Invariant generalized complex and Kähler structures on Lie groups

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September 25, 2008

1 Introduction

Generalized geometry unifies complex and symplectic geometry, by considering both complex and symplectic structures as particular cases of a more general structure, the so called generalized complex structure. A generalized almost complex structure on a smooth manifold M is an endomorphism

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$$

with the following two properties: $\mathcal{J}^2 = -\mathrm{Id}$ (where "Id" denotes the identity endomorphism); $\mathcal{J}$ is skew-symmetric with respect to the standard metric g_{can} of $TM \oplus T^*M$, defined by

$$g_{\mathrm{can}}(X+\xi, Y+\eta) := \frac{1}{2}\left(\xi(Y) + \eta(X)\right), \quad \forall X+\xi, Y+\eta \in TM \oplus T^*M. \quad (1)$$

Almost complex and almost symplectic structures are particular cases of generalized almost complex structures: if J is an almost complex structure and ω is an almost symplectic structure then

$$\mathcal{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \quad \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

are generalized almost complex structures. There is also a notion of integrability of generalized almost complex structures, which reduces to the usual integrability of almost complex and almost symplectic structures: an generalized almost complex structure $\mathcal{J}$ is integrable (or is a generalized complex structure) if the bundle $L \subset (TM \oplus T^*M)^{\mathbb{C}}$ of $(1,0)$-vectors of $\mathcal{J}$ is closed under the complex linear extension of the Courant bracket, defined by

$$[X+\xi, Y+\eta] := [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d\left(\eta(X) - \xi(Y)\right), \qquad (2)$$

for any sections $X+\xi$ and $Y+\eta$ of $TM \oplus T^*M$. We can go further and define a generalized Kähler structure on M to be a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of integrable, commuting, generalized complex structures such that the endomorphism $D := -\mathcal{J}_1\mathcal{J}_2$ of $TM \oplus T^*M$ is positive definite with respect to g_{can}. Note that if (g, J, ω) is Kähler, then $(\mathcal{J}_J, \mathcal{J}_\omega)$ is generalized Kähler. Many other notions and results from complex and symplectic geometry can be extended to generalized complex geometry. For example, there is a Darboux theorem for generalized complex structures, which generalizes the Darboux theorem for symplectic structures and the Newlander-Nirenberg theorem for usual complex structures [2]. Our aim in this paper is to find invariant generalized complex structures and invariant generalized Kähler structures on Lie groups and flag manifolds.

Section 2 is intended for completeness of the text and to fix notations and conventions. In the first part of Section 2 we recall, following [2], basic definitions and results on generalized complex structures and we explain the relation between generalized Kähler geometry and bi-Hermitian geometry. In the second part of Section 2 we describe (usual) invariant complex structures on Lie groups and invariant Kähler structures on flag manifolds. Good references for these topics are [1] and [3].

In Section 3 we prove that there is a bijective correspondence between invariant generalized complex structures on a Lie group G is usual complex structures on the cotangent group T^*G, compatible with the canonical metric of T^*G of split signature. A key role in this correspondence is played by the observation that the Lie algebra bracket of T^*G is precisely the restriction of the Courant bracket to invariant sections of $TG \oplus T^*G$.

The aim of Section 4 is to describe invariant generalized complex structures of semi-simple Lie groups. We first develop an infinitesimal description of the bundle of $(1,0)$-forms of an invariant generalized complex structure of a Lie group (see Proposition 11). We then apply it to show that there are no generalized complex structures on split forms of complex semi-simple Lie groups, but there are many such structures on compact forms and we describe them explicitly, in algebraic terms (see Theorem 20). Moreover, we show that generalized Kähler structures do not exist on compact semi-simple Lie groups (see Corollary 17).

In Section 5 we turn our attention to generalized geometry of flag manifolds. While integrability of invariant generalized almost complex structures on Lie groups has a nice infinitesimal description, there is no such description for invariant generalized complex structures on flag manifolds. For this reason, we restrict our attention to invariant generalized Kähler structures on flag manifolds, which admit an infinitesimal description, due to their relation with bi-Hermitian structures. We show that for any invariant generalized Kähler structure (g, J_1, J_2, b) on a flag manifold, the pairs (g, J_1) and (g, J_2) are Kähler and b is a closed 2-form. Invariant Kähler structures on flag manifolds were described in [1].

2 Preliminary material

In this paper all our manifolds will be smooth. For a smooth manifold M we shall denote by $\mathcal{X}(M)$ and $\Omega^k(M)$ the space of smooth vector fields and the space of smooth k-forms on M. For an almost complex structure J and a 3-form α on M, we shall use the notation $\alpha \circ J^3$ for the 3-form

$$(\alpha \circ J^3)(X, Y, Z) := \alpha(JX, JY, JZ), \quad \forall X, Y, Z \in \mathcal{X}(M).$$

For a vector bundle $E \to M$, we shall denote by $\Gamma(E)$ the space of smooth sections of E. By an "invariant" tensor field on a Lie group G, we will always mean a "left-invariant" tensor field. We will sometimes identify, without further explanations, invariant tensor fields on G with their value at the identity $e \in G$. In particular, we shall use the same notation, namely g_{can}, for the standard metric of split signature of T^*G, defined as in (1), and also for its value at $e \in G$.

2.1 Generalized complex structures on manifolds

In this Section we recall, following [2], a useful characterization of the bundle of $(1,0)$-vectors of a generalized complex structure. We shall use this characterization in our treatment of generalized complex structures on Lie groups.

Let $\mathcal{J}$ be a generalized almost complex structure on a manifold M of dimension n. The bundle L of $(1,0)$-vectors of $\mathcal{J}$ is isotropic with respect to g_{can} and satisfies $L \oplus \bar{L} = (TM \oplus T^*M)^{\mathbb{C}}$. Conversely, any complex subbundle of $(TM \oplus T^*M)^{\mathbb{C}}$ with these properties defines a generalized almost complex structure $\mathcal{J}$, whose complex linear extension to $(TM \oplus T^*M)^{\mathbb{C}}$ satisfies $\mathcal{J}|_L = i$ and $\mathcal{J}|_{\bar{L}} = -i$. Such sub-bundles of $(TM \oplus T^*M)^{\mathbb{C}}$ are of a very special form. More precisely, the following holds [2]:

Proposition 1. *A complex rank n subbundle L of $(TM \oplus T^*M)^{\mathbb{C}}$ is the bundle of $(1,0)$-vectors of a generalized almost complex structure $\mathcal{J}$ if and only if it is of the form*

$$L = \{X + \xi \in E \oplus (T^{\mathbb{C}}M)^*, \quad \xi|_E = \omega(X, \cdot)\}$$

where $E \subset T^{\mathbb{C}}M$ is a complex sub-bundle and $\omega \in \Lambda^2(E^*)$ is such that $\mathrm{Im}\,(\omega|_\Delta)$ *is non-degenerate. Here $\Delta \subset TM$ is the real part of $L \cap \bar{L}$, i.e. $\Delta^{\mathbb{C}} = L \cap \bar{L}$. Moreover, $\mathcal{J}$ is integrable if and only if E is involutive and $d_E(\omega) = 0$, where d_E is the exterior derivative along E, defined by*

$$d_E\omega(X,Y,Z) = -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X), \quad \forall X,Y,Z \in \Gamma(E).$$

2.2 Generalized Kähler structures on manifolds

We now explain the relation between generalized Kähler geometry and bi-Hermitian geometry.

