

Pseudo-horizontally weakly conformal maps on Riemannian manifolds and Riemannian polyhedra

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1 Introduction

Harmonic maps are smooth mappings between Riemannian manifolds that are critical points for a natural functional, called *the energy*. Since its inception in 1964 by Eells and Sampson [ES64], this fundamental notion situated at the border between geometry and analysis provided a field of intensive fruitful research. Relationships with physics, or probabilities increased the interest of the mathematical community in this theory. Several hundreds of papers have been written on the subject, and many excellent monographs treat the theory of harmonic maps in deep details.

In the present paper, based on the talk given at the 8th Workshop in Differential Geometry and its Applications held in Cluj, we give a brief survey of some results obtained in a few personal (joint or not) works. We focus mainly on the class of *pseudo-harmonic morphisms*, (short PHM). Defined by the condition to pull-back germs of holomorphic functions to germs of harmonic functions, it was introduced in [Lou97] (following [BBdBR89]), with the aim of enlarging another remarkable class of harmonic maps, namely the class of *harmonic morphisms*, [F78], [I79]. On the one hand, pseudo-harmonic morphisms generalize harmonic morphisms, and on the other hand, the framework is more restrictive, as the target manifold is asked to be at least *Hermitian* (and sometimes, even Kähler). Loubeau proved that pseudo-harmonic morphisms are harmonic maps with an extra-property called *pseudo-horizontal weakly conformality* (short PHWC). Intuitively, the latter notion (which gives the title of my paper) is the condition of pulling-back the Hermitian structure to a partial Hermitian structure on the horizontal distribution. Asking this partial structure to satisfy a Kähler-type condition leads to the notion of *pseudo-horizontal homotheticity*, introduced in [AAB00]. Pseudo-horizontally homothetic submersions that are moreover harmonic, and take values in Kähler manifolds, behave well with respect to the geometry of the manifolds in question. We mention here the fact that their fibres are minimal submanifolds, and more generally, inverse

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images of complex submanifolds are minimal submanifolds. This provides an effective way of constructing minimal submanifolds, see [AAB00], [AA99]. These are all indications that pseudo-horizontal homotheticity plus harmonicity is very closed to holomorphy, while there is a priori no complex structure on the source manifold. Another indication in this direction is given in [A], where we proved stability of these maps, generalizing the case of holomorphic maps between Kähler manifolds.

The theory of harmonic maps between Riemannian manifolds was extended by Korevaar and Schoen to certain singular spaces, see [KS93]. Admissible Riemannian polyhedra are prototypes of the relevant singular spaces, being harmonic, geodesic, and Dirichlet spaces. Some basic examples are: smooth Riemannian manifolds, Riemannian orbit spaces, normal analytic spaces, Thom spaces etc. Following the developing of harmonic morphism theory in the smooth case, it became apparent that this notion should be expanded to the case of Riemannian polyhedra. This was done by Eells and Fuglede in [EF01]. They have also recovered many of the properties of harmonic maps from the smooth case. However, the most important results were proved only under the assumption that the target was still a Riemannian manifold. In [AB06], we continued these ideas. Our aim is to generalize pseudo harmonic morphisms to maps from an admissible Riemannian polyhedron into a Kähler manifold and to characterize them by geometric criteria and analytic criteria, similarly to Loubeau's result. This generalization hides several difficulties, as for example, the absence of global differential calculus on singular spaces. It is also not so easy to find a geometric condition that characterizes PHM on Riemannian polyhedra, since it makes no sense to talk about horizontal vectors. Another difficulty is in the use of germs of harmonic functions in the sense of Korevaar and Schoen, as the analytic aspect of our construction.

We would like to make a few final comments on a possible research direction along the lines of harmonic maps. Recall that a fundamental ingredient in the theory of harmonic maps is the first variation formula that describes harmonic maps as solutions of the Euler-Lagrange equations. Stable maps are defined by a positivity condition of the Hessian. In the case of manifolds, stable maps are described by the second variation formula, which relied the Hessian and the Jacobi operator. A natural question is to know whether stability can be introduced for maps between a Riemannian polyhedron and a Riemannian manifold. It is a natural question, as stable maps are easier to classify. In the same direction, it would be nice to have a second variation formula in the singular case. This would also allow to introduce PHH maps on Riemannian polyhedra, and to generalize all the results quoted above.

