SOME REMARKS ON SIMPLE CLOSED GEODESICS OF SURFACES WITH ENDS

JIN-ICHI ITOH, FUMIKO OHTSUKA AND TUDOR ZAMFIRESCU

Jin-ichi Itoh
Faculty of Education
University of Kumamoto
Kumamoto, Kumamoto 860-8555, Japan
e-mail: j-itoh@gpo.kumamoto-u.ac.jp

Fumiko Ohtsuka
Department of Mathematical Sciences
Faculty of Science
Ibaraki University
Mito, Ibaraki 310-8512, Japan
e-mail: ohtsuka@mx.ibaraki.ac.jp

Tudor Zamfirescu
Faculty of Mathematics
Dortmund University of Technology, Germany
and IMAR "Simion Stoilow",
Bucharest 14700, Romania
e-mail: tzamfirescu@yahoo.com

Abstract. If a non-compact complete surface M is not homeomorphic to a subset of the plane or of the projective plane, then it has infinitely many simple closed geodesics [6]. In this paper, we consider simple closed geodesics on a surface homeomorphic to such a subset.

2000 *Mathematics Subject Classification.* Primary: 53C , Secondary:
Key words and phrases. Simple closed geodesic, self-intersection.
Acknowledgement. The first author is partially supported by Grant-in-aid for Scientific Research(C) No.17540085, Japan Society for the Promotion of Science.
The second author is partially supported by Grant-in-aid for Scientific Research(C) No.17540060, Japan Society for the Promotion of Science.
The third author thankfully acknowledges partial support from Grant 2-CEx 06-11-22/2006 of the Romanian Government, and from the JSPS research fellowship S-07025, Japan.

1. Introduction

It is well-known that there are closed geodesics on closed manifolds with non-trivial fundamental group. They can be obtained as minimal loops in free homotopy classes of loops. However, in open manifolds topological restrictions do not necessarily imply the existence of closed geodesics. For an arbitrary manifold M, Thorbergsson [6] has constructed a complete Riemannian metric on $\mathbf{R} \times M$ for which there are no closed geodesics. But, especially in the 2-dimensional case, he also showed (Theorem 3.2 of [6]) that if an open surface M is not homeomorphic to a plane, a cylinder or a Möbius strip, then there exist infinitely many closed geodesics on M, any of which is not a covering of another one, and if M is not homeomorphic to a subset of the plane or of the projective plane, then it is possible to choose such closed geodesics without self-intersections. There are many results about geometric conditions which ensure the existence of closed geodesics ([5], [6], [1] are just some examples).

In this paper we consider closed geodesics without self-intersections on a smooth surface S_n with n ends and without handles, called from now on just *surface* (except when otherwise specified). It is easily seen that a cylindrical surface S_2 with 2 ends may have no closed geodesics. As an example, we can take a surface of revolution of funnel type with the generating curve $y = 1/x$ $(x > 0)$. On the other hand, on S_n with $n \geqq 3$, there always are infinitely many closed geodesics.

We shall see here, using elementary arguments, that, for $n \geqq 4$, there are infinitely many simple closed geodesics, while, for $n = 3$, there might be no such geodesics at all. Also, we shall present a classification of surfaces concerning their capability of allowing simple closed geodesics, according to the type of neighbourhoods of their ends.

We would like to express our hearty thanks to Professor Minoru Tanaka for his kind bibliographical suggestions.

2. Surfaces with three ends

Theorem 1. *There exist surfaces with* 3 *ends which have no simple closed geodesics.*

Proof. We define a surface $\widetilde{S}_3$ as the double of the domain

$$B = \{(x_1, x_2) \in \mathbf{R}^2 \mid \frac{1}{|x_1|} \geqq x_2 \geqq 0\}.$$

We will show that there are no simple closed geodesics on $\widetilde{S}_3$. Let b_0, b_1 and b_2 be the three boundary curves of B such that b_0 is the x_1-axis, $b_1(t) = (-t, 1/t)$ and $b_2(t) = (t, 1/t)$ for $t > 0$.