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on a manifold M. Recall that $\mathcal{J}_1$ and $\mathcal{J}_2$ are commuting generalized complex structures and $D := -\mathcal{J}_1\mathcal{J}_2 \in \mathrm{End}(TM \oplus T^*M)$ is positive definite with respect to g_{can}. We write $D := -\mathcal{J}_1\mathcal{J}_2$ in the form

$$D = \begin{pmatrix} A & g^{-1} \\ \sigma & A^* \end{pmatrix}.$$

From the positivity of D, g, σ are Riemannian metrics on TM and $A \in \mathrm{End}(TM)$ is skew-symmetric with respect to g and σ. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, $D^2 = \mathrm{Id}$ and D has two eigenvalues, ± 1. The eigenspaces $C^{\pm}$ of D, with eigenvalues ± 1, are preserved by $\mathcal{J}_1$ and $\mathcal{J}_2$ and the standard projections

$$\pi_{\pm} : C_{\pm} \to TM$$

are isomorphisms. Note that $\mathcal{J}_1 = \mathcal{J}_2$ on C_+ and $\mathcal{J}_1 = -\mathcal{J}_2$ on C_-. The almost complex structure J_1 of M is induced, by means of π_+, by the restriction of $\mathcal{J}_1$ to C_+; similarly, $\mathcal{J}_2$ is induced, by means of π_-, by the restriction of $\mathcal{J}_1$ on C_-. Since A is skew-symmetric with respect to g, the bilinear form $b(X,Y) := g(AX,Y)$ is a 2-form, sometimes called the b-field. It was shown in [2] that the integrability of $\mathcal{J}_1$ and $\mathcal{J}_2$ is equivalent to the integrability of J_1 and J_2, together with the additional condition

$$db = (d\omega_1) \circ J_1^3 = -(d\omega_2) \circ J_2^3, \tag{3}$$

where $\omega_i := g(J_i\cdot, \cdot)$ are the Kähler forms of the pairs (J_i, g). Reversing this argument, (J_1, J_2, g, b) determines $(\mathcal{J}_1, \mathcal{J}_2)$. Therefore, we can state:

Definition 2. *A generalized Kähler structure on a manifold M is a bi-Hermitian structure (J_1, J_2, g), together with a 2-form $b \in \Omega^2(M)$, such that the condition (5) is satisfied.*

We end this Section with the following simple Lemma on the i-eigenbundles of the generalized complex structures forming a generalized Kähler structure.

Lemma 3. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on a manifold M. Denote by $L_1 := L(E_1, \omega_1)$ and $L_2 := L(E_2, \omega_2)$ the bundles of $(1,0)$-vectors of $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively. Then*

$$\overline{E_1 \cap E_2} \subset E_1 + E_2, \quad \bar{E}_1 \cap E_2 \subset E_1 + \bar{E}_2, \quad E_1 \cap \bar{E}_2 \subset \bar{E}_1 + E_2. \tag{4}$$

Proof. The positivity condition of $G := -\mathcal{J}_1\mathcal{J}_2$ with respect to g_{can} is equivalent to the following two conditions (see [2]):

$$g_{\mathrm{can}}(x, \bar{x}) = \frac{1}{2}\mathrm{Re}\xi(\bar{X}) > 0, \quad \forall x = X + \xi \in L_1 \cap L_2, \quad x \neq 0 \tag{5}$$

and

$$-g_{\mathrm{can}}(x, \bar{x}) = -\frac{1}{2}\mathrm{Re}\xi(\bar{X}) > 0, \quad \forall x = X + \xi \in L_1 \cap \overline{L_2}, \quad x \neq 0. \tag{6}$$

Note that $x = X + \xi \in L_1 \cap L_2$ if and only if

$$\xi|_{E_1} = \omega_1(X, \cdot), \quad \xi|_{E_2} = \omega_2(X, \cdot). \tag{7}$$

Similarly, $X + \xi \in L_1 \cap \overline{L}_2$ if and only if

$$\xi|_{E_1} = \omega_1(X, \cdot), \quad \xi|_{\overline{E}_2} = \overline{\omega}_2(X, \cdot), \tag{8}$$

where $\bar{\omega}_2 \in \Lambda^2(\bar{E}_2^*)$ is defined by

$$\bar{\omega}_2(Y, Z) := \overline{\omega_2(\bar{Y}, \bar{Z})}, \quad \forall Y, Z \in \bar{E}_2.$$

In particular, if $X + \xi \in L_1 \cap L_2$ then $\xi \in (T^{\mathbb{C}}M)^*$ is determined by X on $E_1 + E_2$ and can take any value outside this sub-bundle. Condition (5) implies the first equality of (4). The second equality of (5) follows similarly. The third equality of (5) follows from the second. $\square$

2.3 Invariant complex structures on Lie groups

In this Section we briefly recall some facts we will need on the theory of invariant complex structures on Lie groups. Invariant complex structures on semi-simple Lie groups were considered for the first time in [4].

