STOCHASTIC CHARACTERIZATION OF HARMONIC MAPS ON RIEMANNIAN POLYHEDRA.

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Abstract. The aim of this paper is to relate the theory of harmonicity, in the sense of Korevaar-Schoen and Eells-Fuglede, to the notion of Brownian motion in a Riemannian polyhedron, achieved by the second author. We define an exponential map at some singular points. Under the assumption that these exponential maps are totally geodesic (for instance in dimension one), we find the infinitesimal generator of a Brownian motion in a Riemannian polyhedron. We prove that it is uniquely defined on some Banach space. Finally, we show that harmonic maps, in the sense of Eells-Fuglede, with target smooth Riemannian manifolds, are characterized by mapping Brownian motions in Riemannian polyhedra into martingales, while harmonic morphisms are exactly the maps which are Brownian preserving paths.

1. Introduction.

Brownian motions in Riemannian manifolds are intimately related to harmonic functions, maps and morphisms. The origin of this relationship is the definition of a Brownian motion in a Riemannian manifold as a diffusion process generated by the Laplace-Beltrami operator, which is also the basic tool in the theory of harmonic maps. In [8], Darling studied the relation between the behaviour of Brownian motions under maps between Riemannian manifolds and harmonicity.

The theory of harmonic maps between smooth Riemannian manifolds was extended by Gromov, Korevaar and Schoen (see [17], [20]) to the case of maps between certain singular spaces, such as admissible Riemannian polyhedra. Riemannian polyhedra are interesting in many regards. They carry natural harmonic space structures in the sense of Brelot, and they include several geometric objects: smooth Riemannian manifolds, Riemannian orbit spaces, normal analytic spaces, Thom spaces etc. A deeper study of harmonic maps and morphisms between Riemannian polyhedra was done by Eells and Fuglede in [11]. Using the *weak conformality property*, in the case when the target space is a Riemannian manifold, they obtained the same

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characterization for harmonic morphisms as in the smooth case, cf. [13] and [18].

On the other hand, a rigorous construction of a Brownian motion in a Riemannian complex, was given by the second author in [4]. In particular, this construction applies to the case of a Riemannian polyhedron. Previously, Brin-Kifer had constructed a Brownian motion in the particular case of flat 2-dimensional admissible complexes [7].

The aim of this paper is to relate, in the case of Riemannian polyhedra, the theory of harmonic maps and morphisms [11] to the notion of Brownian motions in Riemannian polyhedra [4], in order to generalize Darling's results, [8], [22]. Notice that the second differential calculus on Riemannian manifolds, which is the basis of the theory of stochastic calculus, has no natural generalization on Riemannian polyhedra. Consequently, we are compelled to develop a new approach mixing smooth theory with some hybrid methods.

The outline of the paper is as follows. Section 2, included here for the sake of completeness, is an overview on Riemannian polyhedra, energy of maps, harmonic maps and morphisms on Riemannian polyhedra, Brownian motions in Riemannian manifolds, martingales etc. In Section 3, we prove that the Brownian motion has a unique infinitesimal generator defined on some Banach space. In this section we also study the behaviour of Brownian motions under harmonic functions in the sense of Gromov-Korevaar-Schoen [17], [20]. In order to state this characterization we show that the Brownian motion in a Riemannian polyhedron has the "Laplacian" as an infinitesimal generator Theorem 3.5 (we give a suitable definition of the "Laplacian" in this case). For technical reasons we make some assumption on the exponential maps (being totally geodesic). This condition is realized for example in the one-dimensional case or for flat metrics. In the last section we prove that harmonic maps, with target smooth Riemannian manifolds in the sense of [11], are exactly those which map Brownian motions in Riemannian polyhedra into martingales, Theorem 4.1, while harmonic morphisms are exactly the maps which are Brownian preserving paths, Theorem 4.2.

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2. Preliminaries.

In this section we recall some basic notions and results which will be used throughout the paper.

2.1. **Riemannian admissible complexes.** [2], [5], [6], [9], [11], [28].

Let C be a locally finite simplicial complex, endowed with a piecewise smooth Riemannian metric g, i.e. g is a family of smooth Riemannian metrics

g_S on simplexes S of C, such that the restriction $(g_S)_{|S'} = g_{S'}$, for any simplexes S' and S with $S' \subset S$.

The set of all formal linear combinations $\alpha = \sum_{v \in C} \alpha(v)v$ of vertices of C, such that $0 \leqslant \alpha(v) \leqslant 1$, $\sum_{v \in C} \alpha(v) = 1$ and $\{v; \alpha(v) > 0\}$ is a simplex of C, is denoted by *space*$|C|$. This set is a subset of the linear space $linC$ of all formal finite linear combinations of vertices of C. A vertex v of C will be identified with the formal linear combination $1v$, thus formal linear combinations of vertices become true linear combinations in $linC$.

Let C be a finite dimensional simplicial complex which is connected locally finite. A map f from $[a, b]$ to C is called a broken geodesic if there is a subdivision $a = t_0 < t_1 < \cdots < t_{p+1} = b$, such that $f([t_i, t_{i+1}])$ is contained in some cell and the restriction of f to $[t_i, t_{i+1}]$ is a geodesic inside that cell. Then, define the length of the broken geodesic map f to be:

$$\mathcal{L}(f) = \sum_{i=0}^{p} d(f(t_i), f(t_{i+1})).$$

The length inside each cell is measured with respect to its metric.

For every two points x, y in C, define $\tilde{d}(x, y)$ to be the lower bound of the lengths of broken geodesics from x to y. Note that $\tilde{d}$ is a pseudo-distance.

If C is connected and locally finite, then $(C, \tilde{d})$ is a length space and hence a geodesic space (i.e. a metric space where every two points are connected by a curve with length equal to the distance between them), if complete.

We say that the complex C is *admissible*, if it is dimensionally homogeneous, and for every connected open subset U of C, the open set $U \setminus \{U \cap \{(n-2)-\text{ skeleton }\}\}$ is connected, where n is the dimension of C (i.e. C is $(n-1)$-chainable).

We call an admissible connected locally finite simplicial complex, endowed with a piecewise smooth Riemannian metric, an *admissible Riemannian complex*.

In the sequel we shall denote a *p-skeleton* of a complex C by $C^{(p)}$.

2.2. **Riemannian polyhedra.** [11].

We mean by *polyhedron* a connected locally compact separable Hausdorff space K for which there exists a simplicial complex C and a homeomorphism $\theta : C \to K$. Any such pair (C, θ) is called a *triangulation* of K. The complex C is necessarily countable and locally finite (see [26] page 120) and the space K is path connected and locally contractible. The *dimension* of K is by definition the dimension of C and it is independent of the triangulation.

If K is a polyhedron with specified triangulation (C, θ), we shall speak of vertices, simplexes, i-skeletons (the set of simplexes of dimensions lower or equal to i), stars of K as the image under θ of vertices, simplexes, i-skeletons, stars of C. Thus our simplexes become compact subsets of K.

If for a given triangulation (C, θ) of the polyhedron K, the homeomorphism θ is locally bi-lipschitz then K is said to be a *Lip polyhedron* and θ a *Lip homeomorphism*.

A *null set* in a Lip polyhedron K is a set $Z \subset K$ such that Z meets every maximal simplex S, relative to a triangulation (C, θ) (hence any) in a set whose pre-image under θ has n-dimensional Lebesgue measure 0, with $n = \dim S$. Note that *'almost everywhere'* (a.e.) means everywhere except in some null set.

A *Riemannian polyhedron* $K = (K, g)$ is defined as a Lip polyhedron K with a specified triangulation (C, θ), such that C is a simplicial complex endowed with a covariant bounded measurable Riemannian metric tensor g, satisfying the ellipticity condition below. In fact, suppose that K has homogeneous dimension n and choose a measurable Riemannian metric g_S on the open Euclidean n-simplex $\theta^{-1}(\overset{o}{S})$ of C. In terms of Euclidean coordinates $\{x_1, \ldots, x_n\}$ of points $x = \theta^{-1}(p)$, g_S assigns to almost every point $p \in \overset{o}{S}$ (or x), an $n \times n$ symmetric positive definite matrix $g_S = (g_{ij}^S(x))_{i,j=1,\ldots,n}$, with measurable real entries and there is a constant $\Lambda_S > 0$ such that (ellipticity condition):

$$\Lambda_S^{-2} \sum_{i=0}^{n} (\xi^i)^2 \leq \sum_{i,j} g_{ij}^S(x) \xi^i \xi^j \leq \Lambda_S^2 \sum_{i=0}^{n} (\xi^i)^2$$

for *a.e.* $x \in \theta^{-1}(\overset{o}{S})$ and every $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$. This condition amounts to the components of g_S being bounded and it is independent not only of the choice of the Euclidean frame on $\theta^{-1}(\overset{o}{S})$ but also of the chosen triangulation.

For simplicity of statements, we shall sometimes require that, relative to a fixed triangulation (C, θ) of the Riemannian polyhedron K (uniform ellipticity condition),

$$\Lambda := \sup \{\Lambda_S : S \text{ is simplex of } K\} < \infty.$$

A Riemannian polyhedron K is said to be *admissible* if for a fixed triangulation (C, θ) (hence any) the Riemannian simplicial complex C is admissible.