2 Preliminaries

We recall some basic facts on linear algebra used in the sequel.

Let (V, g) and (W, h) be two euclidean vector spaces and $L : V \rightarrow W$ a linear map.

Definition 2.1 *The adjoint operator of L is the map $L^* : W \rightarrow V$ characterized by*

$$g(v, L^*(w)) = h(L(v), w), \text{ for all } v \in V, w \in W.$$

Definition 2.2 *The L -horizontal component of V is the space defined by*

$$H^L := (\ker L)^\perp = \text{Im } L^*.$$

We denote by g_{H^L} the restriction of the inner product g to H^L .

Remark 2.3 *If W is endowed with a complex structure J such that:*

1. $h(JX, JY) = h(X, Y)$, for any $X, Y \in W$;
2. $\text{Im } L$ is J -invariant;

then on H^L we can define a linear complex structure: $J_H := L^{-1}JL$ and $L : (H^L, J_H) \rightarrow (W, J)$ becomes a complex linear map.

Remark 2.4 *For the complex linear map $L : (H^L, J_H, g_H) \rightarrow (W, J, h)$ defined above, by a simple computation it can be proved that:*

$$g_{H^L}(J_H-, J_H-) = g_{H^L}(-, -) \text{ if and only if } LL^*J = JLL^*.$$

3 PHWC maps on Riemannian manifolds ([Lou97], [BBdBR89]).

Let $\varphi : (M^m, g) \rightarrow (N^{2n}, J, h)$ be a map defined on a Riemann manifold with value in a Kähler one. For any point $x \in M$, we consider $d\varphi_x^* : T_{\varphi(x)}N \rightarrow T_xM$ the adjoint of the tangent map $d\varphi_x : T_xM \rightarrow T_{\varphi(x)}N$ and $H_x^\varphi := H^{d\varphi_x}$ the horizontal space of $d\varphi$ at x .

If $\text{Im } d\varphi_x$ is J -invariant, then one can define an almost complex structure $J_{H,x}$ on the space H_x^φ by

$$J_{H,x} = d\varphi_x^{-1} \circ J_{\varphi(x)} \circ d\varphi_x,$$

see the previous discussion.

Similarly, if the spaces $\text{Im } d\varphi_x$ are J -invariant for all x , then we define the almost complex structure on the horizontal distribution H^φ , by $J_H = d\varphi^{-1} \circ J \circ d\varphi$.

Definition 3.1 *Notation as before.*

- (i) *The map φ is called PHWC (i.e. pseudo-horizontally weakly conformal) at x if and only if $\text{Im}(d\varphi_x)$ is J -invariant and $g|_{H_x^\varphi}$ is $J_{H,x}$ -Hermitian.*
- (ii) *The map φ is called PHWC (pseudo-horizontally weakly conformal) if and only if it is PHWC at any point of X .*

From Remark 2.4 it follows that the PHWC condition at a point x is equivalent to: $d\varphi_x \circ d\varphi_x^*$ commutes with $J_{\varphi(x)}$.

This notion appears for the first time in [BBdBR89] in relation with stability of minimal immersions.

3.1 PHH maps and their stability ([AAB00], [AA99], [A]).

In the joint paper [AAB00], we have introduced a class of harmonic maps, defined on a Riemann manifold, with value in a Kähler manifold, called *PHH harmonic maps* which have a behaviour somewhat similar to that of holomorphic maps. Holomorphic maps between Kähler manifolds are typical examples of PHH harmonic maps, but examples of different flavour have been found in [AA99].

Let (M^m, g) be a Riemannian manifold, (N^{2n}, J, h) be a Kähler manifold, and φ a smooth map from M to N . Denote ∇^M the Levi-Civita connection on M , ∇^N the Levi-Civita connections on N , and $\tilde{\nabla}$ the induced connection in the bundle $\varphi^{-1}TN$.

