Convex polytopes passing through circles

by **Tudor Zamfirescu**

Introduction. Some convex bodies can pass through circles smaller than the section of their circumscribed cylinders. This was observed already by Zindler [3] in 1920, for a certain affine image of the cube.

Here, a *cylinder* will be a set in $\mathbb{R}^3$ congruent to $C \times \mathbb{R}$, where C is a circle in $\mathbb{R}^2$. The radius of C is called *radius* of the cylinder.

Let $K \subset \mathbb{R}^3$ be a *convex body*, i.e., a compact convex set with interior $\mathrm{int}\, K \neq \emptyset$. We say that K *passes through the circle* C if some rigid motion brings K from one side of the plane Π_C of C to the other side, without hitting $\Pi_C \setminus \mathrm{conv} C$ at any time. Let $r_p(K)$ be the radius of the smallest circle through which K can pass.

We say that the cylinder Z *surrounds* the convex body K if $K \subset \mathrm{conv} Z$ and $Z \cap K \neq \emptyset$. Let Ξ_K denote the cylinder with the z-axis as symmetry axis, congruent to a cylinder of smallest radius surrounding K, and $r_c(K)$ its radius.

Goal and preparations. In this note we give sufficient conditions for the existence of circles through which the convex polytope P can pass, of radius smaller than the radius of Ξ_P, i.e., for the inequality $r_p(P) < r_c(P)$ to hold.

How many convex bodies enjoy this property? Many? Few? We proved in [2] that all convex bodies, except those in a nowhere dense subset, enjoy it (the space of all convex bodies being equipped with the usual Pompeiu-Hausdorff metric). But given a single one, how can we decide whether it has this property or not? For example, the regular tetrahedron has it, the cube not.

Since the polytopes in this paper are always convex, we shall omit mentioning it. We denote by $V(P)$ the vertex set of the polytope P.

Suppose that the vertical cylinder Z defined by $x^2+y^2=1$ surrounds the polytope P. Let P_ξ be the intersection of P with the plane $z=\xi$. Consider the horizontal lines meeting both the z-axis and $V(P)\cap Z$. There are just finitely many of them. Call these lines and those parallel to them *critical.*

For every ξ, the set $P_\xi \cap Z$ is finite. If $P_\xi \cap Z \neq \emptyset$, then every point of $P_\xi \cap Z$ (which must be a vertex of P_ξ if the convex polygon P_ξ is not degenerate) is either a vertex of P or an interior point of an edge of P lying entirely on Z.

Let $J = \{\xi : P_\xi \cap Z \neq \emptyset\}$. This set is a finite union of closed intervals, each of which is possibly reduced to a single point.

For $\xi \in \mathrm{bd}J$, $P_\xi \cap Z \subset V(P)$, but the converse is in general not true.

Result. With the above preparations we can formulate our criterium.

Theorem. *If* $(0,0,\xi) \notin \mathrm{conv}(P_\xi \cap Z)$ *for every* $\xi \in \mathrm{int}J$, *then* $r_p(P) < 1$.

Note that the condition in the Theorem is sufficient, but not necessary, for $r_p(P) < 1$. To see this, it suffices to take P to be a regular tetrahedron T_1 with one vertical edge and all vertices on Z. The condition of the Theorem is not verified. However, it will be verified, if the tetrahedron P is the (larger) regular tetrahedron T_2 positioned such that $Z = \Xi_{T_2}$, that is, with two opposite horizontal edges. Hence $r_p(T_1) < r_p(T_2) < 1$ (see also the last Section). More precisely, $r_p(T_1) = 0,844...$

Proof. In this proof we keep fixed the polytope P and move a unit circle C such that $C \cap P = \emptyset$ at all times and $\mathrm{conv}C$ meets all points of P during its movement; the presented path of C is equivalent to moving P through C without even meeting $\Pi_C \setminus \mathrm{intconv}C$.

Once this achieved, it is clear that P also passes through a circle concentric with and slightly smaller than C.

Let us move the circle C, from a position far above P downwards, keeping $C \subset Z$ as long as $P_\xi \cap Z = \emptyset$. Stop short before (above) the largest ξ with $P_\xi \cap Z \neq \emptyset$ would be reached.

Choose $\xi_1 > \xi_2 > ... > \xi_k$ such that

$$\{\xi_1, \xi_2, ..., \xi_k\} = \{\xi : P_\xi \cap Z \cap V(P) \neq \emptyset\}.$$

We stopped at C at $z = \xi$ with ξ slightly larger than ξ_1.

Let E_ξ be the set of those points in $P_\xi \cap Z$ which belong to edges lying in Z, i.e., vertical edges. Of course, E_ξ is topologically closed, but can be empty.