Definition 4. *An almost complex structure on a Lie algebra $\mathfrak{g}$ is an endomorphism $j \in \mathrm{End}(\mathfrak{g})$ such that $j^2 = -\mathrm{Id}$. We say that j is a complex structure if*

$$N_j(v, w) := [jv, jw] - [v, w] - j([jv, w] + [v, jw]) = 0, \quad \forall v, w \in \mathfrak{g}.$$

Alternatively, an almost complex structure j on $\mathfrak{g}$ is defined by a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, where $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are subspaces of $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{g}^{0,1} = \overline{\mathfrak{g}^{1,0}}$: we just take $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ to be the eigenspaces of j with eigenvalues i and $-i$ respectively. Moreover, j is integrable if and only if $\mathfrak{g}^{1,0}$ (hence also $\mathfrak{g}^{0,1}$) is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Proposition 5. *There is a one to one correspondence between invariant almost complex structures on a Lie group G and almost complex structures on its Lie algebra $\mathfrak{g}$. The correspondence associates to an almost complex structure J on G its value $j := J_e : \mathfrak{g} \to \mathfrak{g}$ at the identity $e \in G$. Moreover, J is integrable if and only if j is.*

We now turn our attention to invariant complex structures on semi-simple Lie groups. Let G be a semi-simple real Lie group. Denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of the Lie algebra $\mathfrak{g}$ of G. Since G is semi-simple, the Killing form $\langle X, Y\rangle := \mathrm{tr}\,(\mathrm{ad}(X)\mathrm{ad}(Y))$, where $X, Y \in \mathfrak{g}^{\mathbb{C}}$, is non-degenerate. Let $\mathfrak{h}$ be the complexification of the Cartan subalgebra of $\mathfrak{g}$, and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

the corresponding root decomposition of $\mathfrak{g}^{\mathbb{C}}$, where $R \subset \mathfrak{h}^*$ is the set of roots and $\mathfrak{g}_\alpha$ are the (1-dimensional) root spaces. Following [4], invariant complex structures on G are in one to one correspondence with pairs (j_0, Π) where j_0 is any complex structure of $\mathfrak{g} \cap \mathfrak{h}$ and Π is a system of simple roots: if $W \subset \mathfrak{h}$ is the set of $(1,0)$-vectors with respect to j_0 and $R^+_\Pi := [\Pi]_+ \cap R$ is the system of positive roots determined by Π, define

$$\mathfrak{g}^{1,0} := W + \sum_{\alpha \in R^+_\Pi} \mathfrak{g}_\alpha$$

and

$$\mathfrak{g}^{0,1} := \overline{W} + \sum_{\alpha \in R^+_\Pi} \mathfrak{g}_{-\alpha}.$$

The decomposition $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} = \mathfrak{g}^{\mathbb{C}}$ defines an invariant complex structure on G. Moreover, any invariant complex structure on G is obtained this way.

2.4 Invariant Hermitian structures on flag manifolds

In this Section we describe invariant Hermitian structures on flag manifolds. Details can be found in [3].

Recall that a flag manifold is a quotient $M = G^{\mathbb{C}}/B$, where $G^{\mathbb{C}}$ is a complex semi-simple Lie group and B is a Borel subgroup, i.e. a maximal solvable subgroup. Let $\mathfrak{g}^{\mathbb{C}}$ be the Lie algebra of $G^{\mathbb{C}}$. With respect to a Cartan subalgebra $\mathfrak{h}$, $\mathfrak{g}^{\mathbb{C}}$ has the Cartan decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha \tag{9}$$

and the Lie algebra $\mathfrak{b}$ of B has the form

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha \in R^+} \mathfrak{g}_\alpha, \tag{10}$$

where $R^+ \subset R$ is a system of positive roots. On $\mathfrak{g}^{\mathbb{C}}$ we consider a Weyl basis, which amounts to choosing root vectors $E_\alpha \in \mathfrak{g}_\alpha$ with $\langle E_\alpha, E_{-\alpha} \rangle = 1$ (where $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}^{\mathbb{C}}$) and such that the constants $K_{\alpha\beta}$ (defined by $[E_\alpha, E_\beta] = K_{\alpha\beta} E_{\alpha+\beta}$), satisfy the symmetries:

$$K_{-\alpha,-\beta} = -K_{\alpha\beta}, \quad \forall \alpha, \beta \in R. \tag{11}$$

The restriction of the Killing form $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}$ is non-degenerate. For any $\alpha \in R$, let $H_\alpha \in \mathfrak{h}$ be defined by $\alpha|_{\mathfrak{h}}(\cdot) = \langle H_\alpha, \cdot \rangle$ and let $\mathfrak{h}_{\mathbb{R}}$ be the real subspace of $\mathfrak{h}$ spanned by the vectors H_α. The vectors $A_\alpha := E_\alpha - E_{-\alpha}$ and $B_\alpha := i(E_\alpha + E_{-\alpha})$, for any $\alpha \in R$, together with $i\mathfrak{h}_{\mathbb{R}}$ generate a real subalgebra $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Let G be any connected Lie group, with Lie algebra $\mathfrak{g}$. Then G is compact, with Cartan subalgebra $i\mathfrak{h}_{\mathbb{R}}$. The homogeneous space $M = G^{\mathbb{C}}/B$ is diffeomorphic to G/T, where $T := G \cap B$ is a maximal torus of G, with Lie algebra $i\mathfrak{h}_{\mathbb{R}}$. The tangent space of M at the identity coset $e \in M$ can (and will) be identified with $\mathfrak{q} := \mathfrak{g} \ominus i\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha \in R} \mathrm{Span}_{\mathbb{R}}\{A_\alpha, B_\alpha\}$. Its complexification $T_e^{\mathbb{C}} M$ is $\sum_{\alpha \in R} \mathfrak{g}_\alpha$.

Now, T-invariant complex structures on M are in bijective correspondence with positive roots systems of R, as follows: let $R_\Pi^+ := [\Pi]_+ \cap R$ be a positive root system of R, defined by a system of simple roots Π. Then

$$m_\Pi^{1,0} := \sum_{\alpha \in R_\Pi^+} \mathfrak{g}_\alpha$$

and

$$m_\Pi^{0,1} = \sum_{\alpha \in R_\Pi^+} \mathfrak{g}_{-\alpha}$$

are the $\pm i$-eigenspaces of a complex structure J on $T_e M$, which extends to a T-invariant, integrable complex structure on M. Moreover, any T-invariant complex structure on M is obtained this way.