We underline that, for simplicity, the given definition of a Riemannian polyhedron (K, g) contains already the fact (because of the definition above of the Riemannian admissible complex) that the metric g is *continuous* relative to some (hence any) triangulation (i.e. for every maximal simplex S the metric g_S is continuous up to the boundary). This fact is sometimes omitted in the literature. The polyhedron is said to be *simplexwise smooth* if relative to some triangulation (C, θ) (and hence any), the complex C is simplexwise smooth. Both continuity and simplexwise smoothness are preserved under subdivision.

2.3. **Energy of maps.** [17], [20], [11].

The concept of energy of maps from a Riemannian domain into an arbitrary metric space Y was defined and investigated by Gromov, Korevaar and Schoen [17], [20]. Later on, this concept was extended by Eells and Fuglede [11] to the case of a map from an admissible Riemannian polyhedron K

with simplexwise smooth Riemannian metric. The *energy* $E(\varphi)$ of a map φ from K to the space Y is defined as the limit of suitable approximate energy expressed in terms of the distance function d_Y of Y.

The maps $\varphi : K \to Y$ of finite energy are precisely those quasicontinuous (i.e. have continuous restrictions to closed sets), whose complements have arbitrarily small capacity, (see [11], page 153) whose restriction to each top dimensional simplex of K has finite energy in the sense of Korevaar-Schoen, and $E(\varphi)$ is the sum of the energies of these restrictions.

Consider now an admissible m-dimensional Riemannian polyhedron (K, g) with simplexwise smooth Riemannian metric. It is not required that g is continuous across lower dimensional simplexes. The target (Y, d_Y) is an arbitrary metric space.

Denote $L^2_{loc}(K, Y)$ the space of all μ_g-measurable (μ_g the volume measure of g) maps $\varphi : K \to Y$ having separable essential range and for which the map $d_Y(\varphi(.), q) \in L^2_{loc}(K, \mu_g)$ (i.e. locally μ_g-squared integrable) for some point q (hence by triangle inequality for any point). For $\varphi, \psi \in L^2_{loc}(K, Y)$ define their distance $\mathcal{D}(\varphi, \psi)$ by:

$$\mathcal{D}^2(\varphi, \psi) = \int_K d_Y^2(\varphi(x), \psi(y)) d\mu_g(x).$$

Two maps $\varphi, \psi \in L^2_{loc}(K, Y)$ are said to be *equivalent* if $\mathcal{D}(\varphi, \psi) = 0$, (i.e. $\varphi(x) = \psi(x)$ μ_g-a.e.). If the space K is compact, then $\mathcal{D}(\varphi, \psi) < \infty$ and $\mathcal{D}$ is a metric on $L^2_{loc}(K, Y) = L^2(K, Y)$ which is complete if the space Y is complete [20].

The *approximate energy density* of the map $\varphi \in L^2_{loc}(K, Y)$ is defined for $\epsilon > 0$ by:

$$e_\epsilon(\varphi)(x) = \int_{B_K(x,\epsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\epsilon^{m+2}} d\mu_g(x').$$

The function $e_\epsilon(\varphi) \geq 0$ is locally μ_g-integrable.

The *energy* $E(\varphi)$ of a map φ of class $L^2_{loc}(K, Y)$ is:

$$E(\varphi) = \sup_{f \in \mathcal{C}_c(K,[0,1])} \left(\limsup_{\epsilon \to 0} \int_K f e_\epsilon(\varphi) d\mu_g \right),$$

where $\mathcal{C}_c(K, [0,1])$ denotes the space of continuous functions from K to the interval $[0,1]$ with compact support.

A map $\varphi : K \to Y$ is said to be *locally of finite energy*, and we write $\varphi \in W^{1,2}_{loc}(K, Y)$, if $E(\varphi_{|U}) < \infty$ for every relatively compact domain $U \subset K$, or, equivalently, if K can be covered by domains $U \subset K$ such that $E(\varphi_{|U}) < \infty$.

For example (Lemma 4.4, [11]), every Lip continuous map $\varphi : K \to Y$ is of class $W^{1,2}_{loc}(K, Y)$. In the case when K is compact, $W^{1,2}_{loc}(K, Y)$ is denoted by $W^{1,2}(K, Y)$ the space of all maps of finite energy.

$W_c^{1,2}(K,Y)$ denotes the linear subspace of $W^{1,2}(K,Y)$ consisting of all maps of finite energy of compact support in K.

2.4. **Harmonic maps and harmonic morphisms on Riemannian polyhedra.** [11].

Let (K,g) be an arbitrary admissible Riemannian polyhedron (g is only bounded, measurable, with local elliptic bounds), dim $K=m$ and (Y,d_Y) a metric space .

A continuous map $\varphi: K\to Y$ of class $W^{1,2}_{loc}(K,Y)$ is said to be *harmonic* if it is *bi-locally E-minimizing*, i.e. K can be covered by relatively compact subdomains U for each of which there is an open set $V\supset\varphi(U)$ in Y such that

$$E(\varphi_{|U})\leq E(\psi_{|U})$$

for every continuous map $\psi\in W^{1,2}_{loc}(K,Y)$, with $\psi(U)\subset V$ and $\psi=\varphi$ in $K\backslash U$.

Let (N,h) denote a smooth Riemannian manifold without boundary of dimension n and $\Gamma^k_{\alpha\beta}$ the Christoffel symbols on N. By a *weakly harmonic map* $\varphi: K\to N$ we mean a quasicontinuous map (a map which is continuous on the complement of open sets of arbitrarily small capacity; in the case of the Riemannian polyhedron K, it is just the complement of open subsets of the $(m-2)$-skeleton of K) of class $W^{1,2}_{loc}(K,N)$ with the following property:

For any chart $\eta: V\to\mathbb{R}^n$ on N and any quasiopen set $U\subset\varphi^{-1}(V)$ of compact closure in K, the equation

$$\int\limits_U\langle\nabla\lambda,\nabla\varphi^k\rangle d\mu_g=\int\limits_U\lambda(\Gamma^k_{\alpha\beta}\circ\varphi)\langle\nabla\varphi^\alpha,\nabla\varphi^\beta\rangle d\mu_g,$$

holds for every $k=1,\dots,n$ and every bounded function $\lambda\in W_0^{1,2}(U)$.

When K and Y denote two Riemannian polyhedra (or any harmonic spaces in the sense of Brelot; see Chapter 2, [11]), a continuous map $\varphi:K\to Y$ is a *harmonic morphism* if, for every open set $V\subset Y$ and for every harmonic function v on V, $v\circ\varphi$ is harmonic on $\varphi^{-1}(V)$.

2.5. **Brownian motions in Riemannian manifolds.** [8], [12], [29].

Consider $(\Omega,\mathcal{A},P)$ a probability space, (E,ε) a measurable space, and I an ordered set. By a *stochastic process* on $(\Omega,\mathcal{A},P)$ with values on (E,ε) and I as time interval, we mean a map (see [12], or [29], or [8]):

$$\begin{array}{rccc} X: & I\times\Omega & \to & E\\ & (t,\omega) & \mapsto & X(t,\omega),\end{array}$$

such that for each $t\in I$, $X_t:\omega\in\Omega\mapsto X(t,\omega)\in E$ is measurable from $(\Omega,\mathcal{A})$ to (E,ε).

A family $\mathcal{F}=(\mathcal{F}_t)_{t\in I}$ of σ-subalgebras of $\mathcal{A}$, such that $\mathcal{F}_s\subset\mathcal{F}_t$, for all s, t with $s<t$, is called *a filtration* on $(\Omega,\mathcal{A},P)$ with I time interval.

Given a filtration $\mathcal{F}=(\mathcal{F}_t)_{t\in I}$, a process X, admitting as time interval a part J of I, is said to be *adapted* to $\mathcal{F}$, if for every $t\in J$, X_t is $\mathcal{F}_t$-measurable.

A real-valued process X is said to be a *submartingale*, with respect to a filtration $\mathcal{F}_t$ fixed on $(\Omega, \mathcal{A}, P)$, if it has the following properties : a) X is adapted; b) each random variable X_t is integrable; c) for each pair of real numbers s, t, $s < t$, and every $A \in \mathcal{F}_s$ we have:

$$\int_A X_s dP \leq \int_A X_t dP.$$

When the equality holds we say that X is a *martingale*.

A real-valued process X is said to be a *continuous local martingale* if and only if it is a continuous (with respect to the time variable) adapted process X such that each $X_{t\wedge T_n}\chi_{\{T_n>0\}}$ is a martingale, where χ is the characteristic function and T_n is the stopping time: $inf\{t : |X_t| \geq n\}$.

A *semimartingale* is the sum of a continuous local martingale and a process with finite variation. If the process of the finite variation is an increasing one, the semimartingale is called a *local submartingale*.