Definition 3.2 *A map $\varphi : (M^m, g) \rightarrow (N^{2n}, J, h)$ from a Riemannian manifold to a Kähler manifold is called PHH (i.e. pseudo-horizontally homothetic) if and only if*

1. φ is PHWC;
2. J_H is parallel in horizontal directions (i.e. $\nabla_X^M J_H = 0$ for every $X \in H^\varphi$).

The above Definition 3.2 has a local version. If φ is PHWC at x , we say that φ is *PHH at x* if and only if

$$d\varphi_x \left((\nabla_v^M d\varphi_x^*(JY))_x \right) = J_{\varphi(x)} d\varphi_x \left((\nabla_v^M d\varphi_x^*(Y))_x \right)$$

for any horizontal tangent vector $v \in T_x M$, and any vector field Y , locally defined in a neighbourhood of $\varphi(x)$. By definition, a PHWC map is PHH if and only if

$$d\varphi(\nabla_X^M d\varphi^*(JY)) = Jd\varphi(\nabla_X^M d\varphi^*(Y)),$$

for any horizontal vector field X on M and any vector field Y on N , i.e. φ is PHH if and only if it is PHH at any point x of M .

Several non-trivial examples of PHH harmonic submersions can be found in [AA99], [AAB00] and a general recipe for producing harmonic PHH maps is to solve suitable algebraic systems, see [AA99].

An important property we discuss here is the stability of harmonic submersive maps $\varphi : (M^m, g) \rightarrow (N^{2n}, J, h)$ from a *compact* Riemannian manifold to a Kähler manifold. Recall that this property is controlled by a condition on the Hessian of the energy-functional, [Urk93], p. 155:

$$H(E)_\varphi(V, V) \geq 0,$$

for any section V of the bundle $\varphi^{-1}TN$.

Let R be the curvature tensor field on N , $\{\varepsilon_1, \dots, \varepsilon_m\}$ be an orthogonal vector frame on M , and V be a section in $\varphi^{-1}TN$. Denote by

$$\mathcal{R}_\varphi := \sum_{i=1}^m \frac{1}{\|\varepsilon_i\|^2} R(V, d\varphi(\varepsilon_i))d\varphi(\varepsilon_i),$$

by

$$\bar{\Delta}_\varphi := - \sum_{i=1}^m \frac{1}{\|\varepsilon_i\|^2} \left(\tilde{\nabla}_{\varepsilon_i} \tilde{\nabla}_{\varepsilon_i} - \tilde{\nabla}_{\nabla_{\varepsilon_i} \varepsilon_i} \right)$$

the second-order elliptic differential operator called the *rough Laplacian* of φ , (cf. [Urk93], pp. 155), and by

$$\mathcal{J}_\varphi := \bar{\Delta}_\varphi - \mathcal{R}_\varphi.$$

One of the useful properties of the rough Laplacian is the following, cf. [Urk93], pp. 156.

Proposition 3.3 *The rough Laplacian $\bar{\Delta}_\varphi$ satisfies*

$$\int_M h(\bar{\Delta}_\varphi V, W)v_M = \int_M h(\tilde{\nabla}V, \tilde{\nabla}W)v_M = \int_M h(V, \bar{\Delta}_\varphi W)v_M,$$

where V and W are sections on $\varphi^{-1}TN$, and

$$h(\tilde{\nabla}V, \tilde{\nabla}W) = \sum_{i=1}^m \frac{1}{\|\varepsilon_i\|^2} h(\tilde{\nabla}_{\varepsilon_i} V, \tilde{\nabla}_{\varepsilon_i} W).$$

We have then the following result.

Theorem 3.4 *Let (M^m, g) be a compact Riemann manifold, (N^{2n}, J, h) be a Kähler manifold, and $\varphi : M \rightarrow N$ be a harmonic PHH submersion. Then φ is (weakly) stable.*

4 PHWC maps on polyhedra.

4.1 Riemannian polyhedron ([EF01]).

A *polyhedron* X is a connected locally compact separable Hausdorff space X for which there exists a simplicial complex K and a homeomorphism $\theta : K \rightarrow X$. Any such pair (K, θ) is called a *triangulation* of X .