On the other hand, a T-invariant metric on M is determined by its value the identity $e \in M = G/T$, which is an inner product $(X, Y) = \langle \Lambda X, Y \rangle$ on $\mathfrak{q}$, where $\Lambda : \mathfrak{q} \to \mathfrak{q}$ is symmetric, positive definite with respect to the Killing form $\langle \cdot, \cdot \rangle$. Since $(\cdot, \cdot)$ is T-invariant, the complex linear extension $\Lambda_{\mathbb{C}}$ of Λ to $T_e^{\mathbb{C}} M$ is of the form

$$\Lambda(E_\alpha) = g_\alpha E_\alpha, \quad \forall \alpha \in R,$$

where $g_\alpha > 0$ and $g_\alpha = g_{-\alpha}$. In particular, any T-invariant metric g is compatible with any T-invariant complex structure J. The Kähler form $\omega := g(J\cdot, \cdot)$, applied to a pair of root vectors (E_α, E_β), is zero unless $\alpha + \beta = 0$, in which case

$$\omega(E_\alpha, E_{-\alpha}) = i g_\alpha \epsilon_\alpha, \quad \forall \alpha \in R.$$

Similarly, $(d\omega)(E_\alpha, E_\beta, E_\gamma) = 0$ unless $\alpha + \beta + \gamma = 0$, in which case

$$(d\omega)(E_\alpha, E_\beta, E_\gamma) = i K_{\alpha\beta}(\epsilon_\alpha g_\alpha + \epsilon_\beta g_\beta + \epsilon_\gamma g_\gamma), \tag{12}$$

see Proposition 2.1 of [3]. In particular, a pair (J, g) formed by an invariant complex structure J and an invariant metric g as above is Kähler if and only if for any positive roots $\alpha, \beta \in R^+$ such that $\alpha + \beta$ is a root,

$$g_{\alpha+\beta} = g_\alpha + g_\beta. \tag{13}$$

At the end of this Section we recall the general formula for the exterior derivative of G-invariant forms on M: if $\alpha \in \Omega^k(M)$ is G-invariant, its exterior derivative $d\alpha$ is also G-invariant and is determined by its value at the identity coset $e \in M$, where

$$(d\alpha)(X_0, \cdots, X_k) = \sum_{i,j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k), \tag{14}$$

for any $X_i \in \mathfrak{q}$. In (14), the hat means that the term is omitted; the Lie brackets $[X_i, X_j]$ of vectors $X_i, X_j \in \mathfrak{q}$ is obtained by taking the Lie bracket in $\mathfrak{g}$, followed by the projection $\mathfrak{g} = \mathfrak{q} \oplus i\mathfrak{h}_{\mathbb{R}} \to \mathfrak{q}$.

3 The cotangent Lie group T^*G

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. The aim of this Section is to prove the following Proposition and its Corollary.

Proposition 6. *The value $\mathcal{J}_e$ at the identity $e \in G$ of an invariant generalized complex structure $\mathcal{J}$ on G is a complex structure of the cotangent Lie algebra $\mathfrak{t}^*(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$, compatible with respect to the standard metric g_{can}*

of $\mathfrak{t}^*(\mathfrak{g})$ of split signature. Conversely, any complex structure of $\mathfrak{t}^*(\mathfrak{g})$, compatible with g_{can}, is the value at $e \in G$ of an invariant generalized complex structure $\mathcal{J}$ on G.

Corollary 7. *There is a one to one correspondence between invariant generalized complex structure on G and invariant complex structures on the cotangent group T^*G, compatible with the standard metric g_{can} of T^*G. The correspondence associates to an invariant generalized complex structure $\mathcal{J}$ on G the unique invariant complex structure J on T^*G, with the property that $\mathcal{J}_e = J_0$, where $0 \in T_eG$.*

The proof of Proposition 6 is straightforward, once the notions of cotangent Lie groups and cotangent Lie algebras are understood. We recall then now.

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Then T^*G is also a Lie group, with group operation

$$\alpha_1 \cdot \alpha_2 := R^*_{g_2^{-1}}\alpha_1 + L^*_{g_1^{-1}}\alpha_2, \quad \forall \alpha_1 \in T^*_{g_1}G, \quad \forall \alpha_2 \in T^*_{g_2}G. \tag{15}$$

Note that the cotangent bundle projection $\pi^* : T^*G \to G$ is a Lie group homomorphism.

Definition 8. *The manifold T^*G together with the multiplication (15) is called the cotangent group of G.*

We now determine the Lie algebra of T^*G. For a vector $v \in \mathfrak{g}$ and a covector $b \in \mathfrak{g}^*$, define $L_v b \in \mathfrak{g}^*$ by

$$(L_v b)(w) := -b([v, w]), \quad \forall w \in \mathfrak{g}.$$

It can be checked that the bracket

$$[v + a, w + b] = [v, w] + L_v b - L_w a, \quad \forall v + a, w + b \in \mathfrak{g} \oplus \mathfrak{g}^* \tag{16}$$

defines a Lie algebra structure on $\mathfrak{t}^*(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}^*$. Note that the restriction of the Courant bracket to the space $\Gamma(W^G) \cong \mathfrak{g} \oplus \mathfrak{g}^*$ of invariant sections of $TG \oplus T^*G$ coincides with the Lie bracket of the cotangent Lie algebra $\mathfrak{t}^*(\mathfrak{g})$.

Definition 9. *The vector space $\mathfrak{t}^*(\mathfrak{g})$ together with the Lie bracket (16) is called the cotangent Lie algebra of $\mathfrak{g}$.*

Cotangent Lie groups and cotangent Lie algebras are naturally related as follows.

Lemma 10. *The Lie algebra of the cotangent group T^*G is isomorphic to the cotangent Lie algebra $\mathfrak{t}^*(\mathfrak{g})$.*

Proof. The map

$$F : T^*G \to G \times T_e^*G, \quad F(\alpha_g) := (g, R_g^*\alpha), \quad \forall g \in G, \quad \forall \alpha_g \in T_g^*G. \quad (17)$$

is a diffeomorphism. The group structure of T^*G induces a group structure on $G \times T_e^*G$, with multiplication

$$(g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1 g_2, \alpha_1 + \mathrm{Ad}(g_1) \cdot \alpha_2). \quad (18)$$

Above $\mathrm{Ad}(g) \cdot \alpha \in T_e^*G$ is defined by

$$(\mathrm{Ad}(g) \cdot \alpha)(v) := \alpha\left((R_g \circ L_{g^{-1}})_* v\right), \quad \forall v \in T_eG.$$

It can be easily checked that

$$\begin{aligned} (L_{(g,\alpha)})_*(v + a) &= (L_g)_*(v) + \mathrm{Ad}(g) \cdot a \\ (R_{(g,\alpha)})_*(v + a) &= (R_g)_*(v) + a + L_v(\alpha) \end{aligned}$$

for any $v + a \in T_eG \oplus T_e^*G$. Using these relations, it is easy to check that for any $X + a, Y + b \in T_eG \oplus T_e^*G$,

$$[v + a, w + b] = \frac{d}{dt}\left(\mathrm{Ad}(\exp(t(v + a))(w + b)\right)|_{t=0} = [v, w] + L_v(b) - L_w(a)$$

which is precisely the Lie bracket of the cotangent algebra $\mathfrak{t}^*(\mathfrak{g}^*)$. □

Proposition 6 follows easily from these considerations. Clearly, any invariant generalized almost complex structure $\mathcal{J}$ on G is uniquely determined by its value $\mathcal{J}_e$ at the identity $e \in G$, which is an almost complex structure on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ and determines an invariant almost complex structure J on T^*G, compatible with the standard metric of T^*G. The only issue is to prove that $\mathcal{J}$ is integrable if and only if J is integrable. This follows from Lemma 10 (since $\mathcal{J}$ is invariant, it is integrable if and only if the space of *invariant* sections of the i-eigenbundle L of $\mathcal{J}$ is closed under the Courant bracket).