Let M be a manifold with a connection ${}^M\nabla$, and X be an M-valued process. Following Schwartz characterization [25], a ${}^M\nabla$*-martingale tester*, (U_1, U_2, U_3, f) will consist of:

- open sets U_1, U_2, U_3 in M with $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset U_3$,
- a convex function $f : U_3 \to \mathbb{R}$.

The process $X = (X_t, \mathcal{F}_t)$ is said to be a ${}^M\nabla$*-martingale*, if it is a continuous semimartingale on M (i.e. $\forall f \in \mathcal{C}^2(M)$, $f \circ X$ is a real valued semimartingale), and for all ${}^M\nabla$-martingale tester (U_1, U_2, U_3, f), the process

$$Y = (Y_t, \mathcal{F}_t),\ Y_t = \int_0^t \chi_F(s) d(f \circ X_s),$$

is a local submartingale. F denote the previsible set $\bigcup_{i=1}^{\infty} (\sigma_i, \tau_i]$ where $\sigma_i, \tau_i, i \geq 0$ is the collection of stopping-times, associated to the process X and any ${}^M\nabla$-martingale tester, defined by:

- $\sigma_0 = 0, \tau_0 = 0$
- $\sigma_i = inf\{t > \tau_{i-1} : X_t \in U_1\}; \tau_i = inf\{t > \sigma_i : X_t \notin \overline{U_2}\}, i \geq 1.$

χ_F denotes the characteristic function.

Suppose M is a Riemannian manifold with Levi-Civita connection ${}^M\nabla$. A Brownian motion is characterized as a diffusion $B = (B_t, \mathcal{F}_t)$ with generator $\frac{1}{2}\Delta$; in other words, for all $f : M \to \mathbb{R}$, the process C^f, where $C^f_t = f(B_t) - f(B_0) - \frac{1}{2}\int_0^t \chi_F(s)\Delta f(B_s)ds$, is a local martingale.

3. Brownian motions in admissible Riemannian polyhedra.

In [4], the second author proved the existence of a Brownian motion on a Riemannian polyhedron. In this section, we find explicitly its infinitesimal generator, Theorem 3.5. Furthermore, we give necessary and sufficient conditions for harmonicity in terms of local martingales, see Corollary 3.10.

We begin by recalling some basic results used in the sequel.

3.1. **Boundary normal coordinates,** [19]**,** [27] **and the second fundamental form.** [1], [12], [24].

Let M be a Riemannian manifold with non-empty boundary ∂M. For any point $y \in M$ there is a shortest geodesic to the boundary that is normal to ∂M.

Similarly to the exponential mapping defined on T_xM for x an interior point of M, we can define (see [19]) the *boundary exponential mapping*: $\exp_{\partial M} : \partial M \times \mathbb{R}_+ \to M$, $\exp_{\partial M}(z,t) = \gamma_{z,\nu}(t)$, where $\mathbb{R}_+ = [0,\infty)$ and t sufficiently small such that $\exp_{\partial M}(z,t) \in M$. Here, $\gamma_{z,\nu}$ denotes the normal geodesic to ∂M whose derivative at zero equals ν, the unitary normal vector to ∂M at the point z.

Using the boundary exponential mapping, one introduced (see for example [19] or [27]) the *boundary normal* (or *semi-geodesic*) *coordinates*, analogously to the Riemann normal coordinates. Compared to the classical case of empty boundary, instead of a set of geodesics starting from a point one considers the set of geodesics normal to ∂M.

Consider $\mathcal{U}_\rho = \partial M \times [0,\rho)$ a collar neighbourhood of $\partial M \times \{0\}$ in the boundary cylinder $\partial M \times \mathbb{R}_+$. Denote by

$$\mathcal{V}_\rho = \exp_{\partial M}(\mathcal{U}_\rho) = \{x \in M; d(x,\partial M) < \rho\}$$

a collar neighbourhood of ∂M in M.

Then, for ρ sufficiently small, define $(\mathcal{V}_\rho, x_1, \dots, x_n)$ local coordinates in M (the boundary normal coordinates) in the following way: for $x \in \mathcal{V}_\rho$, $x_n := d(x,\partial M)$, $z \in \partial M$ is the unique boundary point such that $d(x,z) = d(x,\partial M)$ and $(x_1,\dots,x_{n-1})$ on ∂M are local coordinates around z. ρ is chosen small enough such that $\gamma_{z,\nu}(t)$ is the unique shortest geodesic to ∂M for $t < \rho$.

As in the case of Riemannian manifolds without boundary, the Laplacian is given in boundary normal coordinates (see [21]) by:

$$\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{|g|}\frac{\partial}{\partial x_j}\right) = g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial}{\partial x_k}\right),$$

where $|g| = det(g_{kl})$ and Γ^k_{ij} are the Christoffel symbols on M.

Let (M,g), (N,h) two Riemannian manifolds and $\varphi : M \to N$ a smooth map. The Levi-Civita connection ∇^M of M and the pull-back connection ∇^φ of the pull-back bundle $\varphi^{-1}TN$ induce a connection ∇ on the bundle $T^*M \otimes \varphi^{-1}TN$. Applying this connection to $d\varphi$ one obtain the *second* *fundamental form of* φ (also called the *Hessian of* φ) (see [1] or [24]), denoted by Hess φ, or $\nabla d\varphi$, and explicitly given by:

$$(1) \qquad (\text{Hess } \varphi)(X,Y) := \nabla^{\varphi}_X(d\varphi(Y)) - d\varphi(\nabla^M_X Y), \ \forall X,Y \in \Gamma(TM).$$

When $N = \mathbb{R}$ and (M,g) is a manifold with or without boundary then (1) reads:

$$(2) \qquad (\text{Hess } \varphi)(X,Y) := X(Y(\varphi)) - d\varphi(\nabla^M_X Y), \ \forall X,Y \in \Gamma(TM).$$

Using the Hessian, the Laplacian can be also defined as:

$$\Delta f := Tr(\text{Hess } f).$$

A smooth map $\varphi : M \to N$ between two Riemannian manifolds is *totally geodesic* (see [1], [12]) if for every $f : N \to \mathbb{R}$

$$(\text{Hess}^M(f \circ \varphi))(*,*) = (\text{Hess}^N f)(d\varphi(*), d\varphi(*)),$$

where Hess^M, Hess^N denote the second fundamental forms on M, N respectively.

3.2. **The tangent cone and the exponential map.** Let (K,g) be an n-dimensional admissible Riemannian polyhedron and p a point in the ($(n-1)$-skeleton)$\setminus$ ($(n-2)$-skeleton).

We shall slightly reformulate the *definition of the tangent cone* previously introduced in [2].

Suppose that p is in $\overset{o}{\widehat{S_{n-1}}}$, the topological interior of the $(n-1)$-simplex S_{n-1}. Let $S^1_n, S^2_n, \ldots, S^k_n$, $k \geq 2$, denote the n-simplexes adjacent to S_{n-1}. Then each S^l_n, for $l = 1, \ldots, k$, can be viewed as an affine simplex in $\mathbb{R}^n$, that is $S^l_n = \bigcap_{i=0}^n H_i$ where H_i are closed half spaces in $\mathbb{R}^n$. The Riemannian metric $g_{S^l_n}$ is the restriction to S^l_n of a smooth Riemannian metric defined in an open neighbourhood of S^l_n in $\mathbb{R}^n$.

Since $p \in ((n-1) - skeleton) \setminus ((n-2) - skeleton)$, each S^l_n for $l = 1, \ldots, k$, can be viewed, locally around p, as a manifold with boundary, where the boundary is S_{n-1}. Then there exists a unique hyperplane, for $i = 0, \ldots, n$, containing p. Define $T_p S^l_n$ as the half-space H_i which contains the corresponding hyperplane.

Notice that $T_p S^l_n$ can be naturally embedded in $linS^l_n \subset linK$ and

$$(3) \qquad T_p S^l_n = T_p S_{n-1} \times [0, \infty).$$

Define the *tangent cone of* K *over* p as: $T_p K = \bigcup_{l=1}^k T_p S^l_n \subset linK$.

The difference from the original definition (see [2]) is that we do not need to pass to subdivision of K in order to make the point p become a vertex.

Let $\mathfrak{U}_p(S^l_n)$ be the subset of all unit vectors in $T_p S^l_n$ and denote $\mathfrak{U}_p = \mathfrak{U}_p(K) = \bigcup_{S^l_n \ni p} \mathfrak{U}_p(S^l_n)$. The set $\mathfrak{U}_p$ is called the *link of* p *in* K. As S^l_n is a simplex adjacent to p, then $g_{S^l_n}(p)$ defines a Riemannian metric on the

$(n-1)$-simplex $\mathfrak{U}_p(S_n^l)$. The family g_p of Riemannian metrics $g_{S_n^l}(p)$ turns $\mathfrak{U}_p(S_n^l)$ into a simplicial complex with a piecewise smooth Riemannian metric such that the simplexes are spherical.

Having defined the tangent cone, and using the boundary exponential map, we can introduce next the *exponential map* locally around a point p in the topological interior of an $(n-1)$-simplex S_{n-1}.

