized by their distortion)**.** (i) When $m \geq n$ and $r \in \{0, n\}$, if $\lambda_1^2 = ... = \lambda_r^2 = \lambda^2$, we say that our map is **horizontally weakly conformal** (HWC) or **semiconformal**, cf. [2, p. 46]. If moreover $\mathrm{grad}\lambda \in \mathrm{Ker}\,\mathrm{d}\varphi$, then the map is called **horizontally homothetic** (HH).

(ii) When $m \leq n$ and $r \in \{0, m\}$, if $\lambda_1^2 = ... = \lambda_r^2 = \lambda^2$, we say that our map is **(weakly) conformal**, cf. [2, p. 40]. If $m = n$ this notion is equivalent to the above one.

(iii) When the codomain is endowed with an almost Hermitian structure J, a class of mappings that includes the above ones was defined by $[\mathrm{d}\varphi \circ \mathrm{d}\varphi^t, J] = 0$, cf. [12]. These maps are called **pseudo horizontally weakly conformal maps** (PHWC). For the extension of this concept, the discussion on the corresponding notion of **pseudo horizontally homothetic** (PHH) maps see [17] and the references therein. Here we point out only that PHWC condition implies that the eigenvalues of $\varphi^* h$ have multiplicity 2 ($\lambda_1^2 = \lambda_2^2$, $\lambda_3^2 = \lambda_4^2$, ... $\lambda_{r-1}^2 = \lambda_r^2$), the eigenspaces are invariant w.r.t. the induced metric almost f–structure, F^φ, on the domain and that the PHH condition in a broader sense is $F^\varphi\left[(\nabla_X F^\varphi)(X) + (\nabla_{F^\varphi X} F^\varphi)(F^\varphi X)\right] = 0$, $\forall X \in \mathrm{Ker}\left((F^\varphi)^2 + I\right)$.

According to [6], up to a half factor, we shall call σ_p**–energy**, the following functional

$$\mathcal{E}_{\sigma_p}(\varphi) = \frac{1}{2}\int_M \sigma_p(\varphi)\nu_g. \tag{2.1}$$

Therefore, the generalized energy (1.3) reads

$$\mathcal{E}_{\sigma_{1,2}}(\varphi) = \mathcal{E}_{\sigma_1}(\varphi) + \kappa\mathcal{E}_{\sigma_2}(\varphi) = \frac{1}{2}\int_M \left(\sum_i \lambda_i^2 + \kappa\sum_{i<j}\lambda_i^2\lambda_j^2\right)\nu_g. \tag{2.2}$$

Let us recall another type of (higher power) energy-type functional that will be useful for our further discussion. The p**-energy** of a (smooth) map is defined as:

$$\mathcal{E}_p(\varphi) = \frac{1}{p}\int_M \|\mathrm{d}\varphi\|^p\nu_g$$

The corresponding Euler-Lagrange operator/equations are, cf. [19]

$$\tau_p(\varphi) := \|\mathrm{d}\varphi\|^{p-2}\left[\tau(\varphi) + (p-2)\mathrm{d}\varphi(\mathrm{grad}(\ln\|\mathrm{d}\varphi\|))\right] \equiv 0,$$

where $\tau(\varphi) := \mathrm{trace}\nabla \mathrm{d}\varphi$ is the *tension field* of φ (i.e. the Euler-Lagrange operator associated to $\mathcal{E}_{\sigma_1} =: \mathcal{E}$).
In particular, for $p = 4$, we have $\|\mathrm{d}\varphi\|^2 \left[\tau(\varphi) + 2\mathrm{d}\varphi(\mathrm{grad}(\ln \|\mathrm{d}\varphi\|))\right] = 0$, or equivalently

$$e(\varphi)\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}(e(\varphi)) = 0. \tag{2.3}$$

Remark 2.3. It is easy to see that $\mathcal{E}_{\sigma_2}(\varphi) = \frac{1}{4}\int_M (\|\mathrm{d}\varphi\|^4 - \|\varphi^* h\|^2)\nu_g = \mathcal{E}_4(\varphi) - \frac{1}{4}\int_M \|\varphi^* h\|^2 \nu_g$. In fact, the relation with 4-energy is more clear if we point out that, according to *Newton's inequalities*,

$$\mathcal{E}_{\sigma_2}(\varphi) \leq \frac{n-1}{n}\mathcal{E}_4(\varphi).$$

with equality if and only if $\lambda_1 = ... = \lambda_n$. If in addition φ is of bounded dilation of order K, we have also the reversed inequality

$$\frac{2}{n^2 K^2}\mathcal{E}_4(\varphi) \leq \mathcal{E}_{\sigma_2}(\varphi).$$

3 Euler-Lagrange equations for σ_2–energy

Let $\{\varphi_t\}$ a (smooth) variation of φ with variation vector field $v \in \Gamma(\varphi^{-1}TN)$, i.e.

$$v(x) = \frac{\partial \varphi_t}{\partial t}(x)\Big|_{t=0} \in T_{\varphi(x)}N, \qquad \forall x \in M.$$

In this section, we are looking for critical points of σ_2–energy, i.e. mappings that satisfy $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \mathcal{E}_{\sigma_2}(\varphi_t) = 0$, for any variation. For simplicity I call these maps σ_2*–critical.*

Recall that to every $v \in \Gamma(\varphi^{-1}TN)$, we can associate a vector field on M, $X_v \in (\mathrm{Ker}\,\mathrm{d}\varphi)^\perp$, defined by:

$$g(X_v, Y) = h(v, \mathrm{d}\varphi(Y)), \quad \forall Y \in \Gamma(TM).$$

Remark 3.1. Denoting $\alpha_v(Y,Z) := h\left(\nabla^\varphi_Y v, \mathrm{d}\varphi(Z)\right), \forall Y, Z \in \Gamma(TM)$ and $\mathrm{div}^\varphi v := \mathrm{trace}\,\alpha_v$, we can easily check that:

(*i.*) $\frac{\partial}{\partial t}\Big|_{t=0}\varphi_t^* h(Y,Z) = \alpha_v(Y,Z) + \alpha_v(Z,Y)$;

(*ii.*) $\alpha_v(Y,Z) = g\left(\nabla_Y X_v, Z\right) - h(v, \nabla \mathrm{d}\varphi(Y,Z))$;

(*iii.*) $\mathrm{div}^\varphi v = \mathrm{div} X_v - h(v, \tau(\varphi))$;

(*iv.*) φ is harmonic iff $\mathrm{div}^\varphi v = \mathrm{div} X_v, \ \forall v \in \Gamma(\varphi^{-1}TN)$.