4 Invariant generalized complex structures on semi-simple Lie groups

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Let $\mathcal{J}$ be a generalized complex structure on G and $\mathcal{J}_e : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$ its value at the identity $e \in G$.

Let $\mathfrak{w}^{1,0}$ be the complex vector space of $(1,0)$-vectors of $\mathcal{J}_e$ which is isotropic with respect to the standard metric g_{can} of $\mathfrak{g}\oplus\mathfrak{g}^*$ of split signature. Define $\mathfrak{k} := \mathrm{pr}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{w}^{1,0})$, where $\mathrm{pr} : (\mathfrak{g}\oplus\mathfrak{g}^*)^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is the standard projection on the first factor. Since $\mathfrak{w}^{\mathbb{C}} = \mathfrak{w}^{1,0}\oplus\overline{\mathfrak{w}^{1,0}}$, also $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}+\bar{\mathfrak{k}}$. Let $\mathfrak{l}\subset\mathfrak{g}$ be the real part of $\mathfrak{k}\cap\bar{\mathfrak{k}}$, i.e. $\mathfrak{l}$ is a real vector space with $\mathfrak{k}\cap\bar{\mathfrak{k}} = \mathfrak{l}^{\mathbb{C}}$. The following Proposition is a straightforward consequence of Proposition 1 of Section 2.

Proposition 11. *The complex vector space of $(1,0)$-vectors at $e\in G$ of an invariant generalized complex structure $\mathcal{J}$ on G is of the form*

$$\mathfrak{w}^{1,0} = \{X+\omega(X),\quad \forall X\in\mathfrak{k}\}, \tag{19}$$

where $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, with $\mathfrak{k}+\bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$ and $\omega\in\Lambda^2(\mathfrak{k}^*)$ is a 2-form on $\mathfrak{k}$ with the following properties:

i) $\mathrm{Im}\left(\omega|_{\mathfrak{l}}\right)$ *is non-degenerate, where $\mathfrak{l}$ is the real part of $\mathfrak{k}\cap\bar{\mathfrak{k}}$;*

ii) $d_{\mathfrak{k}}(\omega) = 0$.

Conversely, let $\mathfrak{k}\subset\mathfrak{g}^{\mathbb{C}}$ be a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{k}+\bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$ and let $\mathfrak{l}$ be the real part of $\mathfrak{k}\cap\bar{\mathfrak{k}}$. If $\omega\in\Lambda^2(\mathfrak{k}^*)$ is a 2-form with the properties i) and ii), then $\mathfrak{w}^{1,0}$ as in (19) is the vector space of $(1,0)$-vectors at $e\in G$ of an invariant generalized complex structure $\mathcal{J}$ on G.

Remark 12. With the notations of Proposition 11, any 2-form $\omega\in\Lambda^2(\mathfrak{k}^*)$ with properties i) and ii) defines a symplectic form on $\mathfrak{l}$, namely $\mathrm{Im}\left(\omega|_{\mathfrak{l}}\right)$. In particular, the dimension of $\mathfrak{l}$ is even and moreover $\mathfrak{l}$ cannot be semi-simple.

Remark 13. Decompositions $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}+\bar{\mathfrak{k}}$ are in one to one correspondence with invariant complex structures on $F := G/L$, where L is the connected subgroup of G generated by $\mathfrak{l}$. In particular, if G is compact then $F = G/T'L'$ where T' is a subtorus of some torus $T\subset G$ such that $Z_G(T) = T\cdot L'$.

Due to Proposition 11, the problem of finding invariant generalized complex structures on G reduces to the algebraic problem of finding pairs $(\mathfrak{k},\omega)$ formed by a subalgebra $\mathfrak{k}\subset\mathfrak{g}^{\mathbb{C}}$ and a 2-form $\omega\in\Lambda^2(\mathfrak{k}^*)$, with properties stated in Proposition 11. We now consider this problem under the additional assumption that G is a semi-simple Lie group. Let $\sigma : \mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$ be the real structure of $\mathfrak{g}^{\mathbb{C}}$ which determines the real form $\mathfrak{g} = \mathrm{Lie}(G)$ of $\mathfrak{g}^{\mathbb{C}}$. We assume that $\mathfrak{k}$ is σ-regular, i.e. there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ which is σ-invariant and $\mathrm{ad}(\mathfrak{h})(\mathfrak{k})\subset\mathfrak{k}$. Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}+\sum_{\alpha\in R}\mathfrak{g}_\alpha$$

be the Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ associated to the choice of Cartan subalgebra $\mathfrak{h}$. Since $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, it is of the form

$$\mathfrak{k} = \mathfrak{h}_0 + \sum_{\alpha \in Q} \mathfrak{g}_\alpha, \tag{20}$$

where $Q \subset R$ is a closed set of roots and $\mathfrak{h}_0$ is a complex subspace of $\mathfrak{h}$, which contains $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, for any $\alpha \in Q^{\mathrm{s}} := Q \cap (-Q)$. Moreover, since $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$, also $\mathfrak{h}_0 + \overline{\mathfrak{h}}_0 = \mathfrak{h}$.

Definition 14. *The subalgebra $\mathfrak{k}$ is called σ-admissible if it is σ-regular, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} + \bar{\mathfrak{k}}$ and there is a closed 2-form $\omega \in \Lambda^2(\mathfrak{k}^*)$ such that $\mathrm{Im}\left(\omega|_{\mathfrak{l}}\right)$ is non-degenerate, where $\mathfrak{l}$ is the real part of $\mathfrak{k} \cap \bar{\mathfrak{k}}$.*

Our first result about generalized complex structures on semi-simple Lie groups concerns the split forms.