Suppose that the surface $\widetilde{S}_3$ possesses a simple closed geodesic c. Let $p_1, p_2, \cdots, p_n$ be the consecutive points on c where c crosses $b_0 \cup b_1 \cup b_2$ (according to the order induced by arc-length parametrization). For any $i \in N := \{1, \cdots, n\}$, if c crosses b_k at p_i, then we define the map $j : N \to \{0, 1, 2\}$ by $j(i) := k$. Since c is a geodesic, for any i we have $j(i) \neq j(i+1)$.

If there exists $i \in N$ such that $j(i) = j(i+2)$, then $\widetilde{S}_3$ is divided into two domains by $p_i p_{i+1} \cup p_{i+1} p_{i+2} \cup p_{i+2} p_i$, and c comes to p_i from one domain D_1 and leaves p_{i+2} towards the other domain D_2. Since c has no self-intersections, all successors $\{p_j | j > i+2\}$ of p_{i+2} (j taken modulo n) lie in D_2 and will never reach D_1 again. This contradicts the fact that c is closed.

Hence $j(i) = j(i+3)$ for any i. In this case, $p_1p_2, p_2p_3, \ldots, p_4p_5$ must essentially lie as in Figure 1 (where $p_1 \in b_1$ and $p_2 \in b_2$), and no admissible position of p_6 can be found, because p_5 and b_0 are separated by p_1p_2 in B. The surface $\widetilde{S}_3$ is not smooth, but an appropriate slight deformation of $\widetilde{S}_3$ yields a smooth example. □

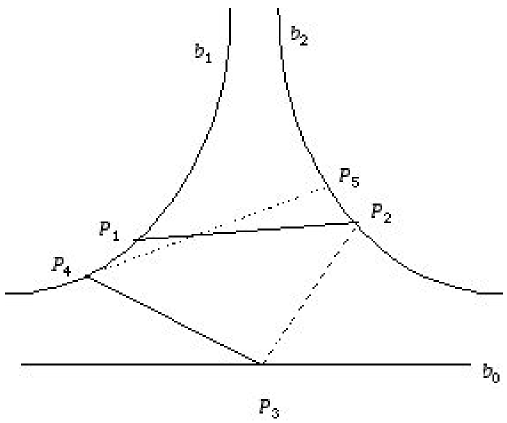

FIGURE 1

3. Surfaces with more than three ends

Theorem 2. *On any surface with at least 4 ends there are infinitely many simple closed geodesics.*

Proof. We choose a large compact set $D \subset S_n$ such that each connected component of $S_n \setminus D$ is homeomorphic to a cylinder and the boundary ∂D consists of n simple closed curves c_i $(i = 1, 2, \ldots, n)$.

Take smooth arcs $\alpha_1, \alpha_2, \alpha_3, ...$ in D each joining a point of c_1 to one of c_2 and having no self-intersections (as illustrated in Figure 2). Note that these arcs form infinitely many homotopy classes, according to the number of times they surround c_2 and c_3.