For further use let us denote by A_v the (1,1)-tensor on M associated to α_v, i.e. $\alpha_v(Y,Z) = g(A_v Y, Z)$.

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \mathcal{E}_{\sigma_2}(\varphi_t) = \frac{1}{2}\int_M \sum_{i<j} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (\|\mathrm{d}\varphi_t(e_i)\|^2 \|\mathrm{d}\varphi_t(e_j)\|^2) = \frac{1}{2}\int_M \sum_i \lambda_i^2 \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (\|\mathrm{d}\varphi_t\|^2 - \|\mathrm{d}\varphi_t(e_i)\|^2)$$
$$= \int_M \sum_i \lambda_i^2 \left\{ \sum_k h(\nabla^{\varphi}_{e_k} v, \mathrm{d}\varphi(e_k)) - h(\nabla^{\varphi}_{e_i} v, \mathrm{d}\varphi(e_i)) \right\} \nu_g$$
$$= \int_M h(v, -2[e(\varphi)\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}e(\varphi))])\nu_g +$$
$$+ \int_M \left\{ h\left(v, \sum_i \lambda_i^2 \nabla \mathrm{d}\varphi(e_i, e_i)\right) - \sum_i \lambda_i^2 g(\nabla_{e_i} X_v, e_i) \right\} \nu_g.$$

Denote $\widetilde{X} = \sum_i \lambda_i^2 g(X_v, e_i) e_i$. Then:

$$\begin{aligned} \mathrm{div}\widetilde{X} - \sum_i \lambda_i^2 g(\nabla_{e_i} X_v, e_i) =& g\left(X_v, \sum_k \left[e_k(\lambda_k^2) + \sum_i (\lambda_i^2 - \lambda_k^2) g(\nabla_{e_i} e_i, e_k)\right] e_k \right) \\ =& h(v, \mathrm{d}\varphi([\mathrm{div}\varphi^* h]^\sharp)). \end{aligned} \tag{3.1}$$

Notice that $[\mathrm{div}\varphi^* h]^\sharp = \mathrm{div}\mathsf{C}_\varphi$ and $\mathrm{trace}(\nabla \mathrm{d}\varphi) \circ \mathsf{C}_\varphi = \sum_i \lambda_i^2 \nabla \mathrm{d}\varphi(e_i, e_i)$.

Definition 3.1. We call σ_2**–tension field** of the map φ the following section of of the pull-back bundle $\varphi^{-1}TN$:

$$\tau_{\sigma_2}(\varphi) = 2[e(\varphi)\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}e(\varphi))] - \mathrm{trace}(\nabla \mathrm{d}\varphi) \circ \mathsf{C}_\varphi - \mathrm{d}\varphi(\mathrm{div}\mathsf{C}_\varphi).$$

We have obtained the following

Proposition 3.1 (The first variation formula)**.**

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \mathcal{E}_{\sigma_2}(\varphi_t) = -\int_M h(v, \tau_{\sigma_2}(\varphi))\nu_g.$$

In particular, a map φ is σ_2–critical if it satisfies the following Euler-Lagrange equations

$$2[e(\varphi)\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}e(\varphi))] - \mathrm{trace}(\nabla \mathrm{d}\varphi) \circ \mathsf{C}_\varphi - \mathrm{d}\varphi(\mathrm{div}\mathsf{C}_\varphi) = 0. \tag{3.2}$$

The following corollary is to be compared with the results in [17].

Corollary 3.1. (*i*) *Any* ***totally geodesic map*** *is σ_2–critical.*

(*ii*) *A* ***harmonic map*** *will be (also) σ_2–critical if and only if:*

$$\mathrm{d}\varphi(\mathrm{grad}e(\varphi)) = \sum_i \lambda_i^2 \nabla \mathrm{d}\varphi(e_i, e_i). \tag{3.3}$$

(*iii*) *A* ***HC submersion*** *is σ_2–critical if and only if:*

$$(n-4)\mathrm{grad}^{\mathcal{H}}(\ln\lambda) + (m-n)\mu^{\mathcal{V}} = 0, \tag{3.4}$$

that is if and only if it is 4-harmonic.

*(iii') A **HH submersion** is σ_2–critical if and only if it has minimal fibres.*

*(iii'') A **HC submersion onto a four-manifold** is σ_2–critical if and only if it has minimal fibres.*

*(iv) A **PHH submersion with minimal fibres** is σ_2–critical if and only if it has constant energy density, $e(\varphi)$, along horizontal directions.*

Proof. (i) By definition, a totally geodesic map satisfies $\nabla \mathrm{d}\varphi = 0$. In this case it is known that $\varphi^* h$ is parallel and its eigenvalues are constant. Consequently every term in (3.2) cancels.