Proposition 15. *Suppose that $\mathfrak{g}$ is a split form of $\mathfrak{g}^{\mathbb{C}}$. There are no σ-admissible subalgebras of $\mathfrak{g}^{\mathbb{C}}$. In particular, there are no invariant generalized complex structures split forms of complex semi-simple Lie groups .*

Proof. Suppose that $\mathfrak{k}$ is a σ-admissible subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Since $\mathfrak{g}$ is a split form of $\mathfrak{g}^{\mathbb{C}}$, the real structure $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ preserves the root spaces, i.e. $\sigma(\mathfrak{g}_\alpha) = \overline{\mathfrak{g}}_\alpha = \mathfrak{g}_\alpha$, for any root $\alpha \in R$. Since $\mathfrak{k}$ is σ-admissible, $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$. We deduce that $Q = R$. This implies that $\mathfrak{k} = \mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{l} = \mathfrak{g}$. On the other hand, from Remark 12 the Lie algebra $\mathfrak{l}$ cannot be semi-simple. We obtained a contradiction. Our claim follows. ☐

4.1 Invariant generalized complex structures on compact semi-simple Lie groups

Let G be a compact semi-simple Lie group, with Lie algebra $\mathfrak{g}$. Our aim in this Section is to describe all invariant generalized complex structures on G. We will also prove that G has no invariant generalized Kähler structures.

As explained in the previous Section, the problem of determining invariant generalized complex structures on G reduces to the problem of finding σ-admissible subalgebras $\mathfrak{k}$ of $\mathfrak{g}^{\mathbb{C}}$, where $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is the real structure which determines the compact form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{h}$ be the σ-invariant Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, such that $\mathrm{ad}(\mathfrak{h})(\mathfrak{k}) \subset\subset \mathfrak{k}$. With respect to the Cartan decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha,$$

the real structure σ maps $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ and $\sigma : i\mathfrak{h}_{\mathbb{R}} \to i\mathfrak{h}_{\mathbb{R}}$ is the identity map.

Proposition 16. *Any admissible subalgebra $\mathfrak{k}$ of $\mathfrak{g}^{\mathbb{C}}$ is of the form*

$$\mathfrak{k} = \mathfrak{h}_0 + \sum_{\alpha \in R^+} \mathfrak{g}_\alpha, \tag{21}$$

where $\mathfrak{h}_0$ is a vector subspace of $\mathfrak{h}$ with $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 = \mathfrak{h}$ and R^+ is a system of positive roots of R.

Proof. Being a subalgebra, $\mathfrak{k}$ is of the form

$$\mathfrak{k} = \mathfrak{h}_0 + \sum_{\alpha \in R_0} \mathfrak{g}_\alpha,$$

where R_0 is a closed subset of R and $\mathfrak{h}_0$ is a subspace of $\mathfrak{h}$, which contains $[E_\alpha, E_{-\alpha}]$, where $E_\alpha \in \mathfrak{g}_\alpha$ is any root vector and $\alpha \in R_0^s := R_0 \cap (-R_0)$. The algebra $\mathfrak{k} \cap \bar{\mathfrak{k}}$ is reductive: its semi-simple part $\mathfrak{g}_{\mathrm{ss}}$ is generated by the vectors E_α, where $\alpha \in R_0^s$, and its center $\mathfrak{c}$ is the orthogonal complement, with respect to the Killing form, of $\{[E_\alpha, E_{-\alpha}], \quad \alpha \in R_0^s\}$ in $\mathfrak{h}_0 \cap \bar{\mathfrak{h}}_0$. On the other hand, since $\mathfrak{k}$ is admissible, $\mathfrak{k} \cap \bar{\mathfrak{k}}$ admits a symplectic form. Since a reductive Lie algebra admits a symplectic form if and only if it is abelian, $\mathfrak{g}_{\mathrm{ss}}$ must be trivial, hence $R_0^s = \emptyset$. Now, any asymmetric subset of roots is included into a set of positive roots. Since $\mathfrak{k}$ is admissible, $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$. This implies that $R_0 = R^+$ is a system of positive roots of R and that $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 = \mathfrak{h}$. Our claim follows.

□

As a consequence of Proposition 16, we obtain:

Corollary 17. *There are no invariant generalized Kähler structures on compact semi-simple Lie groups.*

Proof. Suppose that $(\mathcal{J}_1, \mathcal{J}_2)$ defines an invariant generalized Kähler structure on G. Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be the σ-admissible subalgebras of $\mathfrak{g}^{\mathbb{C}}$ associated to the vector spaces of $(1,0)$-vectors at $e \in G$ of $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively (see Proposition 11), and let R_1^+, R_2^+ be the positive root systems of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ (see Proposition 16). From Lemma 3, the following conditions on the R_1^+ and R_2^+ must hold:

$$\left(-R_1^+\right) \cap \left(-R_2^+\right) \subset R_1^+ \cup R_2^+, \quad \left(-R_1^+\right) \cap R_2^+ \subset R_1^+ \cup \left(-R_2^+\right). \tag{22}$$

The first inclusion of (22) implies that $R_2^+ = -R_1^+$. But then the second inclusion cannot hold. We obtained a contradiction. Our claim follows. □

We now describe invariant generalized complex structures on G. Let $\mathfrak{k}$ be a σ-admissible subalgebra of $\mathfrak{g}^{\mathbb{C}}$. From Lemma 16, $\mathfrak{k}$ is of the form (21), with $\mathfrak{k}_0$ such that $\mathfrak{h}_0 + \bar{\mathfrak{h}}_0 = \mathfrak{h}$. For simplicity, we restrict to the situation when $\mathfrak{h}_0 \oplus \bar{\mathfrak{h}}_0 = \mathfrak{h}$. We begin with the following observation:

Lemma 18. *The complex vector space $\mathfrak{h}_0$ satisfies $\mathfrak{h}_0 \oplus \overline{\mathfrak{h}}_0 = \mathfrak{h}$ if and only if*

$$\mathfrak{h}_0 = \{v + i\rho(v), \quad \forall v \in \mathfrak{h}_{\mathbb{R}}\} \tag{23}$$

where $\rho : \mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}$ is a complex structure of $\mathfrak{h}_{\mathbb{R}}$.

Proof. Since $\mathfrak{h}_0 \cap \bar{\mathfrak{h}}_0 = \{0\}$ and $\sigma : i\mathfrak{h}_{\mathbb{R}} \to i\mathfrak{h}_{\mathbb{R}}$ is the identity, $\mathfrak{h}_0$ and $i\mathfrak{h}_{\mathbb{R}}$ intersect trivially and the restriction of the standard projection $\mathrm{pr} : \mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}$ to $\mathfrak{h}_0$ is bijective. It follows that $\mathfrak{h}_0$ is of the form (23), where ρ is an endomorphism of $\mathfrak{h}_{\mathbb{R}}$. Moreover, since $\mathfrak{h}_0$ is a complex vector space, $\rho : \mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}$ is a complex structure. $\square$

Before stating the main result of this Section – Theorem 20 – we introduce the following convention of notations.