Take V_0 a neighbourhood of 0 in T_pK. The definition of the exponential map $E_p : T_pK \to K$ on each maximal face $V_0 \bigcap T_pS_n^l$, $l = 1, \ldots, k$ is based on the fact that, locally around p, each S_n^l becomes a manifold with boundary, with $\partial(S_n^l) = S_{n-1}$. This allows us to consider the boundary exponential map (see Section 3.1):

$$\exp_{\partial S_n^l} : \mathcal{V}_\rho \to \mathcal{W}_\rho,$$

where U_p is a small neighbourhood of p, $\mathcal{V}_\rho = (U_p \cap S_{n-1}) \times [0, \rho)$ is a collar neighbourhood of $(U_p \cap S_{n-1}) \times 0$ in the boundary cylinder $(U_p \cap S_{n-1}) \times \mathbb{R}_+$ and

$$\mathcal{W}_\rho = \exp_{\partial S_n^l}(\mathcal{V}_\rho) := \{x \in (U_p \cap S_n^l); d(x, (U_p \cap S_{n-1})) < \rho\}.$$

Moreover, on the manifold S_{n-1} we consider the usual exponential map at p:

$$\exp_p : T_pS_{n-1} \to S_{n-1}.$$

Using the decomposition $T_pS_n^l = T_pS_{n-1} \times [0, \infty)$, we define the exponential map

$$E_p : V_0 \cap T_pS_n^l \to S_n^l$$

in the following way. Consider u a tangent vector in $V_0 \cap T_pS_n^l$. We can decompose $u = (v, w)$ where $v \in T_pS_{n-1}$ and w is a normal vector to ∂S_n^l. Then

$$E_p(u) = \exp_{\partial S_n^l}(\exp_p(v), ||w||).$$

3.3. **Brownian motions.** The Brownian motion in a piecewise smooth Riemannian complex, was obtained in [4], as a weak limit of isotropic processes. This construction holds obviously for the piecewise smooth Riemannian polyhedron K.

Let us recall some essential facts about this construction. In [4], the second author defined a process: $Y^\eta = (\Omega, \mathcal{F}_t^0, Y_t^\eta, \theta_t, P)$, for $\eta \in (0, 1]$, in the following way:

$$Y_t^\eta(\omega) = \begin{cases} \Upsilon_{\eta Z_i(\omega)}(\frac{t}{\eta^2} - \tau_i(\omega)) & \text{if} \quad \tau_i(\omega) \le \frac{t}{\eta^2} \le \tau_{i+1}(\omega) \\ D & \text{if} \quad \xi(\omega) \le \frac{t}{\eta^2}, \end{cases}$$

where τ_i are the stopping times such that, for all $i \in \mathbf{N}$, the real random variable $(\tau_i - \tau_{i+1})$ is exponentially distributed and $\tau_0 = 0$; Υ_η is the generalized geodesic flow (see [2]); D is the one point compactification of K (because K is semicompact) and ξ is the life time of Y_t^η; Z is a unit tangent vector randomly chosen in the link of the point $\Upsilon_{\eta Z_{i-1}(\omega)}(\tau_i(\omega))$ with respect to the volume measure (link is viewed as a spherical Riemannian polyhedron), where $Z_0(\omega)$ is also a unit tangent vector randomly chosen in the link of the starting point.

In [4] it is also proved that Y^η_t (for $\eta \in (0,1]$) is a continuous Markov process, for each $\eta \in (0,1]$, Y^η generates a measure μ_η on the space $\mathcal{C}(\mathbb{R}^+, K) := \{f : \mathbb{R}^+ \to K, f - \mathit{continuous}\}$ and μ_η has a subsequence which converges to a measure W on $\mathcal{C}(\mathbb{R}^+, K)$, called *Wiener measure.* This Wiener measure generate a Brownian motion in the Riemannian polyhedron, such that the transition functions of the generated Brownian motion are just the projections of the Wiener measure on K (see for details [4]).

Proposition 3.1. *Let $(B_t)_{t\geq 0}$ denote the Brownian motion in the $n-$ dimensional admissible Riemannian polyhedron K, see* [4]. *Then $(B_t)_{t\geq 0}$ almost surely never hits the $(n-2)-$skeleton and for every point p of the $((n-1)-skeleton) \setminus ((n-2)-skeleton)$, all the maximal simplexes adjacent to p have the same probability to be chosen by B^p_t i.e. $(B_t)_{t\geq 0}$ has equal branch probabilities.*

Proof: The $s-$*skeleton* of K is usually denoted by $K^{(s)}$, [11].

For any $p \in K^{(n-1)} \setminus K^{(n-2)}$, denote by $U \subset K \setminus K^{(n-2)}$ an open neighbourhood of p and by $\tau_U := inf\{t > 0/B^p_t \notin U\}$ the fist exit time of B^p_t from U.

For any maximal simplex S of K adjacent to p and for t close to 0,

$$P(B^p_{t\wedge\tau_U} \in U \cap S) = \lim_{\eta\to 0} P({}^pY^\eta_t \in U \cap S).$$

Denote by τ_1 the first stopping-time associated to the process Y^η_t. Suppose that $P\{\tau_U \leq \tau_1\} = 1$. For any $\eta > 0$,

$$P({}^pY^\eta_t \in U \cap S) = E[\chi_{U\cap S}({}^pY^\eta_t)] =$$

$$= \int\limits_0^\infty e^{-\left(s+\frac{t}{\eta^2}\right)} \int\limits_{\mathfrak{U}_p(U\cap S)} \chi_{U\cap S}\left({}^p\Upsilon_{\eta\xi}\left(s+\frac{t}{\eta^2}\right)\right) d\lambda(\xi)\, ds.$$

So, for all $\eta > 0$, $P({}^pY^\eta_t \in U \cap S)$ depends only on the link $\mathfrak{U}_p(U \cap S)$ which is independent of the choice of the maximal simplex adjacent to p. We conclude that B^t_p has equal branch probabilities.

Let us compute $P(B_t \in K^{(n-2)})$:

$$P(B_t \in K^{(n-2)}) = \lim_{\eta\to 0} P(Y^\eta_t \in K^{(n-2)}).$$

For any $\eta > 0$,

$$P(Y^\eta_t \in K^{(n-2)}) = E[\chi_{K^{(n-2)}}(Y^\eta_t)].$$

Since the process Y^η_t is Markov, to compute the above average, we can suppose that $0 < t < \tau_1$ which does not change the result. Then

$$E[\chi_{K^{(n-2)}}(Y^\eta_t)]$$

is equal to

$$\int_0^{\infty} e^{-\left(s+\frac{t}{\eta^2}\right)} \int_{\mathfrak{U}_p(K(K^{(n-2)}))} \chi_{K^{(n-2)}}\left({}^p\Upsilon_{\eta\xi}\left(s+\frac{t}{\eta^2}\right)\right) d\lambda(\xi)\, ds,$$

where $\mathfrak{U}_{\mathfrak{p}}(K(K^{(n-2)}))$ denotes all the vectors in the link $\mathfrak{U}_p(K)$ pointing into the $K^{(n-2)}$.

Since $\lambda(\mathfrak{U}_p(K(K^{(n-2)}))) = 0$, we have

$$\int_{\mathfrak{U}_p(K(K^{(n-2)}))} \chi_{K^{(n-2)}}\left({}^p\Upsilon_{\eta\xi}\left(s+\frac{t}{\eta^2}\right)\right) d\lambda(\xi) = 0.$$

Therefore, we have proved that for any $\eta > 0$, $P(Y_t^{\eta} \in K^{(n-2)}) = 0$ and consequently $P(B_t \in K^{(n-2)}) = 0$.

Proposition 3.2. *The Brownian motion $(B_t)_{t\geq 0}$ on admissible Riemannian polyhedra has an infinitesimal generator L defined on a Banach subspace $\mathbf{D}_L$ i.e. for every $f \in \mathbf{D}_L$,*

$$Lf := \lim_{t\to 0} \frac{E[f(B_t)] - f(B_0)}{t} \qquad \textit{uniformly.}$$

Proof: Remark that the Brownian motion $(B_t)_{t\geq 0}$ is trajectories continuous [4] so it is stochastically continuous. Then by Dynkin's result (see [10] Theorem 2.3), the existence of L is completely insured. □

Let (K, g) be a n-dimensional admissible Riemannian polyhedron, endowed with a continuous simplexwise smooth metric.

Definition 3.3. Consider $U \subset K$ a domain which meets exactly one $(n-1)$-simplex S_{n-1}. Let $S_n^1, \ldots, S_n^k$ denote the n-simplexes adjacent to S_{n-1} and f a continuous function on U which is of class $\mathcal{C}^2$ in each $\overset{o}{\widehat{S_n^j}} \cap U$ and at least of class $\mathcal{C}^1$ in each $(\overset{o}{\widehat{S_n^j}} \cup \overset{o}{\widehat{S_{n-1}}}) \cap U$. The function f is said to be of *zero normal trace condition* if and only if:

$$\sum_{j=1}^{k} D_j f(x) = 0$$

at almost every point x of $S_{n-1} \cap U$, where $D_j f(x)$ denotes the inner normal derivative of $f_{|S_n^j \cap U}$ at x.