The complex K is necessarily countable and locally finite (see [S66] page 120) and the space X is path connected and locally contractible.

The *dimension* of X is by definition the dimension of K and it is independent of the triangulation.

If X is a polyhedron with specified triangulation (K, θ) , we shall speak of vertices, simplexes, i -skeletons, stars, space of links or tangent cones of X as the image under θ of vertices, simplexes, i -skeletons, stars, space of links or tangent cones of K .

A polyhedron X will be called *admissible*, (see [EF01]), if in some (hence in any) triangulation:

1. X is dimensionally homogeneous (i.e. all maximal simplexes have the same dimension $n = \dim(X)$),
2. X is locally $(n - 1)$ -chainable (i.e. for every connected open set $U \subset X$, the open set $U \setminus X^{(n-2)}$ is connected).

Replacing "homeomorphism" by "Lip homeomorphism" throughout the preceding definitions leads us to the notion of a *Lip polyhedron*.

A *null set* in a Lip polyhedron X is a set $Z \subset X$ such that Z meets every maximal simplex Δ , relative to a triangulation (K, θ) (hence any) in a set whose pre-image under θ has n -dimensional Lebesgue measure 0, $n = \dim(\Delta)$. Note that "*almost everywhere*" (a.e.) means everywhere except in some null set.

A *Riemannian polyhedron*, (see [EF01]), $X = (X, g)$ is defined as a Lip polyhedron X with a specified triangulation (K, θ) , endowed with a covariant bounded measurable Riemannian metric tensor g_Δ on each maximal simplex Δ satisfying the ellipticity condition below. In fact, suppose that X has homogeneous dimension n and choose a measurable Riemannian metric g_Δ on the open euclidean n -simplex $\theta^{-1}(\Delta^\circ)$ of K . In terms of euclidean coordinates $\{x_1, \dots, x_n\}$ of points $x = \theta^{-1}(p)$, g_Δ thus assigns to almost every point $p \in \Delta^\circ$ (or x) an $n \times n$ symmetric positive definite matrix $g_\Delta = (g_{ij}^\Delta(x))_{i,j=1,\dots,n}$ with measurable real entries; and there is a constant $\Lambda_\Delta > 0$ such that (ellipticity condition):

$$\Lambda_\Delta^{-2} \sum_{i=0}^n (\xi^i)^2 \leq \sum_{i,j} g_{ij}^\Delta(x) \xi^i \xi^j \leq \Lambda_\Delta^2 \sum_{i=0}^n (\xi^i)^2$$

for a.e. $x \in \theta^{-1}(\Delta^\circ)$ and every $\xi = (\xi^1, \dots, \xi^n) \in \mathbf{R}^n$. This condition amounts to the components of g_Δ being bounded and it is independent not only of the choice of the euclidean frame on $\theta^{-1}(\Delta^\circ)$ but also of the chosen triangulation.

For simplicity of statements we shall sometimes require that, relative to a fixed triangulation (K, θ) of a Riemannian polyhedron X (uniform ellipticity condition),

$$\Lambda := \sup \{ \Lambda_\Delta : \Delta \text{ is simplex of } X \} < \infty.$$

4.2 Energy of maps [EF01].

The concept of energy of maps from Riemannian domains into arbitrary metric spaces Y was defined and investigated by Korevaar and Schoen [KS93]. Later on,

Eells and Fuglede [EF01] extended this concept to the case of maps from an admissible Riemannian polyhedron X with simplexwise smooth Riemannian metric. Thus, the *energy* $E(\varphi)$ of a map φ from X to the space Y is defined as the limit of suitable approximate energy expressed in terms of the distance function d_Y of Y .

It is proved in [EF01] that the maps $\varphi : X \rightarrow Y$ of finite energy are precisely those quasicontinuous (i.e. has a continuous restriction to closed sets, whose complements have arbitrarily small capacity, (see [EF01] page 153)) whose restriction to each top dimensional simplex of X has finite energy; $E(\varphi)$ is the sum of the energies of these restrictions.