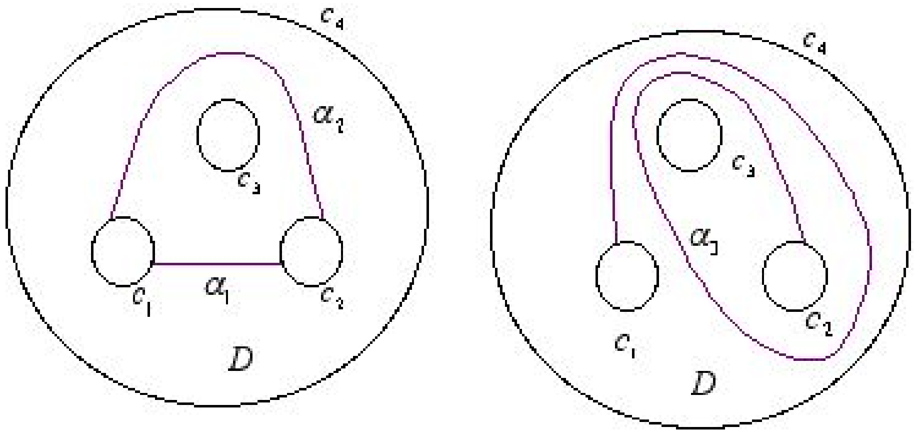

FIGURE 2

Let N_i be a thin tubular neighborhood of α_i on S_n and c_i' be the component of $\partial(D \setminus N_i)$ meeting c_1. Since c_i' is not contractible, we get a closed geodesic s_i in the homotopy class of c_i'. Note that s_i, like c_i', has no self-intersections. □

4. A classification for surfaces with two or three ends

In the previous sections, we have seen that there are infinitely many simple closed geodesics on S_n for $n \geqq 4$, and on S_2 or S_3 there might be no such closed geodesics. For S_n, if D is a large compact set such that each component D_i of $S_n \setminus D$ is homeomorphic to a cylinder ($i = 1, ..., n$), we call the closure U_i of D_i an *endtube* of S_n.

In this section we define three types of endtubes, and relate them to the existence of simple closed geodesics on S_2 or S_3.

For an arc or a closed curve c, we shall denote by $\lambda(c)$ its length.

The following definitions are inspired by and similar to those of Cohn-Vossen [3] (see also Busemann [2]).

Definition 1. *The endtube U is called:*

- contracting if it contains no closed curve freely homotopic to ∂U, of relative minimal length, i.e., minimal in some neighbourhood in U,

- expanding if it includes no contracting subtube and all its closed curves freely homotopic to ∂U, of relative minimal length, meet ∂U.

- bulging if it does not include any contracting or expanding subtube.

We note that every endtube includes a subendtube of one of the above three types, and a subendtube of a contracting (expanding) endtube is contracting (expanding).

The following lemma is essentially proven in [3], where it is shown that a “ Schaft” has a simple closed geodesic or a “ hohles Eineck”, i.e., a geodesic loop with the angle toward the end less than π. The conclusion is slightly different because the definitions are modified compared with Cohn-Vossen's.

Lemma 1. *If S_n has a contracting endtube U, then $U \setminus \partial U$ contains a geodesic loop at some point x, homotopic to ∂U, with the angle at x toward the end of U less than π.*

For the next lemma, see Bangert [1], p. 93.

Lemma 2. *If S_2 has two disjoint locally convex endtubes, then it has a simple closed geodesic.*

Theorem 3. *Let S_n be a surface with n ends $(n = 2, 3)$. Then one of the following situations occurs:*

(1) *If S_n contains a bulging endtube, then it has a simple closed geodesic.*
(2) *If S_3 contains an expanding endtube, then it has a simple closed geodesic.*
(3) *If S_3 contains three pairwise disjoint contracting endtubes, it may not contain any simple closed geodesic.*
(4) *If S_2 has two disjoint contracting endtubes or two disjoint expanding endtubes, then it has a simple closed geodesic.*
(5) *If S_2 has both a contracting endtube and an expanding endtube, it may not contain any simple closed geodesic.*

Proof. (1) It is clear that a bulging endtube U contains a curve of relative minimal length freely homotopic to ∂U, which does not meet ∂U, and is therefore of minimal length in S_n, thus providing a simple closed geodesic.

(2) Assume U_1 is an expanding endtube of S_3.

Let c^* be a shortest curve in U_1 homotopic to ∂U_1 and meeting ∂U_1. Take $a \in c^* \cap \partial U_1$.

If there is a curve $\widetilde{c} \subset D_1 = U_1 \setminus \partial U_1$ of length less than $\lambda(c^*)$, homotopic to ∂U_1, then either there exists such a curve of minimal relative length in D_1 or there is no such curve of minimal relative length in the whole U_1, both contradicting the definition of an expanding endtube.

