Biharmonic and bianalytic maps

Michèle Benyounes, Eric Loubeau, Radu Slobodeanu

June 26, 2008

1 Overture

Biharmonic equation arose through a classical problem of elasticity:

> "In April, 1828, Poisson read before the Paris Academy of Sciences a memoir, destined to become famous, on the Theory of Elasticity. One of the many things that he did in that memoir was to formulate a theory of the equilibrium of elastic plates. If the plate is subjected to a pressure p (per unit of area) at the point (x, y), he showed that the deflexion w (i.e. *the displacement of a point of the plate in the direction of the pressure*) must satisfy the partial differential equation which we now write $d\nabla^4 w = p$, where $\nabla^4 w = \nabla^2(\nabla^2 w)$, ∇^2 denotes the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and d is a constant which we now call the "flexural rigidity" of the plate. It depends on the elastic quality of the material ... and it varies as the cube of the thickness." ([Lo])

Recently it has benefited of much attention from differential geometers, [M-O], mostly from the point of view of *real* Rimannian geometry. This paper fills a gap by dealing with the complex geometry of the biharmonic map problem.

The relation between biharmonic *functions* and complex bianalytic ones is the same as the relation between harmonic and analytic (holomorphic) functions. We shall find a way to recover this relation of biharmonicity with bianalyticity at the most general level, for *mappings* between Hermitian manifolds.

Even if the case of complex bianalytic functions (from the complex plane to itself) is already completely understood (see [B] and the references therein), we prefer to present it *ab initio* for the sake of clarity.

2 A very simple example, due to Love

Let $f : \mathbb{C} \longrightarrow \mathbb{C}, f = u + \mathrm{i}v$ a holomorphic map. It is classical that the holomorphicity condition, $\partial f/\partial \overline{z} = 0$, translates into Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0; \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

In this context, Love, [Lo], noticed that $U = yu - xv : \mathbb{R}^2 \to \mathbb{R}$ is a biharmonic function. Indeed, it is easy to see that:

$$\Delta U = 4\frac{\partial u}{\partial y}.$$

Then it is obvious that $\Delta^2 U = 0$.

Taking $V = xu + yv$, we have:

$$\begin{cases} \dfrac{\partial U}{\partial x} - \dfrac{\partial V}{\partial y} = -2v; \\ \dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x} = 2u \end{cases}$$

In particular, $\Delta V = 4\dfrac{\partial u}{\partial x}$ and therefore $\Delta^2 V = 0$.

Note that the folowing relation holds good for $F : \mathbb{C} \to \mathbb{C},\ F := U + \mathrm{i}V$:

$$\frac{\partial}{\partial \overline{z}}\left(\frac{\partial F}{\partial \overline{z}}\right) = 0 \tag{2.1}$$

3 The generality of Love's example

Definition 3.1. We call **bianalytic** a complex function that satisfies the condition (2.1).

Remark first the real version of (2.1) (where $F = U + \mathrm{i}V$):

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial^2 U}{\partial y^2} = 2\dfrac{\partial^2 V}{\partial x \partial y}; \\ \dfrac{\partial^2 V}{\partial x^2} - \dfrac{\partial^2 V}{\partial y^2} = -2\dfrac{\partial^2 U}{\partial x \partial y} \end{cases} \tag{3.1}$$

It is easy to see that any bianalytic function has the form $F(z) = \overline{z}A(z) + B(z)$, where A, B are holomorphic functions (and obviously $A = \frac{\partial F}{\partial \overline{z}}$).

Theorem 3.1. *A bianalytic complex function is biharmonic. Conversely, any biharmonic function is (locally) the real or imaginary part of a bianalytic function.*

Proof. Remark first that (3.1) implies:

$$\Delta U = 2\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right); \qquad \Delta V = -2\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right).$$

But the bianaliticity assures us that $\frac{\partial F}{\partial \overline{z}} = (\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}) + \mathrm{i}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x})$ is holomorphic, so its components are harmonic functions. Therefore we check imediatedly that $\Delta(\Delta U) = 0$, $\Delta(\Delta V) = 0$.

