1 Objectives

1) To formulate a suitable notion of holomorphicity for maps between manifolds endowed with generalized complex structures and to study the harmonicity of such maps between generalized Kaehler manifolds. Also, to study certain classes of generalized complex structures.
2) To contribute to a better understanding of harmonic and pseudo harmonic morphisms, under given geometrical conditions such as flat and conformally-flat domains.
3) To study the twistorial properties of harmonic morphisms with one-dimensional fibres from four-dimensional Einstein-Weyl spaces.
4) To obtain a cosymplectic reduction.
5) To study harmonic maps between contact metric manifolds.
6) To extend the join construction from Sasakian-Einstein geometry to the non-Einstein case and to relate it to Lerman's construction of contact fibre bundles. This will lead to a wealth of new examples of Sasakian manifolds.
7) To introduce the class of locally conformal generalized Kaehler manifolds and study its properties. This should include complex locally conformal Dirac and complex locally conformally symplectic structures.
8) Construction of complex homogeneous spaces and geometric realizations for specific representations of Banach-Lie groups.
9) To classify the generalized invariant complex and invariant Kahler structures on adjoint orbits of compact semi-simple Lie groups.
10) To find a twistor interpretation of Weyl connections on conformal 3-manifolds.
11) To extend and develop Darling's results [D] to the case of Riemannian poyhedra.
12) The investigation of harmonic maps on polyhedra in relations with foliations (cf. [P3], for the smooth case).
13) The study of submanifolds and submersions in Quaternionic Geometry. Also, to study the hypersurfaces in paraquaternion-Kaehler manifolds.
14) Geodesics and critical points on convex surfaces and Alexandrov surfaces.

2 SCIENTIFIC AND TECHNICAL PRESENTATION OF THE PROJECT

We describe our approach for reaching the objectives (1)-(14):
1) A generalized complex structure is a complex Dirac structure. Therefore holomorphic maps between manifolds endowed with generalized complex structures should be complex Dirac morphisms; to define the notion of Dirac morphism we shall exploit the fact that Dirac structures generalize both presymplectic and Poisson structures. Also, we shall see to what extent the classical fact that holomorphic maps between Kaehler manifolds are harmonic can be generalized to holomorphic maps between generalized Kaehler manifolds. This could lead to new constructions of harmonic maps and morphisms.
2) We attempt to prove that any harmonic morphism with one-dimensional fibres from a conformally-flat Riemannian manifold, of dimension at least four, to a Riemannian manifold is either of Killing type or its horizontal

distribution is integrable. This would be an extension of the known classification results for harmonic morphisms with one-dimensional fibres from Einstein manifolds. Also, we intend to obtain classification results for polynomial pseudo harmonic morphisms.

3) This is motivated by the fact that the twistorial properties of harmonic morphisms with one-dimensional fibres are well-understood in the following two particular cases:

(a) the domain is a four-dimensional Einstein manifold and the codomain is a Riemannian manifold;

(b) the domain and codomain are Einstein-Weyl spaces of dimensions four and three.

4) We attempt to obtain a reduction in the case of almost contact metric geometry, in particular in the cosymplectic case. This would be analogous to the Kaehler reduction.

5) We shall study harmonic maps between contact metric manifolds by using twistorial tools.

6) Boyer and Galicki defined in [BG] a monoid of Sasakian-Einstein manifolds. The operation is called “join”. This permits the construction of a new manifold out of two known ones. The Einstein condition is rather rigid and can be relaxed. We shall do it in collaboration with them. This leads for example to a classification of Sasakian structures on S^2xS^3 viewed as circle bundles over Hirzebruch surfaces. On the other hand, Lerman gave sufficient conditions for the total space of a fibre bundle with contact fibres to be contact. We shall suppose the fibres Sasakian and give the corresponding conditions for the total space to be Sasakian. It is to be expected that the two constructions (join and contact bundle) be related. A special attention will be given to the toric case.

7) As in the Hermitian case, locally conformal generalized Kaehler (lcgK) manifolds should be defined as quotients by a discrete group of conformal automorphisms of a generalized Kaehler manifold. The notion should be related to that of locally conformal bihermitian structure. On the model of the relation between locally conformal hyperkaehler and HKT (hyperkaehler with torsion) geometries, we expect lcgK geometry be related to hyperparakaehler geometry with torsion.