If there is a curve $\widetilde{c} \subset D_1$ of length $\lambda(c^*)$, but not less, homotopic to ∂U_1, then $\widetilde{c}$ itself has minimal relative length, again in contradiction to the definition.

Hence every curve in D_1 homotopic to ∂U_1 has length exceeding $\lambda(c^*)$.

Let γ_i be a geodesic ray starting at a, such that $\gamma_i \cap U_i$ contains a geodesic ray ($i = 1, 2$). We parametrize $\gamma = \gamma_1 \cup \gamma_2$ by arc length, with $\gamma(0) = a$, $\gamma(t) = \gamma_1(-t)$ for $t < 0$, and $\gamma(t) = \gamma_2(t)$ for $t > 0$. For any number t, consider a curve c_t of minimal length in S_3, passing through $\gamma(t)$ and homotopic to ∂U_1.

If $t \to \infty$, then $\lambda(c_t) \to \infty$, because c_t keeps meeting ∂U_2.

Clearly, $\lambda(c_0) \leq \lambda(c^*)$.

For $t \leq -\lambda(c^*)$, either c_t meets ∂U_1 and

$$\lambda(c_t) > |t| \geq \lambda(c^*),$$

or c_t does not meet ∂U_1 and we showed that in this case $\lambda(c_t) > \lambda(c^*)$.

Thus $\lambda(c_t)$ attains a relative minimum for some $t_0 \geq -\lambda(c^*)$, and c_{t_0} has a relative minimal length among all curves homotopic to ∂U_1. Hence c_{t_0} is a simple closed geodesic.

(3) We have seen in Theorem 1 that there exists such a surface with no simple closed geodesics.

(4) Assume S_2 has two contracting endtubes. If U is anyone of them, it has, by Lemma 1, a geodesic loop at some point x, homotopic to ∂U, with the angle at x toward the end of U less than π. Now, the conclusion follows from Lemma 2.

If S_2 has two expanding tubes, we follow the same reasoning as for (2). The difference is that now both halves γ_1 and γ_2 of γ are treated in the same way, namely similar to γ_1 in the proof of (2).

(5) A surface of revolution of funnel type has no simple closed geodesics. $\square$

5. Two examples

In this section we give two examples of surfaces with four ends. The first surface has infinitely many closed geodesics without self-intersections illustrating Theorem 2, while the second is not a smooth surface, has four vertices of negative singular curvature, and on it all closed geodesics except for four do have self-intersections.

