Harmonicity on cosymplectic manifolds[*†]

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Abstract

We prove that an $(\varphi, J)-$holomorphic maps from a compact cosymplectic manifold to a Kähler manifold is not only a harmonic map but also an energy minimizer on its homotopy class. We also prove a converse result.

1 Introduction

Combining both global and local aspects and borrowing both from Riemannian geometry and from analysis, the theory of harmonic maps between Riemannian manifolds has developed in many diverse branches. In particular, there is now a whole battery of deep and interesting results about harmonic maps to or from complex manifolds and Kähler spaces.

Within contact geometry, there are several classes of manifolds that can be considered as odd-dimensional analogs of Kähler spaces, the most important ones being Sasakian and cosymplectic spaces.

The theory of harmonic maps on smooth manifolds endowed with some special structures has its origin in the paper of Lichnerowicz [5], in which he considered holomorphic maps between Kähler manifolds.

In general the construction of energy minimizing maps is much more difficult than to find harmonic ones. The main purpose of this paper is to show that structure-preserving maps on cosymplectic manifolds minimize the energy of maps. We prove that an $(\varphi, J)-$holomorphic map from a compact

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cosymplectic manifold to a Kähler manifold is not only harmonic but also a minimizer for its energy.

We also prove a converse of the previous result, that is, a smooth energy minimizer map from a cosymplectic manifold to a Kahler manifold, which is homotopic with an $(\varphi, J)-$holomorphic one is also $(\varphi, J)-$holomorphic.

2 Cosymplectic manifolds

It is well-known that the odd-dimensional counterpart of Kähler manifolds are cosymplectic manifolds. Let M be a smooth manifold of dimension $2n+1$. We recall that an *almost contact structure* on M is a triple (ξ, η, φ), where ξ is a vector field, η is an one-form and φ is a tensor field of type $(1,1)$ which satisfy (see [1]):

$$\varphi^2 = -Id + \eta \otimes \xi \quad and \;\; \eta(\xi) = 1 \tag{1}$$

where Id is the identity endomorphism on TM. Then we have $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Furthermore, if g is an associated Riemannian metric on M, that is, a metric which satisfies for any $X, Y \in \mathcal{X}(M)$

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

then we say that (ξ, η, φ, g) is an *almost contact metric structure*. A manifold equipped with such a structure is an *almost contact metric manifold*. The existence of an almost contact structure on M is equivalent to the existence of a reduction of the structural group to $U(n) \times 1$.

The *fundamental 2-form* Φ of M is defined by

$$\Phi(X, Y) = g(X, \varphi Y), \tag{3}$$

for $X, Y \in \mathcal{X}(M)$. The almost contact metric structure (ξ, η, φ, g) is said to be *normal* if the Nijenhuis tensor N_φ of φ satisfies the condition (see [1]):

$$N_\varphi + 2d\eta \otimes \xi = 0. \tag{4}$$

An almost contact metric manifold $M(\xi, \eta, \varphi, g)$ is called to be *cosymplectic* if it is normal and $d\eta = 0$, $d\Phi = 0$.

The canonical example of compact cosymplectic manifold is given by the product $B^{2n} \times S^1$ of a compact Kähler manifold $B^{2n}(J, h)$ with the circle

S^1 (see [2]). Nontrivial examples are obtained by using the suspensions technique. We explain for short this construction. Let N be a 2n-dimensional compact Kähler manifold with the Hermitian structure (J, h). Consider an Hermitian isometry $f : N \to N$, that is f is a diffeomorphism and

$$f_* \circ J = J \circ f_*, \ , \ f^* h = h.$$

We define the action A of $\mathbb{Z}$ on the product manifold $N \times \mathbb{R}$ by

$$A(n, (x, z)) = (f^n(z), z - n),$$

for all $n \in \mathbb{Z}$ and $(x, z) \in N \times \mathbb{R}$. This action is free and properly discontinuous. Thus, the orbit space $(N \times \mathbb{R})/A$ of the $\mathbb{Z}-$ action is an (2n+1)-dimensional compact manifold and the canonical projection $p' : N \times \mathbb{R} \to M$ is a covering map. Moreover, we can define a fibration τ of M on $S^1 = \mathbb{R}/\mathbb{Z}$ by $\tau([(x, z)]) = [z]$, for all $(x, z) \in N \times \mathbb{R}$. It is clear that the fibers of τ are diffeomorphic to N.