Remark 3.4. The space $\mathbf{D}_L$ contains the space

$$W^{1,2}_{loc}(K) \bigcap \left\{ \begin{array}{l} \text{function of class } \mathcal{C}^2 \text{ in the interior of the } n\text{-simplexes} \\ \text{and the } (n-1)\text{-simplexes and of zero normal trace condition} \end{array} \right\}$$

which is denoted by D_L.

Let $B = (\Omega, \mathcal{F}_t^0, B_t, \theta_t, P)$ be the K-valued Brownian motion introduced above (see [4]).

Theorem 3.5. *Suppose that the exponential map (Section 3.2) is totally geodesic in each point in the topological interior of any $(n-1)$-simplex. Let $\varphi \in W^{1,2}_{loc}(K)$, i.e. there exists a covering $\mathcal{U}$ of K with relatively compact subdomains, such that φ is of finite energy on each $U \in \mathcal{U}$. Assume that φ is of class $\mathcal{C}^2$ on the interior of each n-simplex and on the interior of each $(n-1)$-simplex, and of zero normal trace condition. Denote by $\tau_U := inf\{t > 0 / B_t \notin U\}$ the first exit time of B_t from U. Then we have:*

$$\frac{1}{2}\widetilde{\Delta}\varphi = \frac{\partial}{\partial t}E[\varphi(B_{t\wedge\tau_U})], \text{ on } U\backslash((n-2) - skeleton). \tag{4}$$

where $E[\varphi(B_{t\wedge\tau_U})]$ is the expectation with respect to $B_{t\wedge\tau_U}$ and

$$\widetilde{\Delta} = \begin{cases} \frac{1}{k}\sum_{l=1}^{k}\Delta_l, & \text{at a point in } (n-1)\text{-skeleton } \backslash(n-2)\text{-skeleton}, \\ & \Delta_l \text{ is the Laplacian in } S_n^l \text{ at a boundary point;} \\ \text{the usual Laplacian}, & \text{in the interior of each simplex.} \end{cases}$$

Proof: Let $p \in K$. There are two cases to investigate:

***Case* 1**: If p is in the topological interior of some n-dimensional simplex, using [12] or [14], the relation (4) clearly holds.

***Case* 2**: Let p be in the $((n-1) - skeleton)\backslash((n-2) - skeleton)$.

The idea in this case is to transfer, locally, the computations from the polyhedron K to its tangent cone over p, T_pK.

Suppose that p is in $\overset{o}{\widehat{S_{n-1}}}$ the topological interior of the $(n-1)$-simplex S_{n-1}. Let $S_n^1, S_n^2, \ldots, S_n^k$, $k \geq 2$, denote the n-simplexes adjacent to S_{n-1}. Take V_0 a neighbourhood of 0 in T_pK, and consider the exponential map

$$E_p : V_0 \cap T_pS_n^l \to S_n^l,$$

defined in Section 3.2.

We shrink V_0 and U_p, if necessary, such that $E_p(V_0) = U_p$. Denote by $\Phi_p : U_p \to V_0$ the inverse map of E_p. By hypothesis, the maps E_p and Φ_p, are locally totally geodesic diffeomorphisms onto their images.

Let (B_t) denotes the Brownian motion in the polyhedron (K, g) (see [4]) and by $X_t = \Phi_p(B_t)$. We remark that $(X_{t\wedge\tau_{U_p}})$ is a Brownian motion in the flat polyhedron $V_0 \bigcap T_pK$ with equal branch probabilities (since Φ_p is totally geodesic map and B_t has equal branch probabilities), where τ_{U_p} is the first exit time of B_t from U_p.

Using the fact that $V_0 \cap T_pS_n^l$ is also a manifold with boundary, we consider the boundary normal coordinates $(x_1, \ldots, x_{n-1}, x_n)$ in the neighbourhood $V_0 \cap T_pS_{n-1}$ of 0, as follows. Pick up coordinates $(x_1, \ldots, x_{n-1})$ on $V_0 \cap$

T_pS_{n-1} and for a point x in the collar neighbourhood of $V_0 \cap T_pS_{n-1}$ in $V_0 \cap T_pS_n^l$ define $x_n := d(x, V_0 \cap T_pS_{n-1})$. Remark that this latest choice of the coordinates chart in the tangent cone over the point p is possible because the metric g in the polyhedron is continuous.

Using the boundary normal coordinates $(x_1, \dots, x_{n-1}, x_{n_l})$ in any $(V_0 \cap T_pS_n^l), l = 1, \dots, k$, the infinitesimal generator Δ^e of the Brownian motion X_t on the flat polyhedron $V_0 \cap T_pK$ has the following properties (see [7]):

1) Δ^e is defined on the space of:

- continuous functions on the flat polyhedron V_0 which are of class C^2 in the interior of both n-simplexes and $(n-1)$-simplexes, have continuous second derivatives in the interior of S_{n-1} which are limits of corresponding directional derivatives from the interior of adjacent faces;
- functions with zero normal trace condition for any point in the $((n-1)$-skeleton$)\setminus$ $((n-2)$-skeleton$)$, i.e. $\sum_{l=1}^{k} \frac{\partial f}{\partial x_{n_l}} = 0$;

2) $2\Delta^e$ is the usual Laplacian in the interior of each simplex;

3) For $q \in (V_0 \overset{o}{\widehat{\cap T_p}} S_{n-1})$, we have

$$\begin{aligned}
\Delta^e f(q) &= \frac{1}{2}\left(\sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial {x_i}^2} + \frac{1}{k}\sum_{l=1}^{k} \frac{\partial^2 f}{\partial {x_{n_l}}^2}\right) \\
&= \frac{1}{2}\left[\frac{1}{k}\left(\sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial {x_i}^2} + \frac{\partial^2 f}{\partial {x_{n_1}}^2}\right) + \cdots + \frac{1}{k}\left(\sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial {x_i}^2} + \frac{\partial^2 f}{\partial {x_{n_k}}^2}\right)\right] \\
&= \frac{1}{2k}\Delta_1^e f(q) + \cdots + \frac{1}{2k}\Delta_k^e f(q) \\
&= \frac{1}{2k}\sum_{l=1}^{k} \Delta_l^e f(q),
\end{aligned}$$

where

$$\Delta_l^e f(q) = \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial {x_i}^2} + \frac{\partial^2 f}{\partial {x_{n_l}}^2}$$

is the usual Laplacian on the manifold $V_0 \cap T_pS_n^l$ at a boundary point.

Now, for a point $v_1 \in (\overset{o}{\widehat{V_0 \bigcap T_p S_n^l}})$, the second order Taylor's development of a function f (as above) in the boundary normal coordinates has the form:

(5)

$$\begin{aligned} f(v_1) = & \ f(0) + \partial_{n_l} f(0) x_{n_l}(v_1) + \tfrac{1}{2}\partial_{n_l}^2 f(0) x_{n_l}^2(v_1) \\ + & \ \tfrac{1}{2}\sum_{j=1}^{n-1} \partial_{n_l}\partial_j f(0) x_{n_l}(v_1) x_j(v_1) + \tfrac{1}{2}\sum_{i=1}^{n-1} \partial_i \partial_{n_l} f(0) x_i(v_1) x_{n_l}(v_1) \\ + & \ \sum_{i=1}^{n-1} \partial_i f(0) x_i(v_1) + \sum_{i,j=1}^{n-1} \partial_i \partial_j f(0) x_i(v_1) x_j(v_1) + o(\varepsilon), \end{aligned}$$

with the notation:

$$\partial_{n_l} = \frac{\partial}{\partial x_{n_l}}, \partial_i = \frac{\partial}{\partial x_i}, \partial_{n_l}^2 = \frac{\partial^2}{\partial {x_{n_l}}^2}.$$

Using the symmetry of the connection, (5) reduces to:

$$\begin{aligned} f(v_1) \ = & \ f(0) + \partial_{n_l} f(0) x_{n_l}(v_1) + \tfrac{1}{2}\partial_{n_l}^2 f(0) x_{n_l}^2(v_1) \\ & + \sum_{j=1}^{n-1} \partial_j \partial_{n_l} f(0) x_{n_l}(v_1) x_j(v_1) \\ & + \sum_{i=1}^{n-1} \partial_i f(0) x_i(v_1) + \sum_{i,j=1}^{n-1} \partial_i \partial_j f(0) x_i(v_1) x_j(v_1) + o(\varepsilon). \end{aligned} \tag{6}$$

Equation (6) evaluated at $X_{t\wedge\tau_U}$ supposed in $(\overset{o}{\widehat{V_0 \bigcap T_p S_n^l}})$ becomes:

(7)

$$\begin{aligned} f(X_{t\wedge\tau_U}) = & \ f(0) + \partial_{n_l} f(0) x_{n_l}(X_{t\wedge\tau_U}) + \tfrac{1}{2}\partial_{n_l}^2 f(0) x_{n_l}^2(X_{t\wedge\tau_U}) \\ + & \ \sum_{j=1}^{n-1} \partial_j \partial_{n_l} f(0) x_{n_l}(X_{t\wedge\tau_U}) x_j(X_{t\wedge\tau_U}) \\ + & \ \sum_{i=1}^{n-1} \partial_i f(0) x_i(X_{t\wedge\tau_U}) + \sum_{i,j=1}^{n-1} \partial_i \partial_j f(0) x_i(X_{t\wedge\tau_U}) x_j(X_{t\wedge\tau_U}) + o(t). \end{aligned}$$

The process $X_{t\wedge\tau_U}$ can be decomposed into a product $(X^{n-1}_{t\wedge\tau_U}, X^n_{t\wedge\tau_U})$ of two independent processes such that $X^{n-1}_{t\wedge\tau_U}$ is $(n-1)-$dimensional Euclidean Brownian motion in the submanifold $V_0 \cap T_p S_{n-1}$ and $X^n_{t\wedge\tau_U}$ is a Brownian motion on a graph Γ with k edges $e_1, \dots, e_k$ of length ε attached to the point 0 with equal branch probabilities at 0 i.e. $X^n_{t\wedge\tau_U}$ is one dimensional Brownian motion on each edge of Γ with equal branching probabilities, see [3].