(ii) Recall that, for any smooth map we have the identity (cf. [2, Lemma 3.4.5]):

$$\mathrm{div} S(\varphi) = \mathrm{d}e(\varphi) - \mathrm{div}\varphi^* h = -h(\tau(\varphi), \mathrm{d}\varphi), \tag{3.5}$$

where $S(\varphi) := e(\varphi)g - \varphi^* h$ is the *stress-energy tensor* of the map.
In particular, for a harmonic map we have $\tau(\varphi) = 0$, so $\mathrm{d}\varphi(\mathrm{grad} e(\varphi)) = \mathrm{d}\varphi(\mathrm{div}\mathsf{C}_\varphi)$, relation that simplifies (3.2) to (3.3).

(iii) For a HC submersion we have $\mathsf{C}_\varphi|_{\mathcal{H}} = \lambda^2 Id$ (where $\mathcal{H}$ is the horizontal distribution). So the terms involving the Cauchy-Green tensor is (3.2) are equal to

$$\mathrm{trace}(\nabla \mathrm{d}\varphi) \circ \mathsf{C}_\varphi + \mathrm{d}\varphi(\mathrm{div}\mathsf{C}_\varphi) = \mathrm{trace}\nabla(\mathrm{d}\varphi \circ \mathsf{C}_\varphi) = \lambda^2 \tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}\lambda^2).$$

Recall that for HC submersions (of dilation λ) the tension field is given by [2, Prop. 4.5.3]:

$$\tau(\varphi) = -\mathrm{d}\varphi\left((n-2)\mathrm{grad}\ln\lambda + (m-n)\mu^{\mathcal{V}}\right).$$

Replacing the two above identities in (3.2) and taking into account that $e(\varphi) = (n/2)\lambda^2$ we get the equation (3.4). Statements (iii') and (iii'') are obvious consequences of this equation.

(iv) PHH submersions with minimal fibres are, in particular, harmonic maps. As for any PHWC mapping the eigenvalues of $\varphi^* h$ have multiplicity equal to 2 and $\varphi^* h$ is F-invariant, according to (ii) such a map must satisfy:

$$\mathrm{d}\varphi(\mathrm{grad} e(\varphi)) = \sum_i \lambda_i^2 [\nabla \mathrm{d}\varphi(e_i, e_i) + \nabla \mathrm{d}\varphi(Fe_i, Fe_i)].$$

But PHH hypothesis assures precisely that $\nabla \mathrm{d}\varphi(X, X) + \nabla \mathrm{d}\varphi(FX, FX) = 0, \forall X \in \Gamma(\mathcal{H})$. Then our conclusion easily follows. ∎

Remark 3.2. The Euler-Lagrange operator of $\mathcal{E}_{\sigma_p}$ that has already been derived in [21]:

$$\tau_{\sigma_p}(\varphi) = \mathrm{trace}\nabla(\mathrm{d}\varphi \circ \chi_{p-1}(\varphi)),$$

where $\chi_{p-1}(\varphi)$ is the *Newton tensor*. In $p = 2$ case, $\chi_1(\varphi) = 2e(\varphi)Id_{TM} - \mathrm{d}\varphi^t \circ \mathrm{d}\varphi$ and then we can obtain easily the equation (3.2). Nevertheless, in this particular case, I have prefered to derive the first variation *ab initio*, for the sake of completeness (as the reader access to the reference [21] might be difficult).

4 Weak σ_2-Stability

Let $\{\varphi_{t,s}\}$ a (smooth) two-parameter variation of φ with variation vector fields $v, w \in \Gamma(\varphi^{-1}TN)$, i.e.

$$v(x) = \frac{\partial \varphi_{t,s}}{\partial t}(x)\Big|_{(t,s)=(0,0)}, \qquad w(x) = \frac{\partial \varphi_{t,s}}{\partial s}(x)\Big|_{(t,s)=(0,0)}, \qquad \forall x \in M.$$

We ask when the following bilinear function is positive semi-definite for a σ_2-critical mapping φ, which will be consequently called *stable* critical point:

$$\mathrm{Hess}^{\sigma_2}_{\varphi}(v,w) = \frac{\partial^2}{\partial t \partial s}\mathcal{E}_{\sigma_2}(\varphi_{t,s})\Big|_{(t,s)=(0,0)}$$

Let us now recall some standard notations: $\langle \cdot, \cdot \rangle$ is the metric induced by the base manifold metric on various tensor bundles on it (and $|\cdot|$ is the corresponding norm); Ric^{φ} is the fiberwise linear bundle map on $\varphi^{-1}TN$ defined by $\mathrm{Ric}^{\varphi}(v) = \sum_{i=1}^{m} R^N(v, \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i)$; $(\nabla^{\varphi})^2$ is the second order operator on $\Gamma(\varphi^{-1}TN)$ defined as $[(\nabla^{\varphi})^2 v](X,Y) = \nabla^{\varphi}_X \nabla^{\varphi}_Y v - \nabla^{\varphi}_{\nabla_X Y} v$; the rough Laplacian along φ is $\Delta^{\varphi} = \mathrm{trace}(\nabla^{\varphi})^2$ and on compactly supported sections has the property: $\int_M h(\Delta^{\varphi} v, v)\nu_g = -\int_M \sum_i h(\nabla^{\varphi}_{e_i} v, \nabla^{\varphi}_{e_i} v)\nu_g := -\int_M \langle \nabla^{\varphi} v, \nabla^{\varphi} v\rangle \nu_g$.