Denote by $\rho : \mathbb{Z} \to Diff(N)$ the representation of $\mathbb{Z}$ on the group of the diffeomorphisms of N, $Diff(N)$, given by $\rho(k) = f^k$, for all $k \in \mathbb{Z}$. Then the manifold M is called the *suspension with fibre N of the representation ρ.*

Next we shall obtain a cosymplectic structure on M. We consider on $N \times \mathbb{R}$ the cosymplectic structure $(\overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ given by

$$\overline{\varphi} = J \circ (pr_1)_*, \ \overline{\xi} = \frac{\partial}{\partial t}, \ \overline{\eta} = pr_2^*(dt), \ \overline{g} = pr_1^*(h) + pr_2^*(dt^2),$$

where $pr_1 : N \times \mathbb{R} \to N$, and $pr_2 : N \times \mathbb{R} \to \mathbb{R}$ are the canonical projections onto the first and the second factor, respectively and t is the usual coordinate on $\mathbb{R}$. Since f is an Hermitian isometry, we deduce that the cosymplectic structure $(\overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ is invariant under the action A of $\mathbb{Z}$ on $N \times \mathbb{R}$. Therefore, it induces a cosymplectic structure (ξ, η, φ, g) on M. For more detailes see [6]. For a generalisation of this construction see [4].

3 Lichnerowicz type invariant on almost contact metric manifolds

Let $M(\varphi, \eta, \xi, g)$ be an almost contact metric manifold. TM^c denotes the complexification of the tangent bundle TM of M. The complex-linear extension of φ on TM^c has eigenvalues $\pm\sqrt{-1}$ and 0, with corresponding eigenspaces

$$T^{\pm}M = \{-\tfrac{1}{2}\varphi^2 X \mp \tfrac{1}{2}\sqrt{-1}\varphi X, X \in TM\},\ T^0 M = \{X + \varphi^2 X, X \in TM\}.$$

The complexification TM^c splits into eigenbundles: $TM^c = T^+M \oplus T^0M \oplus T^-M$, in which the decomposition is orthogonal with respect to the Hermitian metric $\langle X, Y\rangle = g_0(X, \overline{Y})$, (where g_0 denotes the $\mathbb{C}$-bilinear extension on TM^c of the given metric on M). Now let $N(J,h)$ be an almost Hermitian manifolds, and $f : M \to N$ be a smooth map from M to N. Then the decompositions of TM^c and TN^c induce the corresponding splitting of the differential of f, and hence we can define the following three maps

$$\begin{aligned} d^+f &: T^+M \to T^+N \\ d^-f &: T^-M \to T^+N \\ d_0^+f &: T^0M \to T^+N. \end{aligned}$$

The energy density $e(f)$ of f is defined by $e(f)(p) = \|df_p\|^2/2$ for $p \in M$, where $\|df_p\|^2$ is the norm of the differential $df_p \in T_p^*M \otimes T_{f(p)}N$ at $p \in M$. We set

$$e^+(f) = \|d^+f\|^2, e^-(f) = \|d^-f\|^2, e_0^+(f) = \|d_0^+f\|^2$$

which are called the partial energy densities of f. These partial energy densities are useful to give us precise information about how the differential df of f acts on each eigenspace.

Lemma 1 *Let M be an almost contact metric manifold and N be an almost Hermitian manifold. Let f a smooth map from M to N. Then we have the following decomposition for the energy density $e(f)$ of f:*

$$e(f) = e^+(f) + e^-(f) + e_0^+(f).$$

Proof: Let $\{e_k, \varphi e_k, \xi\}_{k=\overline{1,n}}$ be an orthonormal basis (with respect to g) on TM. Then $Z_k = \frac{1}{\sqrt{2}}(e_k - \sqrt{-1}\varphi e_k)$ is an orthonormal basis on TM^+ (with respect to $< \cdot, \cdot >$).