For the converse it sufices to apply *Almansi formula* [A] for f, supposed to be biharmonic:

$$\exists f_{1,2} \text{ harmonic functions, such that} \quad f = |z|^2 f_1 + f_2.$$

As $f_{1,2}$ are (locally) the real or imaginary part of some holomorphic functions, the proof follows. □

4 Geometrization of Love's example

Recall that *the second covariant derivative* (the generalized Hessian) of functions defined on a Riemannian manifold (M, g) is given by:

$$\nabla \mathrm{d}f(X,Y) := (\nabla_X \mathrm{d}f)(Y) = X(Y(f)) - (\nabla_X Y)(f), \forall X, Y \in \Gamma(TM).$$

We can easily check that this is a symmetric 2-covariant tensor. On an almost complex manifold we can naturally extend it to complexified tangent space $T^{\mathbb{C}}M$. Recall also that, for functions on an almost Hermitian manifold, the Laplacean and biLaplacean w.r.t. a complex unitary frame are given by

$$\Delta f = \sum_{j=1}^{m} \nabla \mathrm{d}f(\overline{Z}_j, Z_j), \qquad \Delta^2 f = \Delta(\Delta f).$$

Definition 4.1. A function $f : (M, g, J) \to \mathbb{C}$ on an almost Hermitian manifold is called **bianalytic** if

$$\nabla \mathrm{d}f(\overline{Z}, \overline{W}) = 0, \quad \forall Z, W \in \Gamma(T^{1,0}M). \tag{4.1}$$

Remark 4.1. If J is integrable, we can (locally) translate (4.1) in complex coordinates:

$$\frac{\partial^2 f}{\partial \overline{z}_i \partial \overline{z}_j} - \Gamma_{\overline{i}\,\overline{j}}^{\overline{k}} \frac{\partial f}{\partial \overline{z}_k} = 0, \quad \forall i, j = 1, ..., m. \tag{4.2}$$

In particular, in this case *any holomorphic function is trivially bianalytic.* To have also *bianalytic conjugate coordinate functions* $\overline{z}_\kappa$ we must have $\Gamma_{\overline{i}\,\overline{j}}^{\overline{\kappa}} = 0$ (this will force a Kähler metric on M to be flat).

In the following, we shall study the biharmonicity of such functions.

Remark 4.2. The classical result of Lichnerowicz [Li] tells us that on holomorphic functions (solutions of the *first order* Cauchy-Riemann equations), the (*second order*) Laplace equation reduces to a *first order* equation which is trivially satisfied if the metric (coefficients) g_{ij} satisfy a *first order* equation (cosymplectic condition). On bianalytic functions (solutions of *second order* equations (4.1)), the (*fourth order*) biLaplace equation reduces to a *second order* equation which is trivially satisfied if the metric (coefficients) g_{ij} satisfy a *second order* equation (some curvature condition).

Theorem 4.1. *On a Kähler manifold, every bianalytic function is biharmonic if and only if the metric is Ricci-flat.*

Proof. Using bianalyticity of f, the biLaplacean:

$$\Delta(\Delta f) = \sum_{j,k,l=1}^{m} Z_l \left(\overline{Z}_l \left(Z_j(\overline{Z}_j(f)) - \Gamma_{j\overline{j}}^{k} \cdot Z_k(f) \right) \right) - (\nabla_{Z_l} \overline{Z}_l) \left(Z_j(\overline{Z}_j(f)) - \Gamma_{j\overline{j}}^{k} \cdot Z_k(f) \right)$$

can be puted in the following form:

$$\mathcal{A}_{pq\overline{r}} \cdot Z_p \left(Z_q(\overline{Z}_r(f)) \right) + \mathcal{B}_{p\overline{q}} \cdot Z_p(\overline{Z}_q(f)) + \mathcal{C}_s \cdot \overline{Z}_s(f).$$

We can check that $\mathcal{A}$ vanishes identically and that the cancelation of $\mathcal{B}$ and $\mathcal{C}$ translates in the Ricci-flat condition (N.B. Kähler assumption is crucial). □

Remark 4.3. The above result is analogous to the following one: "On a Hermitian manifold, every holomorphic (local) function is harmonic if and only if the Hermitian structure is cosymplectic".

5 A replica of Lichnerowicz's result

Recall that *the second fundamental form* of a mapping $\varphi : (M, g) \to (N, h)$ given by:

$$\nabla \mathrm{d}\varphi(X,Y) := \nabla^{\varphi}_X (\mathrm{d}\varphi(Y)) - \mathrm{d}\varphi\left(\nabla^M_X Y\right), \forall X, Y \in \Gamma(TM).$$

We can see that that $\nabla \mathrm{d}\varphi \in \Gamma(T^*M \otimes T^*M \otimes \varphi^{-1}TN)$. For mappings between almost Hermitian manifolds we can naturally extend it to complexified tangent spaces. The tension field (analogous to the Laplacean for functions) is $\tau(\varphi) = \mathrm{trace}\nabla \mathrm{d}\varphi$.