Proposition 4.1 (The second variation formula)**.**

$$\begin{aligned}
\frac{\partial^2}{\partial t \partial s}\mathcal{E}_{\sigma_2}(\varphi_{t,s})\Big|_{(0,0)} =& 2\int_M \{\mathrm{div}^{\varphi} v \cdot \mathrm{div}^{\varphi} w + e(\varphi)\left[\langle \nabla^{\varphi} v, \nabla^{\varphi} w\rangle - h(\mathrm{Ric}^{\varphi} v, w)\right]\}\,\nu_g \\
&+ \int_M h\left(w, 2\mathrm{trace}(\nabla \mathrm{d}\varphi)\circ A_v + \mathrm{trace}[(\nabla^{\varphi})^2 v + R^N(v, \mathrm{d}\varphi)\mathrm{d}\varphi]\circ \mathsf{C}_{\varphi}\right)\nu_g \\
&+ \int_M \left\{X_w(\mathrm{div} X_v) + h(\mathrm{trace}(\nabla^{\varphi})^2 v + \mathrm{Ric}^{\varphi} v, \mathrm{d}\varphi(X_w))\right\}\nu_g \\
&+ \int_M \left\{-h\left(\nabla^{\varphi}_{X_w}\tau(\varphi), v\right) + h\left(w, \nabla^{\varphi}_{\mathrm{div}\mathsf{C}_{\varphi}} v\right)\right\}\nu_g.
\end{aligned} \tag{4.1}$$

Proof. We have:

$$\frac{\partial^2}{\partial t \partial s}\mathcal{E}_{\sigma_2}(\varphi_{t,s}) = -\int_M \left\{h\left(\nabla^{\Phi}_{\partial/\partial t}\frac{\partial \Phi}{\partial s}, \tau_{\sigma_2}(\varphi_{t,s})\right) + h\left(\frac{\partial \Phi}{\partial s}, \nabla^{\Phi}_{\partial/\partial t}\tau_{\sigma_2}(\varphi_{t,s})\right)\right\}\nu_g,$$

where $\tau_{\sigma_2}(\varphi) = \tau_4(\varphi) - \mathrm{trace}(\nabla \mathrm{d}\varphi)\circ \mathsf{C}_{\varphi} - \mathrm{d}\varphi(\mathrm{div}\mathsf{C}_{\varphi})$ is the Euler-Lagrange operator calculated in the previous section ($\tau_4(\cdot)$ is the 4-tension field, cf. Remark 2.1).

The first line in (4.1) is derived from $\tau_4(\varphi_{t,s})$ term, cf. [19] (for a detailed proof see [1]).

Let us derive the other two terms. The variation of the term $\mathrm{trace}(\nabla \mathrm{d}\varphi)\circ \mathsf{C}_{\varphi}$ gives us:

$$\begin{aligned}
&h\left(\frac{\partial \Phi}{\partial s}, \nabla^{\Phi}_{\partial/\partial t}\left[\|\mathrm{d}\varphi_{t,s}(e_i)\|^2 \cdot \nabla \mathrm{d}\varphi_{t,s}(e_i, e_i)\right]\right)\Big|_{(t,s)=(0,0)} = \\
&2\alpha_v(e_i, e_i) h(w, \nabla \mathrm{d}\varphi(e_i, e_i)) + h\left(w, \sum_i \lambda_i^2\left[(\nabla^{\varphi})^2_{e_i, e_i} v + R^N(v, \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i)\right]\right).
\end{aligned}$$

The variation of the term $\mathrm{d}\varphi(\mathrm{div}\mathsf{C}_\varphi)$ gives us:

$$
\begin{aligned}
& h\left(\frac{\partial \Phi}{\partial s}, \nabla^{\Phi}_{\partial/\partial t}\left[(\mathrm{div}\varphi^*_{t,s}h)(e_j)\,\mathrm{d}\varphi_{t,s}(e_j)\right]\right)\Big|_{(t,s)=(0,0)} = \\
& h\left(\frac{\partial \Phi}{\partial s}, \nabla^{\Phi}_{\partial/\partial t}\left[e_j(e(\varphi_{t,s})) + h(\tau(\varphi_{t,s}), \mathrm{d}\varphi_{t,s}(e_j)\right]\,\mathrm{d}\varphi_{t,s}(e_j)\right)\Big|_{(t,s)=(0,0)} = \\
& e_j\left[\mathrm{div}X_v - h(\tau(\varphi), v)\right] h(w, \mathrm{d}\varphi(e_j)) + h(w, \nabla^{\varphi}_{\mathrm{grad}\mathcal{H}_{e(\varphi)}}v) + h((\nabla^{\varphi})^2 v + \mathrm{Ric}^{\varphi}v, \mathrm{d}\varphi(e_j))h(w, \mathrm{d}\varphi(e_j)) \\
& + h(\tau(\varphi), \nabla^{\varphi}_{e_j}v)h(w, \mathrm{d}\varphi(e_j)) - h(w, \nabla^{\varphi}_{[\mathrm{div}S(\varphi)]^\sharp}v) = \\
& X_w(\mathrm{div}X_v) + h(\mathrm{trace}(\nabla^{\varphi})^2 v + \mathrm{Ric}^{\varphi}v, \mathrm{d}\varphi(X_w)) - h\left(\nabla^{\varphi}_{X_w}\tau(\varphi), v\right) + h\left(w, \nabla^{\varphi}_{[\mathrm{div}\varphi^* h]^\sharp}v\right),
\end{aligned}
$$

where we have used again (3.5). ■

Remark 4.1. Another version of the second variation formula for σ_2–energy can be obtained from the general formula derived in [21, p. 37], which has the advantage of revelating the associated σ_p–Jacobi operator. Nevertheless one of the terms is rather difficult to handle in general, so we shall work with the above formula which has more explicit terms.