Then by taking the averages and using [3], (7) turns into:

(8)

$$\begin{aligned}
E^0[f(X_{t\wedge\tau_U})] = \ & f(0) + \textstyle\sum_{l=1}^{k} \partial_{n_l} f(0) E^0[d(o, X^n_{t\wedge\tau_U})/\{X^n_{t\wedge\tau_U} \in e_l\}] \\
+ \ & \textstyle\sum_{l=1}^{k} \frac{1}{2}\partial^2_{n_l} f(0) E^0[d(o, X^n_{t\wedge\tau_U})^2/\{X^n_{t\wedge\tau_U} \in e_l\}] \\
+ \ & \textstyle\sum_{l=1}^{k}\sum_{i=1}^{n-1} \partial_i\partial_{n_l} f(0) E^0[x_i(X^{n-1}_{t\wedge\tau_U})] E^0[d(o, X^n_{t\wedge\tau_U})/\{X^n_{t\wedge\tau_U} \in e_l\}]) \\
+ \ & \textstyle\sum_{i=1}^{n-1} \partial_i f(0) E^0[x_i(X^{n-1}_{t\wedge\tau_U})] \\
+ \ & \textstyle\frac{1}{2}\sum_{i,j=1}^{n-1} \partial_i\partial_j f(0) E^0[x_i(X^{n-1}_{t\wedge\tau_U}) x_j(X^{n-1}_{t\wedge\tau_U})] + o(t).
\end{aligned}$$

Since the process X^n_t is one dimensional Brownian motion on each edge of Γ with equal branch probabilities (see [7] or [3]) then :

$$E^0[d(o, X^n_{t\wedge\tau_U})/\{X^n_{t\wedge\tau_U} \in e_l\}] = \frac{1}{k}\sqrt{\frac{2t}{\pi}}$$

and

$$E^0[d(o, X^n_{t\wedge\tau_U})^2/\{X^n_{t\wedge\tau_U} \in e_l\}] = \frac{t}{k}$$

for every l.

So for a function f with zero normal trace condition, we have:

$$\sum_{l=1}^{k} \partial_{n_l} f(0) E^0[d(o, X^n_{t\wedge\tau_U})/\{X^n_{t\wedge\tau_U} \in e_l\}] = \frac{1}{k}\sqrt{\frac{2t}{\pi}} \sum_{l=1}^{k} \partial_{n_l} f(p) = 0,$$

On the other hand, for the $(n-1)$-Euclidean Brownian motion X^{n-1}_t, we have:

$$E^0[x_i(X^{n-1}_{t\wedge\tau_U})] = 0$$

and

$$E^0[x_i(X^{n-1}_{t\wedge\tau_U}) x_j(X^{n-1}_{t\wedge\tau_U})] = \delta_{ij} t,$$

where $\delta_{ij} = 1$ if $i = j$ $\delta_{ij} = 0$ if $i \neq j$.

Then equality (8) reduces to:

$$E^0[f(X_{t\wedge\tau_U})] = \ f(0) + \tfrac{t}{2k} \textstyle\sum_{l=1}^{k} \partial^2_{n_l} f(0) + \tfrac{t}{2} \textstyle\sum_{i,j=1}^{n-1} \delta_{ij}\partial_i\partial_j f(0) + o(t). \tag{9}$$

Which can be written as:

$$\begin{aligned}
E^0[f(X_{t\wedge\tau_U})] &= f(0) + \tfrac{t}{2k} \textstyle\sum_{l=1}^{k} \Delta^e_l f(0) + o(t) \\
&= f(0) + t\Delta^e f(0) + o(t)
\end{aligned} \tag{10}$$

We infer from (10) that:

(11) $$f(X_{t\wedge\tau_U}) = \quad f(0) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U} \Delta_l^e f(X_s)\chi_{\{X_s\in T_pS_n^l\}}ds \\ + \text{ SOME LOCAL MARTINGALE},$$

for a function f defined on V_0 with zero normal trace condition.

Observe that the zero normal trace condition is preserved by exponential map.

Now, for a function φ of class $W^{1,2}_{loc}$, which is of class C^2 in both the topological interior of the n-dimensional faces and the $(n-1)$-dimensional faces of the polyhedron K, the equation (11) reads for the function $\varphi \circ E_p$:

(12)
$$\varphi\circ E_p(X_{t\wedge\tau_U}) = \quad \varphi\circ E_p(0) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U} \Delta_l^e(\varphi\circ E_p)(X_s)\chi_{\{X_s\in T_pS_n^l\}}ds \\ + \text{ SOME LOCAL MARTINGALE}$$

The process X_t is an Euclidean Brownian motion in each maximal face, so we can write (see [12], Proposition 5.18):

(13)
$$\varphi\circ E_p(X_{t\wedge\tau_U}) = \quad \varphi\circ E_p(0) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U} \mathrm{Hess}_l^e(\varphi\circ E_p)(dX,dX)\chi_{\{X_s\in T_pS_n^l\}} \\ + \text{ SOME LOCAL MARTINGALE},$$

where $\mathrm{Hess}_l^e(\varphi\circ E_p)$ denotes the Euclidean Hessian of the function $(\varphi\circ E_p)$ on the face $T_pS_n^l$.

Since the map E_p is totally geodesic on each maximal face and on the $(n-1)$-dimensional face of $V_0\cap T_pK$, using (4.21) and (4.32) from [12], we obtain:

(14)
$$\begin{aligned}\varphi(B_{t\wedge\tau_U}) = & \quad \varphi\circ E_p(X_{t\wedge\tau_U}) \\ = & \quad \varphi(p) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U}(T^*E_p\otimes T^*E_p)\overline{\mathrm{Hess}_l}\varphi(dX,dX)\chi_{\{X_s\in T_pS_n^l\}} \\ & \quad + \text{ SOME LOCAL MARTINGALE} \\ = & \quad \varphi(p) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U}\overline{\mathrm{Hess}_l}\varphi(d(E_p\circ X),d(E_p\circ X))\chi_{\{X_s\in T_pS_n^l\}} \\ & \quad + \text{ SOME LOCAL MARTINGALE},\end{aligned}$$

where $\overline{\mathrm{Hess}_l}\varphi$ denotes the Hessian of the function φ in the boundary normal coordinates defined by $E_p : V_0\bigcap T_pS_n^l \subset \mathbb{R}^n \to U_p\bigcap S_n^l \subset K$ in S_n^l viewed as a manifold with boundary (see Section 3.2).

Relation (14) is equivalent to (see [12] (3.13)):

(15) $$\varphi(B_{t\wedge\tau_U}) = \quad \varphi(p) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U}\overline{\mathrm{Hess}_l}\varphi(dB,dB)\chi_{\{B_s\in U\bigcap S_n^l\}} \\ + \text{ SOME LOCAL MARTINGALE}$$

By [12], Proposition (5.18), we have:

$$(16)\qquad \varphi(B_{t\wedge\tau_U}) = \varphi(p) + \frac{1}{k}\sum_{l=1}^{k}\frac{1}{2}\int_0^{t\wedge\tau_U}\Delta_l\varphi(B_s)\chi_{\{B_s\in U\bigcap S_n^l\}}ds + \text{SOME LOCAL MARTINGALE},$$

where Δ_l denotes the Laplace-Beltrami operator computed by using boundary normal coordinates in a neighbourhood of p on the manifold with boundary $U_p\bigcap S_n^l$.

This concludes the proof. □

Remark 3.6. The extra-condition appearing in the hypothesis, that the exponential map is totally geodesic, even though restrictive, is realized in a certain number of cases. For instance, it holds if K is one-dimensional (tree). Hence we obtain a new proof of the main results in [7]. Another obvious case is that of flat metrics.