W.r.t. a complex unitary frame, the equation for **biharmonic maps** between (almost) Hermitian manifolds is

$$\sum_{k=1}^{m} \left\{ \nabla^{\varphi}_{Z_k} \nabla^{\varphi}_{\overline{Z}_k} \tau(\varphi) - \nabla^{\varphi}_{\nabla^M_{Z_k} \overline{Z}_k} \tau(\varphi) - R^N(\mathrm{d}\varphi(Z_k), \tau(\varphi)) \mathrm{d}\varphi(\overline{Z}_k) \right\} = 0. \tag{5.1}$$

Definition 5.1. A mapping $\varphi : (M^{2m}, g, J) \to (N^{2n}, h, J^N)$ between almost Hermitian manifolds is called **bianalytic** if

$$\nabla \mathrm{d}\varphi(\overline{Z}, \overline{W}) \in \varphi^{-1}T^{(0,1)}N, \quad \forall Z, W \in \Gamma(T^{(1,0)}M). \tag{5.2}$$

Remark 5.1. If J and J^N are integrable, we can (locally) translate (5.2) in complex coordinates $\{z_i\}_{i=1,\dots,m}$ and $\{z_\alpha\}_{\alpha=1,\dots,n}$ on M and N, respectively:

$$\varphi^{\sigma}_{\bar{i}\bar{j}} - \Gamma^{\bar{k}}_{\bar{i}\bar{j}} \varphi^{\sigma}_{\bar{k}} + \widetilde{\Gamma}^{\sigma}_{AB} \varphi^{A}_{\bar{i}} \varphi^{B}_{\bar{j}} = 0, \quad \forall i, j = 1, ..., m, \forall \sigma = 1, ..., n. \tag{5.3}$$

In particular, in this case *any holomorphic map is trivially bianalytic.* Notice that bianalyticity of functions can be seen as a particular case of the above definition.

Let us study biharmonicity for such maps. We'll denote $\mathrm{d}' = \mathrm{d} \circ pr_{T^{(1,0)}N}$ and $\mathrm{d}'' = \mathrm{d} \circ pr_{T^{(0,1)}N}$.

Lemma 5.1. *A bianalytic map, φ, between Kähler manifolds is biharmonic if and only if:*

$$\begin{aligned} &\sum_{j,k=1}^{m} \left\{ \nabla^{\varphi}_{Z_k} \mathrm{d}'\varphi(R^M(Z_j, \overline{Z}_k)\overline{Z}_j) - \mathrm{d}'\varphi(R^M(Z_j, \nabla_{Z_k}\overline{Z}_k)\overline{Z}_j) \right\} - \\ &\quad \left\{ \nabla^{\varphi}_{Z_k} R^N(\mathrm{d}\varphi(Z_j), \mathrm{d}\varphi(\overline{Z}_k)) \mathrm{d}'\varphi(\overline{Z}_j) - R^N(\mathrm{d}\varphi(Z_j), \mathrm{d}\varphi(\nabla_{Z_k}\overline{Z}_k)) \mathrm{d}'\varphi(\overline{Z}_j) \right\} - \\ &\quad R^N(\mathrm{d}\varphi(Z_j), \tau(\varphi)) \mathrm{d}'\varphi(\overline{Z}_j) = 0. \end{aligned} \tag{5.4}$$

Proof. We denote by (I) and (II) the first two terms in (5.1). We shall make repeated use of (4.1) in the form:

$$\nabla^{\varphi}_{\overline{Z}} \mathrm{d}\varphi(\overline{W}) = \mathrm{d}\varphi\left(\nabla_{\overline{Z}} \overline{W}\right) + \varepsilon_{\overline{Z},\overline{W}}, \quad \varepsilon_{\overline{Z},\overline{W}} \in T^{(0,1)}N.$$

in order to reduce the order of the equation (5.1). We'll often use also the definition of being Kähler:

$$\nabla_X T^{(1,0)}M \subseteq T^{(1,0)}M, \quad \forall X \in TM.$$

$$
\begin{aligned}
(I) &= \nabla^{\varphi}_{Z_k}\left[\nabla^{\varphi}_{\overline{Z}_k}\nabla^{\varphi}_{Z_j}\mathrm{d}\varphi(\overline{Z}_j) - \nabla^{\varphi}_{\overline{Z}_k}\mathrm{d}\varphi(\nabla_{Z_j}\overline{Z}_j)\right]\\
&= \nabla^{\varphi}_{Z_k}\left[\nabla^{\varphi}_{Z_j}\nabla^{\varphi}_{\overline{Z}_k}\mathrm{d}\varphi(\overline{Z}_j) + \nabla^{\varphi}_{[\overline{Z}_k,Z_j]}\mathrm{d}\varphi(\overline{Z}_j) + R^N(\mathrm{d}\varphi(\overline{Z}_k),\mathrm{d}\varphi(Z_j))\mathrm{d}\varphi(\overline{Z}_j) - \mathrm{d}\varphi(\nabla_{\overline{Z}_k}\nabla_{Z_j}\overline{Z}_j)\right]\\
&= \nabla^{\varphi}_{Z_k}\left[\nabla^{\varphi}_{Z_j}\mathrm{d}\varphi(\nabla_{\overline{Z}_k}\overline{Z}_j) + \nabla^{\varphi}_{[\overline{Z}_k,Z_j]}\mathrm{d}\varphi(\overline{Z}_j) + R^N(\mathrm{d}\varphi(\overline{Z}_k),\mathrm{d}\varphi(Z_j))\mathrm{d}\varphi(\overline{Z}_j) - \mathrm{d}\varphi(\nabla_{\overline{Z}_k}\nabla_{Z_j}\overline{Z}_j)\right] +\\
&\quad + (0,1) - \text{type terms.}
\end{aligned}
$$

where the second right-hand term can be rewritten:

$$
\nabla^{\varphi}_{[\overline{Z}_k,Z_j]}\mathrm{d}\varphi(\overline{Z}_j) = \nabla^{\varphi}_{\nabla_{\overline{Z}_k}Z_j}\mathrm{d}\varphi(\overline{Z}_j) - \mathrm{d}\varphi\left(\nabla_{\nabla_{Z_j}\overline{Z}_k}\overline{Z}_j\right).
$$

With this we've made optimal use of (4.1) for the term (I).

Let's do the same for (II) (∇ will mean ∇^M).

$$
\begin{aligned}
-(II) =&\nabla^{\varphi}_{\nabla^M_{Z_k}\overline{Z}_k}\tau(\varphi) = \nabla^{\varphi}_{\nabla_{Z_k}\overline{Z}_k}\nabla^{\varphi}_{Z_j}\mathrm{d}\varphi(\overline{Z}_j) - \nabla^{\varphi}_{\nabla_{Z_k}\overline{Z}_k}\mathrm{d}\varphi(\nabla_{Z_j}\overline{Z}_j)\\
=&\nabla^{\varphi}_{Z_j}\nabla^{\varphi}_{\nabla_{Z_k}\overline{Z}_k}\mathrm{d}\varphi(\overline{Z}_j) + \nabla^{\varphi}_{[\nabla_{Z_k}\overline{Z}_k,Z_j]}\mathrm{d}\varphi(\overline{Z}_j) + R^N(\mathrm{d}\varphi(\nabla_{Z_k}\overline{Z}_k),\mathrm{d}\varphi(Z_j))\mathrm{d}\varphi(\overline{Z}_j)\\
&- \mathrm{d}\varphi\left(\nabla_{\nabla_{Z_k}\overline{Z}_k}\nabla_{Z_j}\overline{Z}_j\right) =\\
=&\nabla^{\varphi}_{Z_j}\mathrm{d}\varphi\left(\nabla_{\nabla_{Z_k}\overline{Z}_k}\overline{Z}_j\right) + \nabla^{\varphi}_{[\nabla_{Z_k}\overline{Z}_k,Z_j]}\mathrm{d}\varphi(\overline{Z}_j) + R^N(\mathrm{d}\varphi(\nabla_{Z_k}\overline{Z}_k),\mathrm{d}\varphi(Z_j))\mathrm{d}\varphi(\overline{Z}_j)\\
&- \mathrm{d}\varphi\left(\nabla_{\nabla_{Z_k}\overline{Z}_k}\nabla_{Z_j}\overline{Z}_j\right) +\\
&+ (0,1) - \text{type terms.}
\end{aligned}
$$

where again the second right-hand term can be rewritten:

$$
\nabla^{\varphi}_{[\nabla_{Z_k}\overline{Z}_k,Z_j]}\mathrm{d}\varphi(\overline{Z}_j) = \nabla^{\varphi}_{\nabla_{\nabla_{Z_k}\overline{Z}_k}Z_j}\mathrm{d}\varphi(\overline{Z}_j) - \mathrm{d}\varphi\left(\nabla_{\nabla_{Z_j}\nabla_{Z_k}\overline{Z}_k}\overline{Z}_j\right).
$$

