ON A CONSTRUCTION OF HARMONIC RIEMANNIAN SUBMERSIONS

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In this note we present a (local) construction method of harmonic Riemannian submersions using Killing vector fields, similarly to the one previously developed by Bryant, [8], [17].

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1. INTRODUCTION

Harmonic Riemannian submersions, and more generally harmonic morphisms as defined by Fuglede and Ishihara [15], [16], are solutions of an over-determined differential system. For this reason, to give a general construction method for harmonic morphism with fixed source manifold (M, g_M) is not an easy task. In [8], Bryant used Killing vector fields for contructing harmonic morphisms with one-dimensional fibres, see also [17]. Nonetheless, not all harmonic morphisms of corank one are obtained in this way.

The aim of this short Note is to give a construction method in the spirit of [8] for maps of higher corank. For general facts about harmonic maps and morphisms we refer to [3], [18], [10], [11], [12]; see [14] for an updated account on Riemannian submersions.

In the last Section, we present some global examples, obtained with the help of the Boothby-Wang fibration of a regular Sasaki manifold, see [6], and also [4], [5], [7].

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2. THE GENERAL CONSTRUCTION

Let (M^{n+m}, g_M) be a Riemannian manifold, and $X_1, \dots X_m$ be unitary Killing vector fields on M, two-by-two orthogonal such that the following relations are satisfied

$$[X_i, X_j] = 0, \tag{1}$$

for any $i, j = 1, \dots, m$. Suppose moreover (restricting to open subsets of M if necessary) that $X_1, \dots X_m$ are nowhere vanishing, hence they are linearly independent in any point of M.

The relations (1) imply that the distribution generated by the given fields is integrable, hence from the Frobenius Theorem (see for example [9]) it is associated to an m-dimensional foliation $\mathcal{F}$.

Let $U \subset M$ be an open subset on which the foliation $\mathcal{F}$ is simple, in other words, the leaves of $\mathcal{F}|_U$ are the fibres of a submersion $\pi : U \to N^n$, where N is an n-dimensional manifold.

For any $i = 1, \dots, m$, consider the one form $\omega_i^\flat$, g_M-dual to the field X_i. It is defined by

$$\omega_i^\flat(*) = g_M(*, X_i).$$

We have the following

LEMMA 1. *With the notation as above, we have $\mathcal{L}_{X_i}\omega_j^\flat = 0$, for any $i, j = 1, \dots, m$.*

Proof. For any vector field Y on M, using (1), and the fact that X_i are Killing, we obtain

$$\begin{aligned}
(\mathcal{L}_{X_i}\omega_j^\flat)(Y) &= X_i\omega_j^\flat(Y) - \omega_j^\flat([X_i, Y]) \\
&= X_i g_M(Y, X_j) - g_M([X_i, Y], X_j) \\
&= g_M(\nabla_{X_i}Y, X_j) + g_M(Y, \nabla_{X_i}X_j) \\
&\quad - g_M(\nabla_{X_i}Y, X_j) + g_M(\nabla_Y X_i, X_j).
\end{aligned}$$

Hence

$$(\mathcal{L}_{X_i}\omega_j^\flat)(Y) = (\mathcal{L}_{X_i}g_M)(X_j, Y) = 0. \ \square$$

Put

$$g'_M = g_M - \sum_{i=1}^{m} (\omega_i^\flat)^2. \tag{2}$$

From Lemma 1, we obtain, for any $i = 1, \dots, m$, $\mathcal{L}_{X_i}g'_M = 0$. Consequently, there exists a Riemannian metric g_N on N such that $\pi^*(g_N) = g'_M$. By (2), we have the following relation

$$g_M = \pi^*(g_N) + \sum_{i=1}^{m} (\omega_i^\flat)^2,$$

hence the submersion $\pi : (U, g_M) \to (N, g_N)$ is horizontally conformal with dilation function 1, i.e. it is a Riemannian submersion.

PROPOSITION 2. *The map $\pi : (U, g_M) \to (N, g_N)$ is harmonic.*

Proof. Denoting ω^V the volume form on the fibres and using [3], Proposition 4.6.3, the harmonicity of π is equivalent to

$$\mathcal{L}_Z\omega^V = 0,$$

for any horizontal vector field Z. In our case, we have

$$\omega^V = \omega_1^\flat \wedge \cdots \wedge \omega_m^\flat,$$

and

$$\mathcal{L}_Z\omega^V = \sum_{j=1}^m \omega_1^\flat \wedge \cdots \wedge (\mathcal{L}_Z\omega_j^\flat) \wedge \cdots \wedge \omega_m^\flat$$