Let us notice that, according to the Remark 3.1, we have

$$
\begin{aligned}
\mathrm{div}^{\varphi}v\,\mathrm{div}^{\varphi}w =& [\mathrm{div}X_v - h(v, \tau(\varphi))][\mathrm{div}X_w - h(w, \tau(\varphi))] \\
=& \mathrm{div}X_v \mathrm{div}X_w + h(v, \tau(\varphi))h(w, \tau(\varphi)) - \mathrm{div}X_v h(w, \tau(\varphi)) - \mathrm{div}X_w h(v, \tau(\varphi)) \\
=& \mathrm{div}X_v \mathrm{div}X_w + h(v, \tau(\varphi))h(w, \tau(\varphi)) + h(\nabla^{\varphi}_{X_v}w + \nabla^{\varphi}_{X_w}v, \tau(\varphi)) \\
& + h\left(\nabla^{\varphi}_{X_v}\tau(\varphi), w\right) + h\left(\nabla^{\varphi}_{X_w}\tau(\varphi), v\right) + \text{ divergence terms}
\end{aligned}
$$

Applying the general formula $\mathrm{div}(fX) = X(f) + f\mathrm{div}X$, we get $X(\mathrm{div}Y) + \mathrm{div}X\mathrm{div}Y = \mathrm{div}((\mathrm{div}Y)X)$, so on a closed Riemannian manifold (M, g), for any two vector fields X and Y, the following identity holds

$$
\int_M [X(\mathrm{div}Y) + \mathrm{div}X\mathrm{div}Y]\,\nu_g = 0.
$$

Therefore, using the above observations, we can rewrite (4.1) in a different form. As the simplifications are not enlightening in the general case, we shall apply them only in particular situations, as we shall see below.

Let us start with a particularly important case, the one of harmonic maps that are also σ_2–critical (so they are critical points for the *full* energy (1.3)). When M supports a transitive action of $\mathbb{R}$, then by a Derrick-type argument [4], *all* possible stable critical solutions for the full energy (1.3) must be harmonic *and* σ_2–critical.

Corollary 4.1 (σ_2–Hessian of harmonic σ_2–critical mappings)**.**

$$
\begin{aligned}
\mathrm{Hess}^{\sigma_2}_{\varphi}(v, v) =& \int_M \left\{2e(\varphi)\left[|\nabla^{\varphi}v|^2 - \mathrm{Ric}^{\varphi}(v, v)\right] + (\mathrm{div}X_v)^2\right\}\nu_g \\
& + \int_M h\left(v, 2\mathrm{trace}(\nabla\mathrm{d}\varphi)\circ A_v + \mathrm{trace}[(\nabla^{\varphi})^2 v + R^N(v, \mathrm{d}\varphi)\mathrm{d}\varphi]\circ \mathsf{C}_{\varphi}\right)\nu_g \\
& + \int_M \left\{h\left(\mathrm{trace}(\nabla^{\varphi})^2 v + \mathrm{Ric}^{\varphi}v, \mathrm{d}\varphi(X_v)\right) + h\left(w, \nabla^{\varphi}_{\mathrm{div}\mathsf{C}_{\varphi}}v\right)\right\}\nu_g.
\end{aligned}
$$

(4.2)

If moreover $e(\varphi)$ is constant, the last term vanishes.

Let us now particularize the above result to the most simple case, the one of harmonic horizontally homothetic (HH) maps, i.e. the dilation λ is constant in horizontal directions and fibres are minimal.

Corollary 4.2 (σ_2–Hessian of harmonic HH mappings)**.**

$$\mathrm{Hess}^{\sigma_2}_{\varphi}(v,v) = \int_M \left\{(n-2)\lambda^2\left[|\nabla^{\varphi}v|^2 - h(\mathrm{Ric}^{\varphi}v,v)\right] + (\mathrm{div}X_v)^2\right\}\nu_g. \tag{4.3}$$

In particular, any harmonic HH submersion to a surface is a weakly stable σ_2–critical point.

Example. *The Hopf map* $\mathbb{S}^3 \to \mathbb{S}^2$ is a (weakly) stable critical point for the strong coupling limit of the Faddeev-Hopf model (as proved in [18, Theorem 5.2]).

In the end of this section, let us consider the stability of σ_2–critical mappings given by Corollary 3.1(iv) which could be related to *the rational map ansatz* [8]. To facilitate the exposition consider the simpler case of holomorphic maps between compact Kähler manifolds, $\varphi : (M,J,g) \to (N,J^N,h)$ (which are in particular PHH maps).

Recall that, in this case, we can define the following connexion in the pull-back bundle, cf. [20]:

$$\mathfrak{D}^{\varphi}v(X) := \nabla^{\varphi}_{JX}v - J^N\nabla^{\varphi}_X v, \quad \forall X \in \Gamma(TM),$$

that has the immediate property $\mathfrak{D}^{\varphi}v(JX) + J^N\mathfrak{D}^{\varphi}v(X) = 0,\ \forall X$.

We can check that:

$$\begin{aligned}&(\nabla^{\varphi})^2_{e_k,e_k}v + (\nabla^{\varphi})^2_{Je_k,Je_k}v + R^N(v,\mathrm{d}\varphi(e_k))\mathrm{d}\varphi(e_k) + R^N(v,\mathrm{d}\varphi(Je_k))\mathrm{d}\varphi(Je_k) = \\ &J^N\left(\nabla^{\varphi}_{e_k}\mathfrak{D}^{\varphi}v(e_k) + \nabla^{\varphi}_{Je_k}\mathfrak{D}^{\varphi}v(Je_k) - \nabla^{\varphi}_{\nabla_{e_k}e_k+\nabla_{Je_k}Je_k}\mathfrak{D}^{\varphi}v\right).\end{aligned}$$

From this identity we can deduce:

$$\begin{aligned}&h\left((\nabla^{\varphi})^2_{e_k,e_k}v + (\nabla^{\varphi})^2_{Je_k,Je_k}v + R^N(v,\mathrm{d}\varphi(e_k))\mathrm{d}\varphi(e_k) + R^N(v,\mathrm{d}\varphi(Je_k))\mathrm{d}\varphi(Je_k),\ w\right) = \\ &-h\left(\mathfrak{D}^{\varphi}v(e_k),\ \mathfrak{D}^{\varphi}w(e_k)\right) - [g(\nabla_{e_k}X_0,e_k) + g(\nabla_{Je_k}X_0,Je_k)],\end{aligned} \tag{4.4}$$

where X_0 is defined by $h\left(\mathfrak{D}^{\varphi}v(Y), J^N w\right) := g(X_0,Y),\ \forall Y$.