Remark 3.7. As we have seen, Remark 3.4 gives us the space of functions on which is defined the infinitesimal generator of the Brownian motion on the polyhedron. Moreover from Theorem 3.5, we conclude that the infinitesimal generator is exactly the Laplace-Beltrami operator on the interior of each simplex and for a point in the $(n-1)$-skeleton $\setminus$ $(n-2)$-skeleton it is equal to $\widetilde{\Delta} = \frac{1}{k}\sum_{l=1}^{k}\Delta_l$ where Δ_l is the usual Laplacian in S_n^l defined at a boundary point.

Lemma 3.8. *L is uniquely determined on the space D_L.*

Proof: All the functions considered are supposed to be at least of class $\mathcal{C}^2$ in the interior of each n-simplex and each $(n-1)$-simplex.

Consider $f \in W^{1,2}_{loc}(K)$. For every $\psi \in W^{1,2}_c(K)$, by Theorem 3.5, we have

$$\frac{1}{2}\int_{K\setminus((n-2)-skeleton)}\psi\widetilde{\Delta}f d\mu_g = \int_{K\setminus((n-2)-skeleton)}\psi L f d\mu_g.$$

Consider now an operator $\tilde{L}$ which is weakly defined on the space $W^{1,2}_{loc}(K)$ by:

$$\int_K \psi\tilde{L}f d\mu_g := -\frac{1}{2}\int_K\langle\nabla\psi,\nabla f\rangle d\mu_g,$$

for every $\psi \in W^{1,2}_c(K)$.

Indeed, $\tilde{L}$ is well defined since $W^{1,2}_c(K)$ is a Dirichlet space (see [11] page 20, 21 and Proposition 5.1) in the Sobolev $(1,2)$-norm: $||u||^2 = \int_K(u^2+|\nabla u|^2)$, for $u : K \to \mathbb{R}$.

It is clear that:

$$\int_{K\setminus((n-2)-skeleton)}\psi L f d\mu_g = \int_{K\setminus((n-2)-skeleton)}\psi\tilde{L}f d\mu_g.$$

On the other hand, the Brownian motion almost surely never hits the $(n-2)$-skeleton, so $\tilde{L}$ is also an infinitesimal generator associate to the transition probability W_t of the Brownian motion.

The transition function W_t associate to the K-valued Brownian motion is stochastically continuous, so its infinitesimal generator is uniquely determined (see [10] Lemma 2.2, Theorem 2.3). We infer that $\tilde{L}$ is equal to L on the space D_L, which concludes the proof. $\square$

Theorem 3.9. *Let (K, g) be an admissible Riemannian polyhedron endowed with a simplexwise smooth Riemannian metric and $f \in D_L$. As in Theorem 3.5, suppose that the exponential map is totally geodesic in each point in the topological interior of an $(n-1)$-simplex. Let $(B_t)_{t\geq 0}$ be a K-valued Brownian motion and let U be an open set of K taken as in the hypothesis of Theorem 3.5. Then, for any $p \in U \cap K \backslash((n-2)-skeleton)$ the process*

$$C^{f(p)}_{t\wedge\tau_U} = f(B^p_{t\wedge\tau_U}) - f(p) - \int_0^{t\wedge\tau_U} L(f(B^p_s))ds$$

is a local martingale, where $\tau_U := inf\{t > 0/B_t \notin U\}$ is the first exit time of B_t from U.

Proof: By construction, the Brownian motion $(B_t)_{t\geq 0}$ almost surely never hits the $(n-2)$-skeleton.

For any $p \in U\backslash((n-2)-skeleton)$, consider the process:

$$\begin{aligned}\tilde{C}^{f(p)}_{t\wedge\tau_U} &= \chi_{\{B^p_t \notin ((n-2)-skeleton)\}} f(B^p_{t\wedge\tau_U}) - f(p) \\ &- \int_0^{t\wedge\tau_U} \chi_{\{B^p_s \notin ((n-2)-skeleton)\}} L(f(B^p_s))ds,\end{aligned}$$

where χ denote the characteristic function.

By Theorem 3.5, $\forall p \in U\backslash((n-2)-skeleton)$,

$$\chi_{\{B^p_s \notin ((n-2)-skeleton)\}} L(f(B^p_s)) = \chi_{\{B^p_s \notin ((n-2)-skeleton)\}} \frac{1}{2}\widetilde{\Delta} f(B^p_s).$$

Taking the expectation, we obtain:

$$E[\tilde{C}^{f(p)}_{t\wedge\tau_U}] = E[f(B^p_{t\wedge\tau_U})] - E[f(p)] - \frac{1}{2}E[\int_0^{t\wedge\tau_U} \chi_{\{B^p_s \notin ((n-2)-skeleton)\}} \widetilde{\Delta} f(B^p_s)ds].$$

Moreover,

$$\frac{\partial}{\partial t}E[\tilde{C}^{f(p)}_t] = \frac{\partial}{\partial t}E[f(B^p_t)] - \frac{1}{2}\widetilde{\Delta} f(p).$$

Then, using Theorem 3.5, we obtain $\frac{\partial}{\partial t}E[\tilde{C}^{f(p)}_t] = 0, \forall p \in U\backslash((n-2)-skeleton)$.

Hence, $\tilde{C}^{f(p)}_t$ is a local martingale. Since $\tilde{C}_t$ is equal, almost surely, to the process C_t we conclude that C_t is also a local martingale. $\square$

Corollary 3.10. *Let (K, g) be an admissible Riemannian polyhedron endowed with a simplexwise smooth Riemannian metric, $f \in D_L$ and let U be an open set of K considered as in the hypothesis of Theorem 3.5. Denote by $\tau_U := inf\{t > 0/B_t \notin U\}$ the first exit time of B_t from U. Then f is harmonic if and only if, for any $p \in U \setminus ((n-2) - skeleton)$, $f(B^p_{t\wedge\tau_U})$ is a local martingale ($(B^p_t)_t$ is a K-valued Brownian motion).*

Proof: By Theorem 3.9, the processes:

$$C^{f(p)}_{t\wedge\tau_U} = f(B^p_{t\wedge\tau_U}) - f(p) - \int_0^{t\wedge\tau_U} L(f(B^p_s))ds$$

and

$$\tilde{C}^{f(p)}_{t\wedge\tau_U} = \chi_{\{B^p_t \notin ((n-2)-skeleton)\}} f(B^p_{t\wedge\tau_U}) - f(p) - \int_0^{t\wedge\tau_U} \chi_{\{B^p_s \notin ((n-2)-skeleton)\}} L(f(B^p_s))ds$$

are both local martingales, for every $p \in U \setminus ((n-2) - skeleton)$, where U is taken as in the hypothesis of the theorem.

Suppose that f is harmonic, then:

$$\chi_{\{B^p_t \notin ((n-2)-skeleton)\}} f(B^p_{t\wedge\tau_U}) = \tilde{C}^{f(p)}_{t\wedge\tau_U} + f(p),$$

for every $p \in U \setminus ((n-2) - skeleton)$.

So the process $\chi_{\{B^p_t \notin ((n-2)-skeleton)\}} f(B^p_{t\wedge\tau_U})$ is a local martingale, for every $p \in U \setminus ((n-2) - skeleton)$. But this last process is almost surely equal to $f(B^p_{t\wedge\tau_U})$, so the process $f(B^p_{t\wedge\tau_U})$ is also a locale martingale.

Conversely, suppose that for every $p \in U \setminus ((n-2) - skeleton)$, $f(B^p_{t\wedge\tau_U})$ is a local martingale. Then, by classical theory, this implies that f is harmonic on each $U \setminus ((n-2) - skeleton)$, so is an E-minimizer on each $U \setminus ((n-2) - skeleton)$. Then we have for every $\psi \in W^{1,2}_{loc}(K)$, with $\psi = f$ on $K \setminus U$,

$$\int_U e(f) = \int_{U \setminus ((n-2)-skeleton)} e(f) \leq \int_{U \setminus ((n-2)-skeleton)} e(\psi) = \int_U e(\psi).$$

We infer that f is a continuous locally E-minimizer map on K, which means that f is harmonic on U. $\square$

4. Brownian motions, Harmonic maps and morphisms.

In this section, we extend classical results due to Darling [8] relating Brownian motions and harmonic maps and morphisms to the case of maps defined on a Riemannian polyhedron. We prove that harmonic maps are characterized by mapping Brownian motions into martingales, Theorem 4.1, and harmonic morphisms are exactly the maps which are Brownian preserving paths, Theorem 4.2.

Theorem 4.1. *Let (K, g) be an admissible Riemannian polyhedron as in Theorem 3.5 and (N, h) a smooth Riemannian manifold. Let $\varphi : K \to N$ be a continuous map such that $\varphi \in W^{1,2}_{loc}(K, N)$, i.e. there exists a covering $\mathcal{U}$ of K with relatively compact subdomains, such that φ is of finite energy on each $U \in \mathcal{U}$. Assume that, φ is of class $\mathcal{C}^2$ on the interior of each n-simplex and on the interior of each $(n-1)$-simplex. Suppose that for every function $\psi : N \to \mathbb{R}$ of class $\mathcal{C}^2$, the function $\psi \circ \varphi$ is of zero normal trace condition.*

Then φ is harmonic if and only if for almost all $p \in U$ (with respect to the volume measure), $\varphi(B^p_{t\wedge\tau_U})$ is a ${}^N\nabla$-martingale, where $(B^p_t)_{t\geq 0}$ is a K-valued Brownian motion and τ_U is the first exit time of B_t from U.