Let (X, g) be an admissible m -dimensional Riemannian polyhedron with simplexwise smooth Riemannian metric. It is not required that g is continuous across lower dimensional simplexes. The target (Y, d_Y) is an arbitrary metric space.

Denote $L_{loc}^2(X, Y)$ the space of all μ_g -measurable (μ_g the volume measure of g) maps $\varphi : X \rightarrow Y$ having separable essential range and for which the map $d_Y(\varphi(\cdot), q) \in L_{loc}^2(X, \mu_g)$ (i.e. locally μ_g -squared integrable) for some point q (hence by triangle inequality for any point). For $\varphi, \psi \in L_{loc}^2(X, Y)$ define their distance $D(\varphi, \psi)$ by:

$$D^2(\varphi, \psi) = \int_X d_Y^2(\varphi(x), \psi(y)) d\mu_g(x).$$

Two maps $\varphi, \psi \in L_{loc}^2(X, Y)$ are said to be *equivalent* if $D(\varphi, \psi) = 0$ (i.e. $\varphi(x) = \psi(x)$ μ_g -a.e.). If the space X is compact then $D(\varphi, \psi) < \infty$ and D is a metric on $L_{loc}^2(X, Y) = L^2(X, Y)$ which is complete if the space Y is complete [KS93].

The *approximate energy density* of the map $\varphi \in L_{loc}^2(X, Y)$ is defined for $\epsilon > 0$ by:

$$e_\epsilon(\varphi)(x) = \int_{B_X(x, \epsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\epsilon^{m+2}} d\mu_g(x').$$

The function $e_\epsilon(\varphi) \geq 0$ is locally μ_g -integrable.

The *energy* $E(\varphi)$ of a map φ of class $L_{loc}^2(X, Y)$ is:

$$E(\varphi) = \sup_{f \in C_c(X, [0, 1])} \left(\limsup_{\epsilon \rightarrow 0} \int_X f e_\epsilon(\varphi) d\mu_g \right),$$

where $C_c(X, [0, 1])$ denotes the space of continuous functions from X to the interval $[0, 1]$ with compact support.

A map $\varphi : X \rightarrow Y$ is said to be *locally of finite energy*, and we write $\varphi \in W_{loc}^{1,2}(X, Y)$, if $E(\varphi|_U) < \infty$ for every relatively compact domain $U \subset X$, or equivalently if X can be covered by domains $U \subset X$ such that $E(\varphi|_U) < \infty$.

For example (see [EF01] Lemma 4.4), every Lip continuous map $\varphi : X \rightarrow Y$ is of class $W_{loc}^{1,2}(X, Y)$. In the case when X is compact, $W_{loc}^{1,2}(X, Y)$ is denoted by $W^{1,2}(X, Y)$, the space of all maps of finite energy.

$W_c^{1,2}(X, Y)$ denotes the linear subspace of $W^{1,2}(X, Y)$ consisting of all maps of finite energy of compact support in X .

One can show (see [EF01] Theorem 9.1) that a real function $\varphi \in L_{loc}^2(X)$ is locally of finite energy if and only if there is a function $e(\varphi) \in L_{loc}^1(X)$, named *energy density* of φ , such that (weak convergence):

$$\lim_{\epsilon \rightarrow 0} \int_X f e_\epsilon(\varphi) d\mu_g = \int_X f e(\varphi) d\mu_g, \text{ for each } f \in C_c(X).$$

4.3 Harmonic maps [EF01].

Let (X, g) be an arbitrary admissible Riemannian polyhedron (g just bounded measurable with local elliptic bounds), $\dim(X) = m$ and (Y, d_Y) a metric space.

A continuous map $\varphi : X \rightarrow Y$ of class $W_{loc}^{1,2}(X, Y)$ is said to be *harmonic* if it is *bi-locally E-minimizing*, i.e. X can be covered by relatively compact subdomains U for which there is an open set $V \supset \varphi(U)$ in Y such that

$$E(\varphi|_U) \leq E(\psi|_U)$$

for every continuous map $\psi \in W_{loc}^{1,2}(X, Y)$, with $\psi(U) \subset V$ and $\psi = \varphi$ in $X \setminus U$.