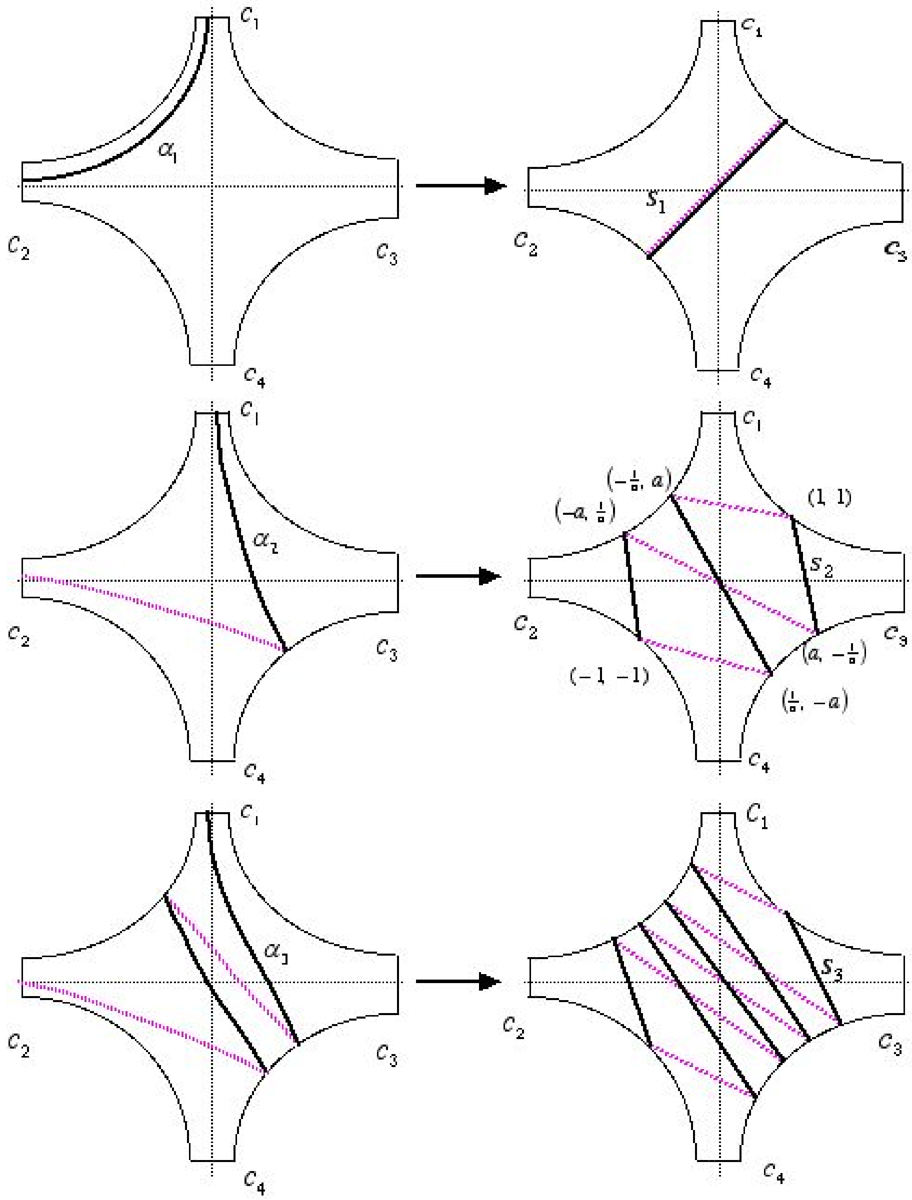

Figure 3

Example 1. Let E be the double of the domain

$$B = \left\{ (x_1, x_2) \in \mathbf{R}^2 \;\middle|\; -\frac{1}{|x_1|} \le x_2 \le \frac{1}{|x_1|} \right\}.$$

Take four simple closed curves c_1, c_2, c_3 and c_4 on E defined by $x_1 = 2$, $x_2 = -2$, $x_1 = -2$ and $x_2 = 2$.

Consider smooth arcs $\alpha_1, \alpha_2, \alpha_3, ...$ in D each joining a point of c_1 to one of c_2 and having no self-intersections (as illustrated in Figure 3). Note that these arcs form infinitely many homotopy classes. Take N_i like in the proof of Theorem 2, then c_i', and eventually find the closed geodesics s_i illustrated in Figure 3.

The coordinates of the points where s_2 passes from one face of E to the other are $(\pm 1, \pm 1)$, $(\pm a, \mp 1/a)$, $(\pm 1/a, \mp a)$, where

$$a = -1 + \sqrt{3} + \sqrt{2 - \sqrt{3}}.$$

If the arcs $\alpha_1, \alpha_2, ...$ had joined c_1 and c_3 instead of c_1 and c_2, we should have obtained another set of simple closed geodesics. But also note that, if $\alpha_1, \alpha_2, ...$ joined c_3 to c_4, we should again obtain $s_1, s_2, ...$

The surface E is not smooth, but an appropriate slight deformation of E yields a smooth example.