Remark 4.2. Recall that the Hessian of a harmonic map, for the (σ_1-) energy, is given by (see e.g. [20], [2, p. 92]):

$$\mathrm{Hess}_{\varphi}(v,w) = -\int_M h\left(\mathrm{trace}[(\nabla^{\varphi})^2v + R^N(v,\mathrm{d}\varphi)\mathrm{d}\varphi],w\right)\nu_g \ := \int_M h\left(\mathfrak{J}_{\varphi}(v),w\right)\nu_g.$$

For a holomorphic map between compact Kähler manifolds, taking the sum in (4.4) gives us:

$$h\left(\mathrm{trace}[(\nabla^{\varphi})^2v + R^N(v,\mathrm{d}\varphi)\mathrm{d}\varphi],v\right) = -\frac{1}{2}|\mathfrak{D}^{\varphi}v|^2 - \mathrm{div}X_1,$$

where X_1 is defined by $h\left(\mathfrak{D}^{\varphi}v(Y), J^N v\right) := g(X_1, Y),\ \forall Y$. Therefore $\mathrm{Hess}_{\varphi}(v,v) = \frac{1}{2}\int_M |\mathfrak{D}^{\varphi}v|^2 \nu_g$ which gives us the stability (as harmonic maps) of holomorphic maps between compact Kähler manifolds, an infinitesimal version of a classical Lichnerowicz result [20].

Now suppose in addition that a holomorphic map between compact Kähler manifolds has $\mathrm{grad}e(\varphi) \in \mathrm{Ker}\,\mathrm{d}\varphi$. Then it becomes a σ_2–critical map. By standard techniques, using (4.4) and a trick similar to (3.1), we obtain

Corollary 4.3 (σ_2–Hessian of holomorphic σ_2–critical maps between Kähler manifolds)**.**

$$\begin{aligned}\mathrm{Hess}_{\varphi}^{\sigma_2}(v,v) = &\int_M \left\{(\mathrm{div}X_v)^2 + e(\varphi)|\mathfrak{D}^{\varphi}v|^2 - \frac{1}{2}\langle \mathfrak{D}^{\varphi}v, \mathfrak{D}^{\varphi}v \circ \mathsf{C}_{\varphi}\rangle - \frac{1}{2}\langle \mathfrak{D}^{\varphi}v, \mathfrak{D}^{\varphi}\mathrm{d}\varphi(X_v)\rangle\right\}\nu_g \\ &+ \int_M \left\{2\sum_{k=1}^{m} h\left(J^N\mathfrak{D}^{\varphi}v(e_k), \mathrm{d}\varphi(e_k)\right) h\left(v, \nabla\mathrm{d}\varphi(e_k,e_k)\right)\right\}\nu_g,\end{aligned} \tag{4.5}$$

where $\langle \mathfrak{D}^{\varphi}v, \mathfrak{D}^{\varphi}v \circ \mathsf{C}_{\varphi}\rangle = 2\sum_k \lambda_k^2 \|\mathfrak{D}^{\varphi}v(e_k)\|^2$*. In particular, a holomorphic map between compact Kähler manifolds with* $\mathrm{grad}e(\varphi) \in \mathrm{Ker}\,\mathrm{d}\varphi$ *is weakly stable under variations that are holomorphic up to first order (i.e.* $\mathfrak{D}^{\varphi}v = 0$*).*

We can check that for the last term we can also use:
$2h\left(v, \nabla\mathrm{d}\varphi(e_k,e_k)\right) = g\left(J\mathfrak{D}X_v(e_k), e_k\right) - h\left(J^N\mathfrak{D}^{\varphi}v(e_k), \mathrm{d}\varphi(e_k)\right)$ and
$\mathfrak{D}^{\varphi}\mathrm{d}\varphi(X_v)(Y) = \mathrm{d}\varphi(\mathfrak{D}X_v(Y)),\ \forall Y$, where $\mathfrak{D} := \mathfrak{D}^{id_M}$.

5 Full generalized Faddeev-Skyrme energy

In this section we shall discuss the full energy (1.3).

From the above discussion, it is clear that a map φ is $\sigma_{1,2}$–critical if it satisfies the following Euler-Lagrange equations

$$[2e(\varphi)+1]\tau(\varphi) + 2\mathrm{d}\varphi(\mathrm{grad}e(\varphi)) - \mathrm{trace}(\nabla\mathrm{d}\varphi)\circ\mathsf{C}_{\varphi} - \mathrm{d}\varphi(\mathrm{div}\mathsf{C}_{\varphi}) = 0.$$

Harmonic HH mappings are clearly the simplest examples of $\sigma_{1,2}$–critical points. From (4.3) we can deduce that the full $\sigma_{1,2}$–Hessian on harmonic HH maps is given by

$$\mathrm{Hess}_{\varphi}^{\sigma_{1,2}}(v,v) = \int_M \left\{(1+\kappa(n-2)\lambda^2)\left[|\nabla^{\varphi}v|^2 - h(\mathrm{Ric}^{\varphi}v,v)\right] + \kappa(\mathrm{div}X_v)^2\right\}\nu_g. \tag{5.1}$$

5.1 The Hopf map

In [15] it was proved the fact that the Hopf map from $\mathbb{S}^3$ into $\mathbb{S}^2$ minimizes the p-energy in its homotopy class for $p \geq 4$ and that it remains true locally for $3 \leq p < 4$. Consequently, for the Hopf map which is obviously 4-harmonic, we have:

$$\mathrm{Hess}_{\varphi}^{\mathcal{E}_4}(v,v) = 2\int_M \left\{|\nabla^{\varphi}v|^2 - h(\mathrm{Ric}^{\varphi}v,v) + (\mathrm{div}X_v)^2\right\}\nu_g \geq 0,$$

where we have used the second variation formula for 4-harmonic maps [19] and the fact that the Hopf map can be regarded as a Riemannian submersion (choosing $\mathbb{S}^2$ to be of radius 1/2).