We replace now the metric space (Y, d_Y) by a smooth Riemannian manifold (N, h) without boundary and $\dim_{\mathbf{R}} N = n$. By $\Gamma_{\alpha\beta}^k$ we denote the Christoffel symbols on N .

A *weakly harmonic map* $\varphi : X \rightarrow N$ is a quasicontinuous map (a map which is continuous on the complement of open sets of arbitrarily small capacity; in the case of the Riemannian polyhedron X it is just the complement of open subsets of the $(m-2)$ -skeleton of X) of class $W_{loc}^{1,2}(X, N)$ with the following property: for any chart $\eta : V \rightarrow \mathbf{R}^n$ on N and any quasiopen set $U \subset \varphi^{-1}(V)$ of compact closure in X , the equation

$$\int_U \langle \nabla \lambda, \nabla \varphi^k \rangle d\mu_g = \int_U \lambda (\Gamma_{\alpha\beta}^k \circ \varphi) \langle \nabla \varphi^\alpha, \nabla \varphi^\beta \rangle d\mu_g,$$

holds for every $k = 1, \dots, n$ and every bounded function $\lambda \in W_0^{1,2}(U)$.

Ishihara characterization on Riemannian manifolds extends to polyhedra.

Theorem 4.1 [EF01] *For a continuous map $\varphi \in W_{loc}^{1,2}(X, N)$ the following are equivalent:*

- (a) φ is harmonic,
- (b) φ is weakly harmonic,
- (c) φ pulls convex functions on open sets $V \subset N$ back to subharmonic functions on $\varphi^{-1}(V)$.

4.4 Horizontally weakly conformal maps [EF01].

Considering maps from a Riemannian polyhedron with values into a Riemannian manifold we recover, as in the smooth case, the notion of horizontally weak conformality.

Let (N, g_N) denote an n -Riemannian manifold without boundary and suppose that the polyhedron X is admissible. A continuous map $\varphi : X \rightarrow N$ of class $W_{loc}^{1,2}(X, N)$ is called *horizontally weakly conformal* (see [EF01]) if there exist a scalar λ , defined *a.e.* in X , such that:

$$\langle \nabla(v \circ \varphi), \nabla(w \circ \varphi) \rangle = \lambda[g_N(\nabla_N v, \nabla_N w) \circ \varphi] \text{ a.e. in } X$$

for every pair of functions $v, w \in C^1(N)$. Henceforth ∇_N denote the gradient operator on N and ∇ the gradient operator defined *a.e.* on the domain space (X, g) .

The property of horizontally weak conformality is a local one, thus it reads in terms of local coordinates (y^α) in N ,

$$\langle \nabla \varphi^\alpha, \nabla \varphi^\beta \rangle = \lambda(g_N^{\alpha\beta} \circ \varphi) \text{ a.e. in } X$$

for $\alpha, \beta = 1, \dots, n$. Taking $\alpha = \beta$, λ is uniquely determined and $\lambda \geq 0$ *a.e.* in X .

4.5 PHWC maps [EF01].

Replacing the Riemannian manifold with an admissible Riemannian polyhedron but keeping as target a Kähler manifold, also in the polyhedra case the class of harmonic maps can be enlarged. For this, the notion of *pseudo-horizontally weakly conformal* maps was extended to Riemannian polyhedra, see [AB06]. (The terminology used in [AB06] is similar to the one used in [Lou97].)

Let (X, g) be an n -dimensional admissible Riemannian polyhedron and (N, J^N, g_N) a Hermitian manifold of $\dim_{\mathbf{R}} N = 2n$, without boundary.

We denote by $Holom(N) = \{f : N \rightarrow \mathbf{C}, f \text{ local holomorphic function}\}$. In what follows, the gradient operator and the inner product in (X, g) are well defined *a.e.* in X and will be denoted by ∇ and \langle, \rangle respectively.