With this we've made optimal use of (4.1) for the term (II).

Let us organize all the terms after the order of derivatives upon φ:
* terms with 3^{rd}-order derivatives := $\mathfrak{A}$,
* terms with 2^{nd}-order derivatives := $\mathfrak{B}$,
* terms with 1^{st}-order derivatives := $\mathfrak{C}$.
We have:

$$
\mathfrak{A} = \nabla^{\varphi}_{Z_k}\nabla^{\varphi}_{Z_j}\mathrm{d}\varphi(\nabla_{\overline{Z}_k}\overline{Z}_j) + \nabla^{\varphi}_{Z_k}\nabla^{\varphi}_{\nabla_{\overline{Z}_k}Z_j}\mathrm{d}\varphi(\overline{Z}_j) + \text{ all terms in } R^N.
$$

$$
\begin{aligned}
\mathfrak{B} &= -\nabla^{\varphi}_{Z_k}\mathrm{d}\varphi\left(\nabla_{\nabla_{Z_j}\overline{Z}_k}\overline{Z}_j\right) - \nabla^{\varphi}_{Z_k}\mathrm{d}\varphi(\nabla_{\overline{Z}_k}\nabla_{Z_j}\overline{Z}_j) - \nabla^{\varphi}_{Z_j}\mathrm{d}\varphi\left(\nabla_{\nabla_{Z_k}\overline{Z}_k}\overline{Z}_j\right) - \nabla^{\varphi}_{\nabla_{\nabla_{Z_k}\overline{Z}_k}Z_j}\mathrm{d}\varphi(\overline{Z}_j)\\
&= (\text{not.})\ \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \mathfrak{B}_4.
\end{aligned}
$$

$$
\mathfrak{C} = \mathrm{d}\varphi\left(\nabla_{\nabla_{Z_k}\overline{Z}_k}\nabla_{Z_j}\overline{Z}_j\right) + \mathrm{d}\varphi\left(\nabla_{\nabla_{Z_j}\nabla_{Z_k}\overline{Z}_k}\overline{Z}_j\right).
$$

We can rewrite:

$$
\begin{aligned}
\mathfrak{A} =&\nabla^{\varphi}_{Z_k}\left[\nabla\mathrm{d}\varphi(Z_j,\nabla_{\overline{Z}_k}\overline{Z}_j) + \nabla\mathrm{d}\varphi(\nabla_{\overline{Z}_k}Z_j,\overline{Z}_j)\right] + \nabla^{\varphi}_{Z_k}\mathrm{d}\varphi(\nabla_{Z_j}\nabla_{\overline{Z}_k}\overline{Z}_j) + \nabla^{\varphi}_{Z_k}\mathrm{d}\varphi(\nabla_{\nabla_{\overline{Z}_k}Z_j}\overline{Z}_j)\\
&+ \text{all terms in } R^N.
\end{aligned}
$$

But the first right-hand side term (which is a sum after j,k) is zero. Indeed:

$$\sum_{j,k} \nabla^{\varphi}_{Z_k} \left[\nabla \mathrm{d}\varphi(Z_j, \nabla_{\overline{Z}_k} \overline{Z}_j) + \nabla \mathrm{d}\varphi(\nabla_{\overline{Z}_k} Z_j, \overline{Z}_j)\right] = \sum_{j,k,l} \nabla^{\varphi}_{Z_k} \left[\Gamma^{\overline{l}}_{\overline{kj}} \nabla \mathrm{d}\varphi(Z_j, \overline{Z}_l) + \Gamma^{l}_{\overline{k}j} \nabla \mathrm{d}\varphi(Z_l, \overline{Z}_j)\right]$$

$$\sum_{j,k,l} \nabla^{\varphi}_{Z_k} \left[\Gamma^{\overline{l}}_{\overline{kj}} + \Gamma^{j}_{\overline{k}l}\right] \nabla \mathrm{d}\varphi(Z_j, \overline{Z}_l) = 0,$$

where we have re-indexed the second sum in j, l.