Example 2. Let ξ be positive, let

$$B_0 = \{(x_1, x_2) \in \mathbf{R}^2 \mid -e^{-x_1^2/5} \leq x_2 \leq e^{-x_1^2/5}\},$$

and let $a_0 = (0, -1)$, $a_0' = (0, 1)$, $\overline{b}_0 = (-\xi, -e^{-\xi^2/5})$, $\overline{b}_0' = (-\xi, e^{-\xi^2/5})$, $b_0 = (\xi, -e^{-\xi^2/5})$, $b_0' = (\xi, e^{-\xi^2/5})$.

We construct the non-smooth surface S with four ends from six pieces $B_1, B_2, \ldots, B_6$, each isometric to B_0, by identifying their boundaries as follows.

First determine points a_i, b_i, $\overline{b}_i$, a_i', b_i', $\overline{b}_i'$ on B_i $(i = 1, ..., 6)$ corresponding to a_0, b_0, $\overline{b}_0$, a_0', b_0', $\overline{b}_0'$ (which are shown on Figure 4, left) via the mentioned isometries. Then place B_1 as shown on Figure 4, right. Then place analogously the other pieces $B_2, ..., B_6$ between all other pairs of cycles c_k, c_l. Thereby we glue one of the arcs $a_i b_i$, $a_i' b_i'$, $a_i \overline{b}_i$, $a_i' \overline{b}_i'$, to one of the arcs $a_j b_j$, $a_j' b_j'$, $a_j \overline{b}_j$, $a_j' \overline{b}_j'$, for any pair of neighbouring pieces B_i, B_j, and let $\xi \to \infty$.

Then S has four singular points of negative singular curvature, in the sense of [4], all corresponding to a_0 or a_0'.

We now prove that S has exactly three simple closed geodesics, one of which is illustrated in Figure 4; thus, this example shows that the smoothness condition in Theorem 2 is essential.

Let $trqr't'$ be a piece of a geodesic in the double of B_0, as shown on Figure 5.

Suppose a simple closed geodesic γ surrounding c_1 and c_2 does not pass through a_1 or a_1' (see Figure 4). Then γ either

1) consists of two broken lines isometric to $trqr't'$ (see Figure 5), or

2) includes two consecutive arcs isometric to the line-segment $pp' \subset B_0$, where p, p' lie on the arcs a_0b_0, $a_0'b_0'$, respectively.

Case 1). Let $v \in \partial B_0$, and m_v be the (oriented) angle between the tangent at v to ∂B_0 and the positive x_1-axis.

Putting $f(x) = e^{-x^2/5}$, we find that, on $[0, \infty[$, $f''(x) = 0$ for $x = \sqrt{5}/\sqrt{2}$, where f' reaches its minimal value $-\sqrt{2}/\sqrt{5e}$. Since $\sqrt{2}/\sqrt{5e} < \sqrt{2}/\sqrt{10} < 1/\sqrt{3}$, the absolute value of the slope of ∂B_0 is everywhere less than $1/\sqrt{3}$. Hence $|m_v| < \pi/6$ for all v.

Let α (α') be the (oriented) angle between tr (respectively $t'r'$) and the positive x_1-axis.

The reflection law implies that the angles β, β' between rq, respectively $r'q$, and the positive x_1-axis are $\beta = 2m_r - \alpha$, $\beta' = 2m_{r'} - \alpha'$.

Again the reflection law (at q) implies $\pi + 2m_q - \beta = \beta'$, whence

$$\alpha + \alpha' = 2(m_r + m_{r'} - m_q) - \pi < 0.$$

This means that $\nu > \nu'$ (see Figure 4, right), because $\nu = (\pi/2) - \alpha$ and $\nu' = (\pi/2) + \alpha'$. And this cannot happen for both pieces composing the closed geodesic γ.

Case 2). Let p^* be the intersection point of pp' with a_0a_0'.