8) Several specific representations of Banach-Lie groups arise in connection with representations of operator algebras, so that we are going to achieve our objective by employing operator theoretic methods which have been developed recently by members of the present team. On the other hand, we are going to focus on Banach-Lie groups which provide rigorous descriptions for certain symmetry groups of mathematical physics. Thus, by constructing complex homogeneous spaces and geometric realizations for representations of these groups, we shall set up new links between various fields like the representation theory of Lie groups, the theory of operator algebras, infinite-dimensional holomorphy and differential geometry, and certain areas of mathematical physics.

9) Alekseevksy and Perelomov (1986) classified all invariant complex structures on adjoint orbits G/K of a compact semi-simple Lie group G in terms of the pair (g,k) of the Lie algebras of G and K respectively. On the other hand, from the Kirillov-Konstant-Souriau theorem, invariant symplectic structures on such orbits are in one to one correspondence with elements of g with stabiliser K under the adjoint representation of G onto its Lie algebra g. Both complex and symplectic structures are particular cases of generalised complex structures. It is therefore natural to unify the Kirilov-Konstant-Souriau theorem with the Alekseevsky-Perelomov results and to find a classification, in terms of (g,k), of all invariant generalised complex structures on G/K. Alekseevksy and Perelomov also studied invariant Kahler and Kahler-Einstein structures on adjoint orbits. It would be interesting to generalize their results to generalized Kahler structures.

10) Gauduchon (1991) showed that every Weyl connection on a conformal self-dual 4-manifold M determines an holomorphic, natural and real, structure on the tangent vertical bundle of twistor projection from the twistor space of

M onto M. Moreover, this correspondence induces a bijection between equivalence classes of Weyl connections on M and equivalence classes of holomorphic, real and natural structures on this vertical bundle. It can be easily seen that we can adapt the metod of Gauduchon to the context of twistor CR spaces of LeBrun (1984), which are associated to 3-dimensional conformal manifolds. More precisely, we can asociate, by a method similar to the one employed by Gauduchon, to a Weyl connection on a 3-dimensional smooth conformal manifold N a d-bar operator on the vertical tangent bundle of the twistor projection from the CR twistor space of N onto N. This could be a first step of a new interpretation of Weyl connections of conformal 3-manifolds on the CR twistor spaces. This project is justified by the Weyl-Bogomolny equations on three-dimensional conformal manifolds: using the twistorial intepretation of Weyl connections on CR twistor spaces, we expect to obtain also a twistorial interpretation of the Weyl-Bogomolny equations of conformal 3-manifolds (without a fixed compatible Einstein-Weyl structure) on CR twistor spaces. This interpretation is missing from
the literature precisely because Weyl connections on 3-dimensional conformal manifolds do not encode so easily into the CR twistor spaces.

11) In the first instance, we aim to extend Darling's results to the case of Riemannian poyhedra. Several technical problems, due to the existence of singularities are envisaged, making it a harder problem compared to the smooth case. Remark that the second differential calculus on Riemannian manifolds has no natural generalisation on Riemannian polyhedra, and on the other hand all the theory of stochastic calculus is based on the second order differential calculus. Consequently, we are obliged to develop a new approach combining smooth theory with some hybrid methods.

12) Other developments of the harmonic theory on polyhedra are taken into account, notably an investigation of harmonic morphisms in relations with foliations, attempting the classification in the case of one-dimensional fibres. A suitable theory of foliations on polyhedra needs to be settled first.

13) We shall consider the conditions under which such hypersurfaces are Einstein, by using the Gauss and Codazzi equations. A significant particular case, at least from the point of view of Mathematical Physics, is when the hypersurface is isolated or lightlike. In [IanVil], it is shown that totally-umbilical hypersurfaces in paraquaternion-Kaehler manifolds of nonnull constant Q-sectional curvature can never be Einstein. By a classical result, any hypersurface in a quaternionic manifold has an almost contact 3-structure. Recently, D. Alekseevski and Y. Kamishima [AK] have shown that certain submanifolds of paraquaternion-Kaehler manifolds carry pseudo-Sasaki 3-structures and applied techniques from submersion theory to study them. We shall study the curvature of these classes of manifolds and, also, their submanifolds, by taking into account the foliations with which they are naturally endowed.

14) The Alexandrov theory concerning geodesics on convex surfaces (with singularities) will be used to develop the theory of critical points on such surfaces. Further, the Burago-Gromov-Perelman introduction of Alexandrov spaces and more recent developments (of Perelman, Ballman, and Shiohama-Tanaka) will be complemented by an investigation in this general context.

3 References

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