Notation 19. Let $\mathfrak{g}^{\mathbb{C}}$ be a semi-simple Lie algebra and $\mathfrak{h}$ a fixed Cartan subalgebra, with root system R. Let E_α be root vectors of a Weyl basis of $\mathfrak{g}^{\mathbb{C}}$. For any $\alpha \in R$, we define the covector $\omega_\alpha \in \mathfrak{g}^*$ by the conditions:

$$\omega_\alpha(E_\alpha) = 1, \quad \omega_\alpha(E_\beta) = 0, \forall \alpha \neq \beta, \quad \omega_\alpha|_{\mathfrak{h}} = 0.$$

We shall use this conventions in the statement of Theorem 20, where we assume implicitly that a Weyl basis corresponding to the Cartan decomposition (24) is chosen.

Theorem 20. *Let G be a compact semi-simple Lie group, with Lie algebra $\mathfrak{g}$. Chose a Cartan decomposition*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha \tag{24}$$

and a subalgebra

$$\mathfrak{k} = \mathfrak{h}_0 + \sum_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

where R^+ is a positive root system of R and $\mathfrak{h}_0 \subset \mathfrak{h}$ is a vector space such that $\mathfrak{h}_0 \oplus \bar{\mathfrak{h}}_0 = \mathfrak{h}$. Let $\omega \in \Lambda^2(\mathfrak{k}^*)$ be a 2-form on gk. The pair $(\mathfrak{k}, \omega)$ defines an invariant generalized complex structure on G if and only if

$$\omega = \omega_0 + \frac{1}{2} \sum_{\alpha, \beta \in R^+} \mu_{\alpha+\beta} K_{\alpha\beta} \omega_\alpha \wedge \omega_\beta + \sum_{\alpha \in R^+} \mu_\alpha \alpha \wedge \omega_\alpha, \tag{25}$$

where $\omega_0 \in \Lambda^2(\mathfrak{h}_0^*)$ is any 2-form on $\mathfrak{h}_0$ (extended trivially to $\mathfrak{k}$) and $\mu_\alpha, \mu_{\alpha+\beta} \in \mathbb{C}$, for any $\alpha, \beta \in R^+$.

Proof. Write ω in the form

$$\omega = \omega_0 + \sum_{\alpha,\beta\in R^+} b_{\alpha\beta}\omega_\alpha\wedge\omega_\beta + \sum_{\alpha\in\Pi}\sum_{\gamma\in R^+} c_{\alpha\gamma}\alpha\wedge\omega_\gamma, \tag{26}$$

where $\omega_0\in\Lambda^2(\mathfrak{h}_0)$, Π is the system of simple roots of R^+ and $b_{\alpha\beta}$, $c_{\alpha\gamma}$ are constants. Since $\mathfrak{h}_0\cap\bar{\mathfrak{h}}_0=\{0\}$, the condition i) of Proposition 11 is trivially satisfied. We will show that $d_{\mathfrak{k}}\omega=0$ if and only if ω is as in (25). For a root $\alpha\in R$, define

$$\begin{aligned} S_\alpha &= \{(\beta,\gamma)\in R:\quad \beta+\gamma=\alpha\}\\ S_\alpha^+ &= \{(\beta,\gamma)\in R^+:\quad \beta+\gamma=\alpha\}\end{aligned}$$

(clearly, $S_\alpha^+=\emptyset$ if $\alpha\notin R^+$). It is easy to check that

$$\begin{aligned} d\alpha &= -\sum_{\beta\in R^+}\alpha(H_\beta)\omega_\beta\wedge\omega_{-\beta}\\ d\omega_\alpha &= -\alpha\wedge\omega_\alpha-\frac{1}{2}\sum_{(\beta,\gamma)\in S_\alpha}\omega_\beta\wedge\omega_\gamma\end{aligned}$$

where, for any $\alpha\in R$, $H_\alpha:=[E_\alpha,E_{-\alpha}]$. It follows that

$$\begin{aligned}(d\omega)|_{\mathfrak{k}} = {}&-2\sum_{\alpha,\beta\in R^+} b_{\alpha\beta}\alpha\wedge\omega_\alpha\wedge\omega_\beta-\sum_{\alpha,\beta\in R^+} b_{\alpha\beta}\sum_{(\gamma,\delta)\in S_\alpha^+}K_{\gamma\delta}\omega_\gamma\wedge\omega_\delta\wedge\omega_\beta\\ &+\sum_{\alpha\in\Pi}\sum_{\beta\in R^+}c_{\alpha\beta}\alpha\wedge\beta\wedge\omega_\beta\\ &+\frac{1}{2}\sum_{\alpha\in\Pi}\sum_{\beta\in R^+}c_{\alpha\beta}\sum_{(\gamma,\delta)\in S_\beta^+}K_{\gamma\delta}\alpha\wedge\omega_\gamma\wedge\omega_\delta.\end{aligned}$$

We deduce that $d_{\mathfrak{k}}\omega=0$ if and only if the restriction to $\mathfrak{k}$ of the following three forms:

$$\begin{aligned}\eta_1 &:= -2\sum_{\alpha,\beta\in R^+} b_{\alpha\beta}\alpha\wedge\omega_\alpha\wedge\omega_\beta+\frac{1}{2}\sum_{i\in\Pi}\sum_{\beta\in R^+}c_{\alpha\beta}\sum_{(\gamma,\delta)\in S_\beta^+}K_{\gamma\delta}\alpha\wedge\omega_\gamma\wedge\omega_\delta,\\ \eta_2 &:= \sum_{\alpha,\beta\in R^+} b_{\alpha\beta}\sum_{(\gamma,\delta)\in S_\alpha^+}K_{\gamma\delta}\omega_\gamma\wedge\omega_\delta\wedge\omega_\beta\\ \eta_3 &:= \sum_{\alpha\in\Pi}\sum_{\beta\in R^+}c_{\alpha\beta}\alpha\wedge\beta\wedge\omega_\beta\end{aligned}$$

are zero. Recall that the roots take real values on $\mathfrak{h}_{\mathbb{R}}$. From Lemma 18, no root vanishes identically on $\mathfrak{h}_0$. Moreover, if $\alpha,\beta\in R$ are such that