Then we have

$$e^+(f) = \sum_{k=1}^{n} h_0(d^+f(Z_k), \overline{d^+f(Z_k)}).$$

$$e^-(f) = \sum_{k=1}^{n} h_0(d^-f(Z_k), \overline{d^-f(Z_k)}).$$

and

$$e_0^+(f) = h_0(d_0^+ f(\xi), \overline{d_0^+ f(\xi)}).$$

If we make the notations $X_k = df(e_k)$, $Y_k = df(\varphi e_k)$ and $Z = df(\xi)$, a direct computation gives:

$$e^+(f) = \frac{1}{4}\sum_{k=1}^{n}\{h(X_k, X_k) + h(Y_k, Y_k)\} + \frac{1}{2}\sum_{k=1}^{n} h(JX_k, Y_k), \tag{5}$$

$$e^-(f) = \frac{1}{4}\sum_{k=1}^{n}\{h(X_k, X_k) + h(Y_k, Y_k)\} - \frac{1}{2}\sum_{k=1}^{n} h(JX_k, Y_k), \tag{6}$$

and

$$e_0^+(f) = \frac{1}{2}h(Z, Z). \tag{7}$$

Finally we add the above three relations to abtain the decomposition in Lemma. □

Now let Ω and ω be the fundamental 2-forms of the almost contact metric manifold M and almost Hermitian manifold N respectively. That is $\Omega(X, Y) = g(X, \varphi Y)$, $X, Y \in \mathcal{X}(M)$ and $\omega(V, W) = h(V, JW)$, $V, W \in \mathcal{X}(N)$. We have the folowing Lemma:

Lemma 2 *Let M be almost contact metric manifold, N be an almost contact Hermitian manifold and f a smooth map from M to N. Then it holds that*

$$e^+(f) - e^-(f) = (f^*\omega, \Omega),$$

where the right side above means the inner product of 2-forms on M induced by g.

Proof: From the relations 5 and 6 we obtain

$$e^+(f) - e^-(f) = \sum_{k=1}^{n} h(Jdf(e_k), df(\varphi e_k)).$$

On the other hand

$$(f^*\omega, \Omega) = \sum_{p<q} f^*\omega(V_p, V_q)\Omega(V_p, W_q),$$

for any orthonormal frame $\{V_p\}_{p=1}^{2n+1}$ on M. If we choose the frame $\{e_k, \varphi e_k, \xi\}$ for $k = \overline{1,n}$, we obtain the formula. $\square$

We recall here the homotopy lemma given in [7]:

Lemma 3 *Let $f_t : M \to N$ be a smooth family of maps between smooth manifolds M and N parametrized by the real number t, ω a closed two-form on N and $\delta f_t/\delta t$ the variation field of f_t. Then*

$$\frac{\partial}{\partial t}(f_t^*\omega) = d\left(f_t^* i\left(\frac{\delta f_t}{\delta t}\right)\omega\right),$$

where $i(X)$ is the interior product with respect to a vector field X on N.

In the complex case, Lichnerowicz defined a smooth invariant associated with a smooth map [5]. Let ϑ_g denotes the volume measure on M associated to the metric g. Assuming M being compact, we can define a similar one for the case when the source manifold is endowed with an almost contact metric structure (see also ([8]):

$$K(f) = E^+(f) - E^-(f),$$

where $E^+(f)$ denotes the partial energy of f defined by integrating $e^+(f)$ on M for ϑ_g. Likewise, $E^-(f)$ and $E_0^+(f)$ are also defined:

$$E^\pm(f) = \int_M e^\pm(f)\vartheta_g,\ E_0^+(f) = \int_M e_0^+(f)\vartheta_g.$$

We are now able to prove the following theorem:

Theorem 1 *Let $f : (M, \varphi, g) \to (N, J, h)$ be a smooth map between an almost contact metric manifold and an almost Hermitian manifold. Suppose that M is compact, ω closed and Ω coclosed. Then $K(f)$ is a smooth invariant.*