Definition 4.2 [AB06] Let $\varphi : X \rightarrow N$ be a continuous map of class $W_{loc}^{1,2}(X, N)$. φ is called *pseudo-horizontally weakly conformal (shortening PHWC)*, if for any pair of local holomorphic functions $v, w \in Holom(N)$, such that $v = v_1 + iv_2$, $w = w_1 + iw_2$, we have:

$$\begin{cases} \langle \nabla(w_1 \circ \varphi), \nabla(v_1 \circ \varphi) \rangle - \langle \nabla(w_2 \circ \varphi), \nabla(v_2 \circ \varphi) \rangle = 0 & \text{a.e. in } X \\ \langle \nabla(w_2 \circ \varphi), \nabla(v_1 \circ \varphi) \rangle + \langle \nabla(w_1 \circ \varphi), \nabla(v_2 \circ \varphi) \rangle = 0 & \text{a.e. in } X \end{cases} \quad (1)$$

Since the above definition is a local one, it is sufficient to check the identities (1) in local complex coordinates (z_1, z_2, \dots, z_n) in N . Taking $z_A = x_A + iy_A, \forall A = 1, \dots, n$, the relations (1), $\forall A, B = 1, \dots, n$, read:

$$\begin{cases} \langle \nabla \varphi_1^B, \nabla \varphi_1^A \rangle - \langle \nabla \varphi_2^B, \nabla \varphi_2^A \rangle = 0 & \text{a.e. in } X \\ \langle \nabla \varphi_2^B, \nabla \varphi_1^A \rangle + \langle \nabla \varphi_1^B, \nabla \varphi_2^A \rangle = 0 & \text{a.e. in } X \end{cases} \quad (2)$$

where

$$\begin{cases} \varphi_1^A := x_A \circ \varphi, \varphi_2^A := y_A \circ \varphi, & \forall A = 1, \dots, n, \\ \varphi_1^B := x_B \circ \varphi, \varphi_2^B := y_B \circ \varphi, & \forall B = 1, \dots, n. \end{cases}$$

If the source manifold is a smooth Riemannian one, without boundary, we obtain from the above definition exactly the definition of PHWC maps gave in [Lou97] or [BW03], [AAB00].

The next proposition justifies the use of the term 'horizontally weakly conformal'. When the dimension of the target is one, we obtain, as in the smooth case, an equivalence between horizontally weak conformality and pseudo-horizontally weak conformality.

Proposition 4.3 [AB06] *Let $\varphi : X \rightarrow N$ be a horizontally weakly conformal map (see subsection 4.4) from a Riemannian admissible polyhedron (X, g) into a Hermitian manifold (N, J^N, g_N) . Then φ is pseudo-horizontally weakly conformal. If the complex dimension of N is equal to one, then the two conditions are equivalent.*

PHWC maps on Riemannian polyhedra can be characterised using germs of holomorphic functions on the target Hermitian manifolds as follows:

Proposition 4.4 [AB06] *Let $\varphi : X \rightarrow N$ be a continuous map of class $W_{loc}^{1,2}(X, N)$. Then φ is pseudo-horizontally weakly conformal if and only if for any local holomorphic function $\psi : N \rightarrow \mathbf{C}$, $\psi \circ \varphi$ is also pseudo-horizontally weakly conformal.*

The next proposition makes clear the relation between PHWC maps on Riemannian polyhedra and holomorphic maps on target Hermitian manifolds.

Proposition 4.5 [AB06] *Let $\varphi : X \rightarrow N$ be a continuous map of class $W_{loc}^{1,2}(X, N)$ and (P, J^P, g_P) another Hermitian manifold of $\dim_{\mathbf{R}} P = 2p$. Then φ is pseudo-horizontally weakly conformal if and only if for every local holomorphic map $\psi : N \rightarrow P$, $\psi \circ \varphi$ is also pseudo-horizontally weakly conformal.*

Remark 4.6 *It would be interesting to know what means stability in the polyhedra case. In the smooth case the stability condition is controlled by the Hessian of the energy (cf. the second variation formula [Urk93], pg.155). For maps defined on Riemannian polyhedra with values in Riemannian manifolds an equivalent formula for the first variation of the energy functional is given in [EF01], pg.224. A natural question arises: is it possible to find an equivalent "second variation formula" in the polyhedra case and to find a class of maps defined on Riemannian polyhedra with values in Riemannian manifolds such that this maps are "stable"?*

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