Therefore we have a simpler version of the formula for $\mathfrak{A}$:

$$\mathfrak{A} = \nabla^{\varphi}_{Z_k} \mathrm{d}\varphi(\nabla_{Z_j} \nabla_{\overline{Z}_k} \overline{Z}_j) + \nabla^{\varphi}_{Z_k} \mathrm{d}\varphi(\nabla_{\nabla_{\overline{Z}_k} Z_j} \overline{Z}_j) + \text{all terms in } R^N.$$

Remark now that the first two terms combine perfectly with $\mathfrak{B}_1$ and $\mathfrak{B}_2$ to give us:

$$\mathfrak{A} + \mathfrak{B}_1 + \mathfrak{B}_2 = \nabla^{\varphi}_{Z_k} \mathrm{d}\varphi \left(R^M(Z_j, \overline{Z}_k) \overline{Z}_j\right) + \text{all terms in } R^N. \tag{5.5}$$

The remaining terms in $\mathfrak{B}$ can be rewritten:

$$\begin{aligned} \mathfrak{B}_3 + \mathfrak{B}_4 = & -\nabla^{\varphi}_{Z_j} \mathrm{d}\varphi \left(\nabla_{\nabla_{Z_k} \overline{Z}_k} \overline{Z}_j\right) - \nabla^{\varphi}_{\nabla_{\nabla_{Z_k} \overline{Z}_k} Z_j} \mathrm{d}\varphi(\overline{Z}_j) \\ = & -\nabla \mathrm{d}\varphi \left(Z_j, \nabla_{\nabla_{Z_k} \overline{Z}_k} \overline{Z}_j\right) - \nabla \mathrm{d}\varphi \left(\nabla_{\nabla_{Z_k} \overline{Z}_k} Z_j, \overline{Z}_j\right) \\ & - \mathrm{d}\varphi \left(\nabla_{Z_j} \nabla_{\nabla_{Z_k} \overline{Z}_k} \overline{Z}_j\right) - \mathrm{d}\varphi \left(\nabla_{\nabla_{\nabla_{Z_k} \overline{Z}_k} Z_j} \overline{Z}_j\right). \end{aligned}$$

Again the first two right-hand side terms (which are sums after j,k) are zero. Indeed:

$$\begin{aligned} & -\sum_{j,k} \nabla \mathrm{d}\varphi \left(Z_j, \nabla_{\nabla_{Z_k} \overline{Z}_k} \overline{Z}_j\right) + \nabla \mathrm{d}\varphi \left(\nabla_{\nabla_{Z_k} \overline{Z}_k} Z_j, \overline{Z}_j\right) \\ = & -\sum_{j,k,p,q} \Gamma^{\overline{p}}_{k\overline{k}} \Gamma^{\overline{q}}_{\overline{pj}} \nabla \mathrm{d}\varphi \left(Z_j, \overline{Z}_q\right) + \Gamma^{\overline{p}}_{k\overline{k}} \Gamma^{q}_{\overline{p}j} \nabla \mathrm{d}\varphi \left(Z_q, \overline{Z}_j\right) \\ = & -\sum_{j,k,p,q} \Gamma^{\overline{p}}_{k\overline{k}} (\Gamma^{\overline{q}}_{\overline{pj}} + \Gamma^{j}_{\overline{p}q}) \nabla \mathrm{d}\varphi \left(Z_j, \overline{Z}_q\right) = 0, \end{aligned}$$

where we have re-indexed the second sum in j and q.

Taking the above fact into account, we observe that:

$$\begin{aligned} \mathfrak{B}_3 + \mathfrak{B}_4 + \mathfrak{C} = & -\mathrm{d}\varphi \left(\nabla_{Z_j} \nabla_{\nabla_{Z_k} \overline{Z}_k} \overline{Z}_j\right) - \mathrm{d}\varphi \left(\nabla_{\nabla_{\nabla_{Z_k} \overline{Z}_k} Z_j} \overline{Z}_j\right) \\ & + \mathrm{d}\varphi \left(\nabla_{\nabla_{Z_k} \overline{Z}_k} \nabla_{Z_j} \overline{Z}_j\right) + \mathrm{d}\varphi \left(\nabla_{\nabla_{Z_j} \nabla_{Z_k} \overline{Z}_k} \overline{Z}_j\right) \\ = & -\mathrm{d}\varphi \left(R^M \left(Z_j, \nabla_{Z_k} \overline{Z}_k\right) \overline{Z}_j\right). \end{aligned} \tag{5.6}$$