For any vertical vector fields $Y_1, \ldots, Y_m$ on M, we have

$$\begin{aligned}(\mathcal{L}_Z\omega^V)(Y_1, \ldots, Y_m) &= \sum_{j=1}^m (\omega_1^\flat \wedge \cdots \wedge (\mathcal{L}_Z\omega_j^\flat) \wedge \cdots \wedge \omega_m^\flat)(Y_1, \ldots, Y_m) \\ &= \sum_{j=1}^m \sum_{\sigma\in\mathfrak{S}_m} \omega_1^\flat(Y_{\sigma(1)}) \ldots (\mathcal{L}_Z\omega_j^\flat)(Y_{\sigma(j)}) \ldots \omega_m^\flat(Y_{\sigma(m)}).\end{aligned}$$

Since X_i are Killing, and $Y_{\sigma(j)}$ can be written as

$$Y_{\sigma(j)} = \sum_{i=1}^m \alpha_i^j X_i$$

we compute

$$\begin{aligned}(\mathcal{L}_Z\omega_j^\flat)(Y_{\sigma(j)}) &= d\omega_j^\flat(Z, Y_{\sigma(j)}) + Y_{\sigma(j)}(\omega_j^\flat(Z)) \\ &= Zg_M(X_j, Y_{\sigma(j)}) - g_M(X_j, [Z, Y_{\sigma(j)}]) \\ &= g_M(\nabla_Z X_j, Y_{\sigma(j)}) + g_M(\nabla_{Y_{\sigma(j)}} Z, X_j) \\ &= -2g_M(\nabla_{Y_{\sigma(j)}} X_j, Z) = -2\sum_{i=1}^m \alpha_i^j g_M(\nabla_{X_i} X_j, Z).\end{aligned}$$

If $i \neq j$, using $[X_i, X_j] = 0$, and the fact that X_i are Killing and orthogonal, we obtain $g_M(\nabla_{X_i} X_j, Z) = 0$.

If $i = j$, the condition that X_i are Killing and unitary yields

$$g_M(\nabla_{X_i} X_i, Z) = 0.$$

Consequently, the map π is a harmonic Riemannian submersion, and the foliation generated by the fields $X_1, \ldots, X_m$ is a Riemannian foliation. □

From Proposition 2 we obtain immediately

COROLLARY 3. *Let (M^{m+n}, g_M) be a Riemannian manifold, and let $X_1, \ldots, X_m$ be unitary Killing vector fields, two-by-two orthogonal, and two-by-two commuting. Then the integral submanifolds of the distribution generated by $X_1, \ldots, X_m$ are minimal submanifolds.*

3. **GLOBAL EXAMPLES**

The classical situation of the Hopf fibration can be used to obtain other examples of Riemannian submersions with the construction of the previous Section. These examples are *global* ones, whereas in general, our construction will work only locally.

It is well-known (cf. [5]) that the odd-dimensional sphere S^{2n+1} carries a natural Sasaki structure; let us denote by η the contact form, and by ξ the Killing vector field. Consider $s \geq 1$ an integer, put $p : S^{2n+1} \to \mathbb{CP}^n$ the Hopf fibration, and define the iterated fibred product over $\mathbb{CP}^n$

$$H^{2n+s} = \{(x_1, \ldots, x_s) \in S^{2n+1} \times \cdots \times S^{2n+1},\ p(x_1) = \cdots = p(x_s)\}.$$

Using the diagonal map

$$\delta : \mathbb{CP}^n \to \mathbb{CP}^n \times \cdots \times \mathbb{CP}^n,$$

where the product is taken s times, and the natural projection

$$S^{2n+1} \times \cdots \times S^{2n+1} \to \mathbb{CP}^n \times \cdots \times \mathbb{CP}^n$$

one can interpret H^{2n+s} as a principal torus bundle on $\mathbb{CP}^n$; denote by $\pi : H^{2n+s} \to \mathbb{CP}^n$ the induced map. On H^{2n+s} one defines, for all $i = 1, \ldots, s$ the 1-forms $\eta_i = (0, \ldots, \eta, \ldots, 0)$ (η is on the i-th position), and the dual vector fields $\xi_i = (0, \ldots, \xi, \ldots, 0)$; note that $\eta(\xi) = 1$.

Using [4], one verifies that the vector fields ξ_i are Killing, unitary, orthogonal, and satisfy (1). By construction, the distribution generated by the vector fields ξ_i, $i = 1, \ldots, s$ coincides with the vertical distribution of the map π, hence the space of leaves is $\mathbb{CP}^n$. We obtain $\pi : H^{2n+s} \to \mathbb{CP}^n$ is a harmonic Riemannian submersion.

Remark. The previous construction can be generalized as follows. Start with a regular Sasaki manifold $(M^{2n+1}, g_M, \eta, \xi, \Phi)$, and consider $p : M \to N^{2n}$ its Boothby-Wang fibration, [6]. It is known that p is a circle bundle, [6]. For an integer s, consider as before the iterated fibred product $\pi : H^{2n+s} \to N$, which becomes a torus bundle. The above argument goes through to prove that π is a harmonic Riemannian submersion.

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