But, for the Hopf map formula (5.1) gives us:

$$\begin{aligned}\mathrm{Hess}_{\varphi}^{\sigma_{1,2}}(v,v) &= \int_M \left\{|\nabla^{\varphi}v|^2 - h(\mathrm{Ric}^{\varphi}v,v) + \kappa(\mathrm{div}X_v)^2\right\}\nu_g \\ &= \int_M \left\{|\nabla^{\varphi}v|^2 - h(\mathrm{Ric}^{\varphi}v,v) + (\mathrm{div}X_v)^2\right\}\nu_g + \int_M (\kappa-1)(\mathrm{div}X_v)^2\nu_g\end{aligned}$$

which is clearly positive if $\kappa \geq 1$. Therefore the Hopf map is a stable critical point for the full $\sigma_{1,2}$–energy if $\kappa \geq 1$ (as it has been already established in [18, Theorem 5.3] by computing the spectrum of the Jacobi operator). Notice that in this case $\sigma_{1,2}$–energy coincides with (1.2) and with the energy introduced in [18].

5.2 Homothetic local diffeomorphisms

Let us particularize further to the case of HH maps between spaces of equal dimensions $m = n$ (if $n \geq 3$ they are homothetic local diffeomorphisms, cf. [2, Theorem 11.4.6]). As $v = \lambda^{-2}\mathrm{d}\varphi(X_v)$, we can check that:

$$|\nabla^{\varphi}v|^2 - h(\mathrm{Ric}^{\varphi}v,v) = \lambda^{-2}\left(|\nabla X_v|^2 - \mathrm{Ric}^M(X_v,X_v)\right).$$

Therefore, in this case we have:

$$\mathrm{Hess}_{\varphi}^{\sigma_{1,2}}(v,v) = \int_M \left\{(\lambda^{-2}+\kappa(n-2))\left[|\nabla X_v|^2 - \mathrm{Ric}^M(X_v,X_v)\right] + \kappa(\mathrm{div}X_v)^2\right\}\nu_g.$$

Employing now the general **Yano identity** [23]

$$\int_M \left\{|\nabla X|^2 - \mathrm{Ric}(X,X) + (\mathrm{div}X)^2 - \frac{1}{2}|\mathcal{L}_X g|^2\right\}\nu_g = 0, \tag{5.2}$$

we get the following expression

$$\mathrm{Hess}_{\varphi}^{\sigma_{1,2}}(v,v) = \int_M \left\{\frac{\lambda^{-2}+(n-2)\kappa}{2}|\mathcal{L}_{X_v}g|^2 - (\lambda^{-2}+(n-3)\kappa)(\mathrm{div}X_v)^2\right\}\nu_g$$

Notice now that, according to Newton inequality

$$\frac{1}{2}|\mathcal{L}_{X_v}g|^2 \geq 2\sum_i g(\nabla_{e_i}X_v,e_i)^2 \geq \frac{2}{n}\left[\sum_i g(\nabla_{e_i}X_v,e_i)\right]^2 = \frac{2}{n}(\mathrm{div}X_v)^2,$$

where equality is reached when X_v is a conformal vector field.

Therefore our homothetic map (between equidimensional manifolds) is (weakly) stable critical point for the full generalized Skyrme energy, i.e. $\mathrm{Hess}_{\varphi}^{\sigma_{1,2}}(v,v) \geq 0$, provided that:

$$\frac{2}{n}(\lambda^{-2}+(n-2)\kappa) \geq \lambda^{-2}+(n-3)\kappa.$$

This inequality can be satisfied (by non-constant maps) only when $n = 2$ (trivially) and $n = 3$. In the latter case we get the condition:

$$\lambda \geq \frac{1}{\sqrt{2\kappa}}, \tag{5.3}$$

that coincides with the condition found in [11, 13] (for $\kappa = 1$). Recall that when $m = n = 3$ and $\kappa = 1$, it has been proved [11] that if $\lambda \geq 1$, then diffeomorphic homotheties are, up to isometries, the only absolute minimizers of the Skyrme energy among all maps of a given degree.

5.3 Constrained stability

The original Skyrme model (1.1) requires solutions of a constrained variational problem: we must search stable critical solutions of *fixed* degree (see [13]).

Remark 5.1. Recall that the *degree of a map* between closed Riemannian manifolds $\varphi : (M^n, g) \to (N^n, h)$ can be computed as:

$$\deg\varphi = \frac{\int_M \varphi^*(\nu_h)}{\mathrm{Vol}(N)}.$$

Moreover, when $N = \mathbb{S}^n$, *Hopf theorem* tells us that two smooth mappings have the same degree if and only if they are homotopic.