$(\alpha+\beta)|_{\mathfrak{h}_0} = 0$, then $\alpha = -\beta$. Using these observations, we can rewrite the conditions $\eta_1|_{\mathfrak{k}} = 0$, $\eta_2|_{\mathfrak{k}} = 0$ and $\eta_3|_{\mathfrak{k}} = 0$ as follows: $\eta_3|_{\mathfrak{k}} = 0$ is equivalent to

$$\sum_{\alpha\in\Pi} c_{\alpha\beta}\alpha = \mu_\beta \beta, \quad \forall \beta \in R^+,$$

for some constants $\mu_\beta \in \mathbb{C}$. Similarly, $\eta_2|_{\mathfrak{k}} = 0$ is equivalent to the following statement: for any positive roots $\alpha, \beta, \gamma \in R^+$,

$$b_{\beta+\gamma,\alpha} K_{\beta\gamma} + b_{\alpha+\beta,\gamma} K_{\alpha\beta} + b_{\gamma+\alpha,\beta} K_{\gamma\alpha} = 0 \tag{27}$$

where $b_{\alpha+\beta,\gamma} = 0$ when $\alpha+\beta$ is not a root, and similarly for $b_{\alpha+\gamma,\beta} = 0$ and $b_{\beta+\gamma,\alpha} = 0$. Finally, the condition $\eta_1|_{\mathfrak{k}} = 0$ is equivalent to $b_{\alpha\beta} = \frac{\mu_{\alpha+\beta}}{2} K_{\alpha\beta}$, for any $\alpha, \beta \in R^+$ with $\alpha+\beta \in R$, and to $b_{\alpha\beta} = 0$ for any $\alpha, \beta \in R^+$ with $\alpha+\beta \notin R$. With the constants $b_{\alpha\beta}$ of this form, the compatibility condition (27) is just the Iacobi identity, hence it is satisfied.

□

5 Generalized Kähler structures on flag manifolds

Using the bi-Hermitian point of view on generalized Kähler geometry, we prove in this Section that invariant generalized Kähler structures on flag manifolds reduce essentially to usual invariant Kähler structures. More precisely, our aim in this Section is to prove the following Theorem.

Theorem 21. *Let $M = G/T$ be a flag manifold, where G is a compact simple Lie group and T a maximal torus of G. For any G-invariant generalized Kähler structure (J_1, J_2, g, b) on M, the pairs (g, J_1) and (J, g_2) are Kähler and $b \in \Omega^2(M)$ is closed.*

Theorem 21 is a consequence of the following Proposition.

Proposition 22. *Let $M = G/T$ be a flag manifold and (g, J) a G-invariant Hermitian structure on M, such that*

$$d\left(d\omega \circ J^3\right) = 0. \tag{28}$$

Then (g, J) is Kähler.

Proof. Recall that $M = G/T$ has an alternative description as an homogeneous complex manifold $G^{\mathbb{C}}/B$, where B is a Borel subgroup of $G^{\mathbb{C}}$ and $G \cap B$ is a maximal torus of G. We employ the notations of Section 2. In particular,

$\mathfrak{g}^{\mathbb{C}} = \mathrm{Lie}(G^{\mathbb{C}})$ has the Cartan decomposition (9) and the Lie algebra of B is of the form (10). Recall that $T_e^{\mathbb{C}}M$ is spanned by root vectors E_α of a Weyl basis of $\mathfrak{g}^{\mathbb{C}}$ and

$$J_e E_\alpha = i\epsilon_\alpha E_\alpha, \tag{29}$$

where $\epsilon_\alpha \in \{-1, +1\}$. Moreover, the set of roots $\alpha \in R$ for which $\epsilon_\alpha = 1$ is a system of positive roots of R. Similarly, the complexification of g at e satisfies $g(E_\alpha, E_{-\alpha}) = g_\alpha > 0$ and $g(E_\alpha, E_\beta) = 0$ when $\alpha + \beta \neq 0$. It is easy to check, using (11), (12) and (29), that for any $\alpha, \beta \in R^+$,

$$\begin{aligned}\frac{1}{2} d(d\omega \circ J^3)(E_\alpha, E_\beta, E_{-\alpha}, E_{-\beta}) &= K^2_{\alpha,\beta}\left(g_{\alpha+\beta} - g_\alpha - g_\beta\right) \\ &\quad + \epsilon_{\alpha-\beta} K^2_{\alpha,-\beta}\left(\epsilon_{\alpha-\beta} g_{\alpha-\beta} - g_\alpha + g_\beta\right).\end{aligned}$$

Our hypothesis $d\left(d\omega \circ J^3\right) = 0$ implies that

$$K^2_{\alpha,\beta}\left(g_{\alpha+\beta} - g_\alpha - g_\beta\right) + \epsilon_{\alpha-\beta} K^2_{\alpha,-\beta}\left(\epsilon_{\alpha-\beta} g_{\alpha-\beta} - g_\alpha + g_\beta\right) = 0. \tag{30}$$

We will now show that (30) implies that for any $\alpha, \beta \in R^+$ such that $\alpha + \beta \in R$,

$$g_{\alpha+\beta} = g_\alpha + g_\beta. \tag{31}$$

We will prove (31) by induction on the length of roots. If $\alpha, \beta \in R^+$ are both of length one, $\alpha - \beta$ is not a root and $K_{\alpha,-\beta} = 0$. Since $K_{\alpha,\beta} \neq 0$, condition (30) implies (31). Suppose now that (31) holds for all $\alpha, \beta \in R^+$ of length less or equal to k, such that $\alpha + \beta$ is a root. Let $\gamma, \delta \in R^+$ be two positive roots of length less or equal to $k+1$, such that $\gamma + \delta$ is a root. We will show that

$$g_{\gamma+\delta} = g_\gamma + g_\delta. \tag{32}$$

If $\gamma - \delta$ is not a root, our previous argument shows that (32) holds. Suppose now that $\gamma - \delta$ is a root. Without loss of generality, we can assume moreover that $\gamma - \delta \in R^+$. Since $\gamma = \delta + (\gamma - \delta)$ and γ has length at most $k+1$, both δ and $\gamma - \delta$ have length less or equal to k. From our induction hypothesis,

$$g_\gamma = g_\delta + g_{\gamma-\delta}. \tag{33}$$

Using (30) for $\alpha := \gamma$ and $\beta := \delta$, together with (33), we obtain (32). We proved that (31) holds, for any $\alpha, \beta \in R^+$ such that $\alpha + \beta \in R$. It follows that (g, J) is Kähler.

$\square$

Acknowledgement L. David acknowledges partial support by the grant 2-Cex 06-11-22/2006 of the Romanian Government.

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