If $E_{\xi_1} \neq \emptyset$ and $(0,0,\xi_1) \in \text{conv}E_{\xi_1}$ then, for any $\xi \in (\xi_2, \xi_1)$, $P_\xi \cap Z \neq \emptyset$ and $(0,0,\xi) \in \text{conv}E_\xi$, in contadiction with the hypotheses. Hence $(0,0,\xi_1) \notin \text{conv}E_{\xi_1}$ if $E_{\xi_1} \neq \emptyset$.

Now choose a horizontal non-critical line L_1 through $(0,0,\xi_1)$, which – in case $E_{\xi_1} \neq \emptyset$ – does not meet conv E_{ξ_1}. Let Z' be the position of Z after a slight rotation ρ around L_1 leaving all edges with endpoints in E_{ξ_1} inside Z'. An entire neighbourhood of P_{ξ_1} in P lies inside Z' (note that both Z and Z' touch the ball of diameter $L_1 \cap \text{conv}Z$ along great circles).

Case 1. $E_{\xi_1} = \emptyset$. In this case the moves of C are as follows. At ξ slightly larger than ξ_1 we apply to C the same rotation ρ, then translate it along Z' until it comes below $z = \xi_1$ and then rotate it back (by ρ^{-1}) around L_1 to a position on Z again. Thus C passed P_{ξ_1} without touching P.

Then C goes downwards keeping its horizontal position until just above ξ_2.

Case 2. $E_{\xi_1} \neq \emptyset$. In this case we proceed with C as above, but perform the rotation back around the diameter Δ of C parallel to L_1 instead of L_1 itself. Thus C does not touch P, particularly the vertical edges ending in E_{ξ_1}.

Then C, which now crosses Z, is translated downwards until just above ξ_2 (not meeting P if Δ was close enough to L_1).

At $z = \xi_2$, still assuming $E_{\xi_1} \neq \emptyset$, it may well happen that $P_{\xi_2} \cap Z$ does not contain only vertices of P. Or, it may happen that $P_{\xi_2} \cap Z \subset V(P)$ and, however, $\xi_2 \in \text{int}J$ (this is the case if some point in E_{ξ_2} is the lower endpoint of a vertical edge and some other point in E_{ξ_2} is the upper endpoint of another vertical edge). Then, in both cases, $(0,0,\xi_2) \notin \text{conv}(P_{\xi_2} \cap Z)$, and we choose a horizontal non-critical line $L_2 \ni (0,0,\xi_2)$ which does not meet $\text{conv}(P_{\xi_2} \cap Z)$.

The circle on Z at $z = \xi_2$, slightly translated in horizontal direction orthogonal to L_2 toward $P_{\xi_2} \cap Z$, comes to a position C^* disjoint from P_{ξ_2}.

Now we continue our movement of C: We rotate it around the z-axis until C^* becomes its orthogonal projection on $z = \xi_2$. Then C moves straight downward, and passes $z = \xi_2$ through the position C^* without hitting P_{ξ_2}.

If $\xi_2 \in \mathrm{bd}J$, then we move C following the procedure at $z = \xi_1$ in Case 2, in reversed order. Note that the two slight rotations can be performed around lines parallel to L_1, since the line through $(0, 0, \xi_2)$, parallel to L_1 does not meet $V(P) \cap Z$.

After reaching a horizontal position on Z, C continues its way downwards.

At each level ξ_i, one of the two Cases appears, and we proceed as described above.

Examples. Applied to the circumscribed cylinder of a polytope P, the Theorem offers a strong instrument to recognize whether $r_p(P) < r_c(P)$ or $r_p(P) = r_c(P)$.

So, for example, all regular polyhedra except for the cube satisfy the condition of the Theorem with respect to their circumscribed cylinders. More precisely, it can be checked that, for all regular polyhedra except the cube, no edge lies on the circumscribed cylinder. So the set J in the Theorem is finite, and the hypothesis is trivially verified; thus $r_p < r_c$. For the cube of side-length 1, $r_p = r_c = \sqrt{2}/2$. For the regular tetrahedron of side-length 1, $r_p = 0,4478...$ and $r_c = 0,5$ (see [1]).

As another example, consider a regular pyramid P_n with an n-gon as basis. If its height is small (compared with the basis), then it has two parallel edges on Ξ_{P_n}, for even n, and just one edge and a vertex on Ξ_{P_n} for odd n, but in both cases the hypothesis of our Theorem is not satisfied, and indeed $r_p(P_n) = r_c(P_n)$. For large height, P_n has no edges on Ξ_{P_n}, and so $r_p(P_n) < r_c(P_n)$ for all n.

For all right prisms, regular or not, $r_p = r_c$. For other prisms, another story...

Acknowledgements. The author wishes to thank a careful and friendly referee, whose remarks helped remove several ambiguities and improve the presentation. Also, the author thankfully acknowledges partial support from Grant 2-CEx 06-11-22/2006 of the Romanian Government, and from the JSPS research fellowship S-07025, Japan.

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