Now (5.5) and (5.6) give us the desired formula (5.4). □

Remark 5.2. Recall that is the Ricci operator $\mathfrak{Ric}$ is defined by:

$$\mathrm{Ricci}(Z, \overline{W}) = g(\mathfrak{Ric} Z, \overline{W}).$$

Therefore (5.4) can be read as

$$\mathrm{trace} \nabla (\mathrm{d}\varphi \circ \mathfrak{Ric}^M) - \mathrm{trace} \nabla (\mathfrak{Ric}^{\varphi^{-1}TN} \circ \mathrm{d}\varphi) = \mathfrak{Ric}^{\varphi^{-1}TN}(\tau(\varphi)) \bmod T^{(0,1)}N. \tag{5.7}$$

In particular, we can state the following

Theorem 5.1. *A bianalytic map from a Ricci-flat Kähler manifold to a a flat one is biharmonic.*

Remark 5.3. Consider *normal* coordinates at $p \in M$ and $\varphi(p) \in N$. Then (5.4) translates as:

$$\begin{aligned}
\tau_2(\varphi)^\sigma\Big|_p &= -\varphi^\sigma_{i\bar{j}} \cdot Ric_{\bar{i}j} - \varphi^\sigma_{\bar{i}} \cdot (Ric_{i\bar{j}})_j + \\
&+ \varphi^C_k\, \varphi^\gamma_{\bar{j}} \left(\varphi^{\overline{\alpha}}_{\bar{k}}\, \varphi^\beta_j - \varphi^\beta_{\bar{k}}\, \varphi^{\overline{\alpha}}_j\right) \cdot \left(\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_{\overline{\alpha}C} + \\
&+ 2\left[\varphi^{\overline{\alpha}}_{j\bar{j}}\, \varphi^\beta_k\, \varphi^\gamma_{\bar{k}} \cdot \left(\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_{\overline{\alpha}} - \varphi^\alpha_{j\bar{j}}\, \varphi^{\overline{\beta}}_k \varphi^\gamma_{\bar{k}} \cdot \left(\tilde{\Gamma}^\sigma_{\alpha\gamma}\right)_{\overline{\beta}}\right] + \\
&+ \varphi^\gamma_{j\bar{k}}\, \varphi^{\overline{\beta}}_{\bar{j}}\, \varphi^\alpha_k \cdot \left(\tilde{\Gamma}^\sigma_{\alpha\gamma}\right)_{\overline{\beta}} - \varphi^\gamma_{j\bar{k}}\, \varphi^\beta_{\bar{j}}\, \varphi^{\overline{\alpha}}_k \cdot \left(\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_{\overline{\alpha}} + \\
&+ \varphi^\alpha_{jk}\, \varphi^{\overline{\beta}}_{\bar{j}}\, \varphi^\gamma_{\bar{k}} \cdot \left(\tilde{\Gamma}^\sigma_{\alpha\gamma}\right)_{\overline{\beta}} = 0.
\end{aligned}$$

6 Appendix

The following proposition interprets the standard classes of almost Hermitian manifolds (M^{2m}, J, g) in terms of the "Christoffel symbols" Γ^A_{BC} with respect to a unitary frame of the form: $\{Z_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - \mathrm{i}Je_\alpha),\ Z_{\overline{\alpha}} = \frac{1}{\sqrt{2}}(e_\alpha + \mathrm{i}Je_\alpha)\}_{\alpha\in\overline{1,m}}$, where $g(e_\alpha, e_\beta) = \delta_{\alpha\beta}$.