The normal line through the point of $a_0'b_0'$ of abscise $x_1 > 0$ cuts the x_2-axis at a point of ordinate

$$x_2 = e^{-x_1^2/5} - \frac{5}{2}e^{x_1^2/5}.$$

Since $e^{-x_1^2/5} < 1$ and $e^{x_1^2/5} > 1$ for all positive x_1, we have $x_2 < -3/2$. Since p^* has ordinate at least -1, the halfline $p'p$ and the arc $p'b_0'$ make an obtuse angle δ, see Figure 4, right. But this cannot happen for both consecutive pieces of γ isometric to pp'.

On S we have a closed geodesic formed by four arcs analogous to a_1a_1', surrounding c_1 and c_2, passing through a_1 (and the other three singular points), and of length 8. As we saw above, there are no further closed geodesics surrounding c_1 and c_2.

Hence every simple closed geodesic is composed by four arcs isometric to a_0a_0' and passes through all four singular points of negative singular curvature. There are only three such geodesics.

References

[1] V. Bangert, *Closed geodesics on complete surfaces,* Math. Ann. **251** (1980) 83-96.

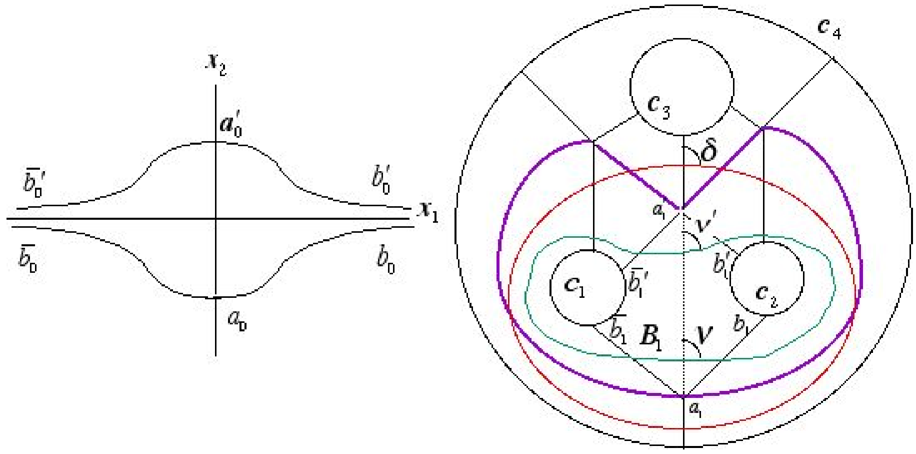

FIGURE 4

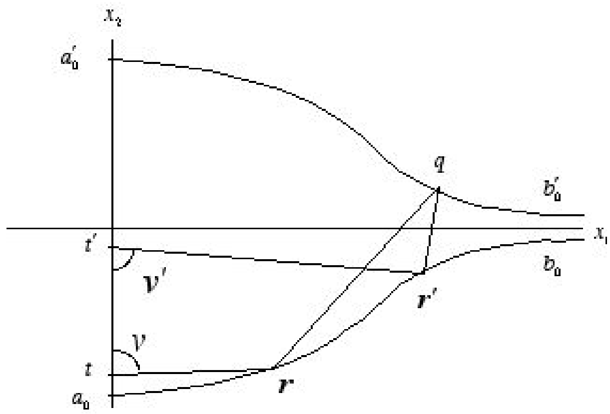

FIGURE 5

[2] H. Busemann, The Geometry of Geodesics, Academic Press, New York-London, 1955.
[3] S. Cohn-Vossen, *Kürzeste Wege und Totalkrümmung auf Flächen,* Compositio Math. **2** (1935) 63-133.
[4] J. Itoh and F. Ohtsuka, *Total curvature of noncompact piecewise Riemannian 2-polyhedra,* Tsukuba J. **29** (2005) 471-493.
[5] S. Kobayashi, *On conjugate and cut loci*, in "Global Differential Geometry", MAA Studies in Math. **27** (1989), 140-169.
[6] G. Thorbergsson, *Closed geodesics on non-compact Riemannian manifolds,* Math. Z. **159** (1978) 249-258.