Proof: By Lemma 2 we have

$$K(f) = \int_M (f^*\omega, \Omega)\,\vartheta_g.$$

Let f_t be a smooth variation of f. Then, it follows from Lemma 3 that

$$\frac{d}{dt}K(f_t) = \int_M \left(\frac{\partial}{\partial t}(f_t^*\omega), \Omega\right)\vartheta_g$$
$$= \int_M \left(d\left(f_t^* i\left(\frac{\delta f_t}{\delta t}\right)\omega\right), \Omega\right)\vartheta_g = \int_M \left(f_t^* i\left(\frac{\delta f_t}{\delta t}\right)\omega, \delta\Omega\right)\vartheta_g = 0,$$

proving the Theorem 1. $\square$

4 Harmonic maps on cosymplectic manifolds

Let M and N be Riemannian manifolds with Riemannian metrics g and h respectively. Suppose that M is compact. A smooth map $f : M \to N$ is said to be harmonic if it is a critical point of the energy functional

$$E(f) = \textstyle\int_M e(f)\vartheta_g \ .$$

The Euler-Lagrange equation of this variational problem is $Tr_g \nabla' df = 0$ where ∇' is the connection on $T^\star M \otimes f^{-1}TN$ induced by the Levi-Civita connection ∇^M of M and the f-pullback $\widetilde{\nabla}$ of ∇^N. We denote the left side of the last equation by $\tau(f)$. This is the section of $f^{-1}TN$ called the tension field of f. For more information concerning harmonic maps see [7].

Let $M(\eta, \xi, \varphi, g)$ be an almost contact metric manifold and and $N(J, h)$ an almost Hermitian manifold. A smooth map $f : M \to N$ is called to be (φ, J)-holomorphic if its differential intertwines the structures: $df \circ \varphi = J \circ df$. We know (see [3]) that the tension field $\tau(f)$ of an (φ, J)-holomorphic map f satisfies the equation

$$J(\tau(f)) = df(div\varphi) - Tr_g\beta \ ,$$

where $\beta(X, Y) = (\widetilde{\nabla}_X J)dfY$ for $X, Y \in \Gamma(TM)$. If M is a cosymplectic manifold then $div\varphi = 0$, and N being Kählerian implies $\beta = 0$. Hence any (φ, J)-holomorphic map from a cosymplectic manifold to a Kähler manifold is harmonic. A natural question to ask is whether such a map is also an absolute minimum of its energy functional or not. With the invariant $K(f)$ on hand we are able to answer the question as follows:

Theorem 2 *Let (M, φ, g) be a compact cosymplectic manifold and (N, J, h) a Kähler manifold. Then any (φ, J)-holomorphic map f from M to N attains an absolute minimum of the energy functional in its homotopy class.*

Proof: First we have to remark that as N is a Kähler manifold the fundamental 2-form ω is closed and because M is cosymplectic Ω is coclosed. Then if $\widetilde{f} : M \to N$ is a smooth map homotopic to f, by Theorem 1, $K(f) = K(\widetilde{f})$. Since f is (φ, J)-holomorphic it is easy to see that $df(T^\pm(M)) \subset T^\pm N$ (see

[8]) and that $df(\xi)=0$, and thus the partial energies $E^-(f)$ and $E^+_0(f)$ of f vanish. So Lemma 1 implies :

$$\begin{aligned} E(f) &= E^+(f)+E^-(f)+E^+_0(f) \\ &= E^+(f) \\ &= E^+(f)-E^-(f) \\ &= E^+(\widetilde{f})-E^-(\widetilde{f}) \leq E^+(\widetilde{f})+E^-(\widetilde{f})+E^+_0(\widetilde{f}) \\ &\leq E(\widetilde{f}), \end{aligned}$$

proving that f attains an absolute minimum of E in its homotopy class. $\square$

Example 1 *It is clear that if B is a compact Kähler manifold and S^1 is the unit circle then the projection on the first factor of the cosymplectic manifold $B\times S^1$ (see Section 1) is (φ,J) holomorphic and thus is a harmonic map which is also a minimizer of the functional energy.*

After the conclusion of Theorem 2, a good question to ask is if the converse is true, that is, when a harmonic map from a cosymplectic manifold into a Kähler manifold is (φ,J)-holomorphic. We can prove the following

Theorem 3 *Let (M,φ,ξ,η,g) be a compact cosymplectic manifold, (N,J,h) Kählerian and $f_0 : M\to N$ a harmonic map minimizing the energy functional E on its homotopy class. If f_0 is homotopic to a (φ,J)-holomorphic map then it is also (φ,J)-holomorphic .*