Let us consider again the case of harmonic HH mappings. The condition (6.1) is trivially satisfied, so the volume is preserved up to first order by any variation of φ. Using (6.2), we can deduce that the (constrained) $\sigma_{1,2}$–Hessian of a harmonic HH map, w.r.t. variations that preserve $V(\varphi)$ (or equivalently, the degree) up to second order, is given by

$$\begin{aligned}\widetilde{\mathrm{Hess}}_{\varphi}^{\sigma_{1,2}}(v,v) &= \int_M \left\{ \frac{1}{2}(\lambda^{-2} + (n-2)\kappa) \sum_i [(\mathcal{L}_{X_v} g)(e_i, e_i)]^2 - (\lambda^{-2} + (n-3)\kappa)(\mathrm{div} X_v)^2 \right\} \nu_g \\ &\geq \int_M \left\{ \frac{2}{n}(\lambda^{-2} + (n-2)\kappa) - (\lambda^{-2} + (n-3)\kappa) \right\} (\mathrm{div} X_v)^2 \nu_g.\end{aligned}$$

So, in order to have constrained stability for such maps (i.e. $\widetilde{\mathrm{Hess}}_{\varphi}^{\sigma_{1,2}}(v,v) \geq 0$), we are leaded to the same condition as in the non-constrained case, namely (5.3) for $n = 3$ (and no condition for $n = 2$).

6 Appendix - Variations of the volume functional

The volume functional on (smooth) maps $\varphi : (M^m, g) \to (N^n, h)$ with M compact is given by:

$$V(\varphi) = \int_M \sqrt{\det(\varphi^* h)}\, \nu_g = \int_M \lambda_1 \lambda_2 \cdots \lambda_m \, \nu_g.$$

This quantity is non-zero at points where φ is an immersion.

Let us see when this quantity is preserved up to first order under a variation $\{\varphi_t\}$ with variation vector v.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 V(\varphi_t) = \int_M \sum_i \frac{\alpha_v(e_i,e_i)}{\lambda_i} \prod_{j\neq i} \lambda_j \ \nu_g.$$

If φ is a (weakly) conformal map, then λ_i's are all equal ($\lambda_i = \lambda, \forall i$) and we get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 V(\varphi_t) &= \int_M \lambda^{m-2} \sum_i \alpha_v(e_i,e_i)\ \nu_g = \int_M \lambda^{m-2}[\mathrm{div} X_v - h(v,\tau(\varphi))]\ \nu_g \\ &= -\int_M h\left(v, \mathrm{d}\varphi(\mathrm{grad}\lambda^{m-2}) + \lambda^{m-2}\tau(\varphi)\right)\ \nu_g.\end{aligned}$$

We can interpret the above identity either as a first variation formula or as follows:

> A variation $\{\varphi_t\}$ of a weakly conformal map preserves the volume $V(\varphi)$ up to first order if and only if its variation vector field satisfies:
>
> $$v \perp (m-2)\mathrm{d}\varphi(\mathrm{grad}\ln\lambda) + \tau(\varphi). \qquad (6.1)$$

Now let us see when the volume is preserved up to second order:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_0 V(\varphi_t) = \int_M \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \frac{h(\nabla^\Phi_{d/dt}\mathrm{d}\varphi_t(e_i), \mathrm{d}\varphi_t(e_i))}{\sqrt{\varphi_t^* h(e_i,e_i)}} \prod_{k\neq i}\lambda_k + 2\sum_{i<j}\frac{\alpha_v(e_i,e_i)\alpha_v(e_j,e_j)}{\lambda_i\lambda_j}\prod_{k\neq i,j}\lambda_k\ \nu_g.$$

By simple derivation the first right hand term takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \frac{h(\nabla^\Phi_{d/dt}\mathrm{d}\varphi_t(e_i), \mathrm{d}\varphi_t(e_i))}{\sqrt{\varphi_t^* h(e_i,e_i)}} = \frac{\lambda_i^2\left[\|\nabla^\varphi_{e_i} v\|^2 + \alpha_u(e_i,e_i) - R^N(v,\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i),v)\right] - \alpha_v(e_i,e_i)^2}{\lambda_i^3},$$

where $u = \nabla^\Phi_{d/dt}\mathrm{d}\varphi_t(\frac{d}{dt})|_0$.

Again, if φ is a (weakly) conformal map ($\lambda_i = \lambda, \forall i$), then the above formula becomes:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_0 V(\varphi_t) &= \int_M \lambda^{m-4}\left\{\lambda^2\left[|\nabla^\varphi v|^2 + \mathrm{div}^\varphi u - \mathrm{Ric}^\varphi(v,v)\right] - \sum_i \alpha_v(e_i,e_i)^2 + 2\sum_{i<j}\alpha_v(e_i,e_i)\alpha_v(e_j,e_j)\right\}\nu_g \\ &= \int_M \lambda^{m-4}\left\{\lambda^2\left[|\nabla^\varphi v|^2 + \mathrm{div}^\varphi u - \mathrm{Ric}^\varphi(v,v)\right] + (\mathrm{div}^\varphi v)^2 - 2\sum_i \alpha_v(e_i,e_i)^2\right\}\nu_g.\end{aligned}$$

If, in addition, λ is constant (φ is harmonic) the above relation simplifies to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_0 V(\varphi_t) = \lambda^{m-4}\int_M \left\{\lambda^2\left[|\nabla^\varphi v|^2 - \mathrm{Ric}^\varphi(v,v)\right] + (\mathrm{div} X_v)^2 - \frac{1}{2}\sum_i [\mathcal{L}_{X_v}(e_i,e_i)]^2\right\}\nu_g.$$

Once more we can interpret the above relations either as second variation formulae or as follows:

A variation $\{\varphi_t\}$ of a homothetic map preserves the volume $V(\varphi)$ up to second order if and only if its variation vector field satisfies:

$$\int_M \left\{ \lambda^2 \left[|\nabla^\varphi v|^2 - \mathrm{Ric}^\varphi(v,v) \right] + (\mathrm{div} X_v)^2 - \frac{1}{2} \sum_i [(\mathcal{L}_{X_v} g)(e_i, e_i)]^2 \right\} \nu_g = 0. \tag{6.2}$$

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