As usual we denote $\Gamma^C_{AB} = g(\nabla_{Z_A} Z_B,\ Z_{\overline{C}})$, where $A = \alpha, \overline{\alpha}$ $(\alpha = 1, ..., n)$ and so does B and C. We have the decomposition:

$$\nabla_{Z_A} Z_B = \Gamma^\gamma_{AB} Z_\gamma + \Gamma^{\overline{\gamma}}_{AB} Z_{\overline{\gamma}}.$$

It is easy to check that:

$$\overline{\Gamma^C_{AB}} = \Gamma^{\overline{C}}_{\overline{AB}}; \qquad \Gamma^C_{AB} = -\Gamma^{\overline{B}}_{A\overline{C}}.$$

We can prove by direct check:

Proposition 6.1. *An almost Hermitian manifold (M, J, g) is*

(i) ***integrable*** *if and only if:* $\Gamma^{\overline{\alpha}}_{\beta\gamma} = 0,\ \forall \alpha, \beta, \gamma \in \overline{1, m}.$

(ii) $(1, 2)$-***symplectic*** *if and only if:* $\Gamma^{\overline{\alpha}}_{\overline{\beta\gamma}} = 0,\ \forall \alpha, \beta, \gamma \in \overline{1, m}.$

(iii) ***Kähler*** *if and only if:* $\Gamma^{\overline{\alpha}}_{\beta\gamma} = \Gamma^{\overline{\alpha}}_{\overline{\beta}\gamma} = 0,\ \forall \alpha, \beta, \gamma \in \overline{1, m}.$

(iv) ***cosymplectic*** *if and only if:* $\sum_\beta \Gamma^{\overline{\alpha}}_{\overline{\beta}\beta} = 0,\ \forall \alpha \in \overline{1, m}.$

Proof. (i) It is known that J is integrable if and only if:

$$(\nabla_{JX} J)\, Y = J\, (\nabla_X J)\, Y, \quad \forall X, Y \in TM.$$

This translates in complex terms as:

$$\nabla_Z T^{(1,0)}M \subseteq T^{(1,0)}M, \quad \forall Z \in T^{(1,0)}M.$$

(*ii*) Direct consequence of the definition of being (1,2)-symplectic:

$$\nabla_{\overline{Z}} T^{(1,0)}M \subseteq T^{(1,0)}M, \quad \forall Z \in T^{(1,0)}M.$$

(*iii*) Direct consequence of the definition of being Kähler:

$$\nabla_X T^{(1,0)}M \subseteq T^{(1,0)}M, \quad \forall X \in TM,$$

in particular $\Gamma^{\overline{\alpha}}_{B\gamma} = 0, \forall B$.

(*iv*) Direct consequence of the definition of being cosymplectic:

$$\sum_{\beta=1}^{n} \nabla_{Z_{\overline{\beta}}} Z_\beta \in T^{(1,0)}M.$$

□

So in the Kähler case, the only possibly non-zero "Christoffel symbols" are $\Gamma^{\alpha}_{\beta\gamma}$, $\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ and $\Gamma^{\overline{\alpha}}_{\beta\overline{\gamma}}$, $\Gamma^{\alpha}_{\overline{\beta}\gamma}$

Acknowledgements. I gratefully acknowledge the support by the CEx grant no. 2-CEx 06-11-22/25.07.2006.

References

[A] Almansi, E., *Sull integrazione dellequazione differenziale* $\Delta^{2n}u = 0$, Ann. Mat. Pura Appl., Suppl. 3, **2** (1898), 1–51.

[B] Balk, M. B., *Polyanalytic functions and their generalizations*, in *Complex analysis, I*, 195-253, Encyclopaedia Math. Sci., **85**, Springer, Berlin, 1997.

[Li] Lichnerowicz, A., *Applications Harmoniques et Variétés Kähleriennes*, Sympos. Math. **III** (1970), 341–402.

[Lo] Love, A.E.H., *Biharmonic analysis, especially in a rectangle, and its applications to the theory of elasticity*, Journal of the London Mathematical Society, **s1-3(2)** (1928), 144–156.

[M-O] Montaldo, S., Oniciuc, C., *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina **47** (2006), 1–22.

M. Benyounes Département de Mathématiques,
Université de Bretagne Occidentale
6, Avenue Victor Le Gorgeu CS 93837, 29238 Brest Cedex 3, France.
Email: Michele.Benyounes@univ-brest.fr

E. Loubeau Département de Mathématiques,
Université de Bretagne Occidentale
6, Avenue Victor Le Gorgeu CS 93837, 29238 Brest Cedex 3, France.
Email: loubeau@univ-brest.fr

R. Slobodeanu Faculty of Physics, Bucharest University,
405 Atomiştilor Str., CP Mg-11, RO - 077125 Bucharest, Romania.
Email: radualexandru.slobodeanu@g.unibuc.ro