Proof: Let f_1 be an $(\varphi,J)-$holomorphic map homotopic with the map f_0. By the previous Theorem we have $E(f_1)\leq E(f_0)$. On the other hand f_0 atains the minimum of the energy functional on its homotopy class, and thus $E(f_0)=E(f_1)$. Now by the Theorem 1, as f_0 and f_1 are homotopic we have $K(f_0)=K(f_1)$ and thus $E^+(f_0)-E^-(f_0)=E^+(f_1)-E^-(f_1)$.

As we heve seen, because f_1 is (φ,J)-holomorphic we have $E^-(f_1)=E^+_0(f_1)=0$. We have just obtained the following two relations:

$$E^+(f_0)-E^-(f_0)=E^+(f_1)$$

and

$$E^+(f_0) + E^-(f_0) + E_0^+(f_0) = E^+(f_1).$$

Thus we have:

$$2E^-(f_0) + E_0^+(f_0) = 0$$

which implies

$$E^-(f_0) = E_0^-(f_0) = 0.$$

Now, from the definition of the partial energy $E^-(f_0)$ of f_0, as $e^-(f_0)$ is continuous we obtain $e^-(f_0) = 0$. On the other hand, using the relation 6 we have

$$e^-(f_0) = \frac{1}{4}\sum_{k=1}^{n} h(Jdfe_k - df(\varphi e_k), Jdfe_k - df(\varphi e_k)) \geq 0.$$

Thus $e^-(f_0) = 0$ if and only if $df_0(\varphi e_k) = Jdf_0(e_k)$ for any $k = \overline{1,n}$. It is is easy to se that we also have $df_0(\varphi(\varphi e_k)) = Jdf_0(\varphi e_k)$ for any $k = \overline{1,n}$.

Similar $e_0^+(f_0)$ is a positive continuous function as

$$e_0^+(f) = \frac{1}{2}h(df_0\xi, df_0\xi).$$

But $e_0^+(f_0) = 0$ and thus $df_0(\xi) = 0 = df_0(\varphi\xi) = Jdf_0(\xi)$. So we have just obtained that $df_0(\varphi X) = Jdf_0(X)$ for any X in an orthonormal basison M, so f_0 is (φ, J)-holomorphic. □

A nice geometric interpretation of the homotopy invariant $K(f)$ is given by the following proposition:

Proposition 1 *Let $f : M \to N$ be a smooth map from a compact cosymplectic manifold M of dimension $2n+1$ into a Kähler manifold N. If there exist $c \in \mathbb{R}$ such that $[f^*\omega] = c[\Omega]$, then*

$$K(f) = n \cdot c \cdot vol(M).$$

Proof: We hnow that $K(f) = \int\limits_M (f^*\omega, \Omega)\,\vartheta_g$. As $[f^*\omega] = c[\Omega]$, there exist an 1-form $\theta \in \Omega^1(M)$ such that $f^*\omega - c\Omega = d\theta$ so

$$
\begin{aligned}
K(f) &= \int_M (c\Omega + d\theta, \Omega)\, \vartheta_g \\
&= c \int_M (\Omega, \Omega)\, \vartheta_g + \int_M (d\theta, \Omega)\, \vartheta_g = \\
&= c \cdot n \cdot vol(M) + \int_M (\theta, \delta\Omega)\, \vartheta_g \\
&= c \cdot n \cdot vol(M).
\end{aligned}
$$

□

From the previous proposition we can obtain the following Corollary:

Corollary 1 *For a map as in the previous proposition which is also* $(\varphi, J)-$ *holomorphic and if* $[f^*\omega] = 0$*, then* $K(f) = 0$ *and* f *is constant*

Proof: From the Proposition 1 we have $K(f) = n \cdot c \cdot vol(M)$, and thus $E^+(f) - E^-(f) = 0$. On the other hand, as f is (φ, J)-holomorphic, we have $E^-(f) = E_0^+(f) = 0$ and so we also have $E^+(f) = 0$. But in this case the total energy $E(f)$ vanishes and thus the map f is constant. □

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