Proof: Let (U_N, V_N, W_N, f) be a ${}^N\nabla$-martingale tester on N, such that $\varphi^{-1}(W_N) \subset U$.

Suppose that φ is a harmonic map.

By Theorem 3.9, for all $p \in U \setminus ((n-2) - skeleton)$, the processes:

$$C^{f\circ\varphi(p)}_{t\wedge\tau_U} := f \circ \varphi(B^p_{t\wedge\tau_U}) - f \circ \varphi(B^p_0) - \int_0^{t\wedge\tau_U} \chi_F(s) L(f \circ \varphi)(B^p_s) ds$$

and

$$\begin{aligned}\tilde{C}^{f\circ\varphi(p)}_{t\wedge\tau_U} := \; & \chi_{\{B^p_t \notin ((n-2)-skeleton)\}} f \circ \varphi(B^p_{t\wedge\tau_U}) - f \circ \varphi(B^p_0) - \\ & \int_0^{t\wedge\tau_U} \chi_{\{B^p_s \notin ((n-2)-skeleton)\}} \chi_F(s) L(f \circ \varphi)(B^p_s) ds\end{aligned}$$

are local martingales, where $F = \bigcup_{i=1}^{\infty}](\sigma_i, \tau_i]$, with

$$\begin{aligned}\sigma_i &= inf\{t > \tau_{i-1};\ \varphi(B^p_t) \in U_N\} \\ \tau_i &= inf\{t > \sigma_i;\ \varphi(B^p_t) \notin \overline{V}_N\} \\ \sigma_0 &= 0 \\ \tau_0 &= 0.\end{aligned}$$

The map φ is supposed to be harmonic and f is a convex function, hence by Eells-Fuglede's result ([11], Theorem 12.1), $f \circ \varphi$ is a subharmonic function on $\varphi^{-1}(W_N)$. But in our case the subharmonicity can be translated by:

$$\widetilde{\Delta}(f \circ \varphi)(p) \geq 0, \forall p \in U \setminus ((n-2) - skeleton)$$

The process $\chi_{\{B^p_t \notin ((n-2)-skeleton)\}}(f \circ \varphi)(B^p_{t\wedge\tau_U})$ is then the sum of a local martingale and an increasing process, so it is a local submartingale $\forall p \in U \setminus ((n-2) - skeleton)$.

Since the process $\chi_{\{B^p_t \notin ((n-2)-skeleton)\}}(f \circ \varphi)(B^p_{t\wedge\tau_U})$ is equal almost surely to $(f \circ \varphi)(B^p_{t\wedge\tau_U})$, this last process is also a local submartingale for every $p \in U \setminus ((n-2) - skeleton)$.

Conversely, suppose that for any $p \in U \setminus ((n-2) - skeleton)$, ($U$ is as in the hypothesis of the theorem), $\varphi(B^p_{t\wedge\tau_U})$ is a ${}^N\nabla$-martingale.

Therefore, for any ${}^N\nabla$-tester function $f : W_N \to \mathbf{R}$, $(f \circ \varphi)(B^p_{t\wedge\tau_U})$ is a local submartingale.

By Theorem (3.9), the process:

$$H^{f\circ\varphi(p)}_{t\wedge\tau_U} := f \circ \varphi(B^p_{t\wedge\tau_U}) - f \circ \varphi(B^p_0) - \int_0^{t\wedge\tau_U} \chi_F(s) L(f \circ \varphi)(B^p_s) ds,$$

is a local martingale, for any $p \in U \backslash ((n-2)-skeleton)$, where $F = \bigcup_{i=1}^{\infty} (\sigma_i, \tau_i]$, with

$$\begin{gathered}\sigma_i = inf\{t > \tau_{i-1}; \varphi(B^p_t) \in U_N\} \\ \tau_i = inf\{t > \sigma_i; \varphi(B^p_t) \notin \overline{V}_N\} \\ \sigma_0 = 0 \\ \tau_0 = 0.\end{gathered}$$

Since for any $p \in U \setminus ((n-2) - skeleton)$ we have

$$\frac{\partial}{\partial t} E[(f \circ \varphi)(B^p_{t\wedge\tau_U})] = \frac{1}{2} \widetilde{\Delta}(f \circ \varphi)(p).$$

and $(f \circ \varphi)(B^p_{t\wedge\tau_U})$ is a local submartingale, then $\widetilde{\Delta}(f \circ \varphi)(p) \geq 0$, for any $p \in U \setminus ((n-2) - skeleton)$.

Hence, by Eells-Fuglede (see [11], Theorem 12.1), we obtain that φ is a harmonic map on $U \setminus ((n-2) - skeleton)$.

Using the same arguments as in the proof of Corollary 3.10, for harmonic functions, we conclude that φ is harmonic on each U. $\square$

Theorem 4.2. *Notation as in Theorem 4.1. Then φ is a harmonic morphism if and only if φ maps K-valued Brownian motions $(B^p_{t\wedge\tau_U})_{t\geq 0}$, for any $p \in U \cap K \backslash ((n-2)-skeleton)$, to a Brownian motion on N, i.e. if $({}^N B^p_t)_{t\geq 0}$ denote the Brownian motion on the manifold N then there exist a continuous increasing process $(A_{t\wedge\tau_U})_{t\geq 0}$ such that: ${}^N B^p_t \circ A_{t\wedge\tau_U} = \varphi \circ B^p_{t\wedge\tau_U}$. τ_U denote the first exit time of B_t from U.*

Remark 4.3. We shall suppose the dim $K \geq$ dim N. Otherwise, φ is constant. Indeed, if dim $K <$ dim N, by smooth theory (see [1], p.46), φ is constant on each interior of maximal simplex (of a chosen fine triangulation of K). On the other hand, φ is continuous and K is $(n-1)$-chainable, so φ is constant on K.

Proof of Theorem 4.2: For the proof of the Theorem 4.2, we will adapt and complete the proof given by Darling (see [8]) in the smooth case.

" $\Rightarrow$ " Suppose $\varphi : K \to N$ is a harmonic morphism. Then by Eells-Fuglede's result ([11], Theorem 13.2), there exists a function $\lambda \in L^1_{loc}(K)$ called the *dilation*, such that:

$$-\int_K \langle \nabla\psi, \nabla(v \circ \varphi)\rangle = \int_K \psi\lambda[(\Delta_N v) \circ \varphi], \tag{17}$$

for every $v \in \mathcal{C}^2(N)$ and $\psi \in Lip_c(K)$.

Let $B^p = (B_t^p)_{t\geq 0}$ a K-valued Brownian motion, for any $p \in K \setminus ((n-2)-skeleton)$. Take U as in the hypothesis of the theorem and suppose $p \in U$.

Define a continuous increasing process $(A_{t\wedge\tau_U})_{t\geq 0}$ by:

$$A_{t\wedge\tau_U} := \int_0^{t\wedge\tau_U} \lambda(B_s)ds, \tag{18}$$

and it's inverse as: $C_{t\wedge\tau_U} = inf\{s; A_{s\wedge\tau_U} > s\}$.

Denote $\varphi \circ B^p$ by $X^{\varphi(p)} = (X_t^{\varphi(p)})_{t\geq 0}$ on N. For any function $f : N \to \mathbb{R}$ of class $\mathcal{C}^2$ and any $p \in U \cap K \setminus ((n-2) - skeleton)$ we have:

$$\int_0^{t\wedge\tau_U} \Delta_N f(X^{\varphi(p)} \circ C_s)ds = \int_0^{C_{t\wedge\tau_U}} \Delta_N f(X_u^{\varphi(p)})dA_u,$$

where Δ_N denote the Laplace-Beltrami operator on the manifold N.

From (18) we obtain

$$\int_0^{t\wedge\tau_U} \Delta_N f(X^{\varphi(p)} \circ C_s)ds = \int_0^{C_{t\wedge\tau_U}} \lambda(B_u^p)\Delta_N f(X_u^{\varphi(p)})du. \tag{19}$$

Using (17), the right hand side of the equality (19) is equal to

$$2\int_0^{C_{t\wedge\tau_U}} L(f\circ\varphi)(B_u^p)du,\ \mu_g \text{ a.e. },$$

for every $p \in U \cap K \setminus ((n-2) - skeleton)$.

On the other hand, by Theorem 3.9, the process

$$\tilde{H}_{t\wedge\tau_U}^{f\circ\varphi(p)} := \chi_{\{B_t^p \notin ((n-2)-skeleton)\}}(f\circ\varphi)(B_{t\wedge\tau_U}^p) - (f\circ\varphi)(p) - \int_0^{t\wedge\tau_U} \chi_{\{B_s^p \notin ((n-2)-skeleton)\}} L(f\circ\varphi)(B_s^p)ds$$

is a continuous local martingale for any $p \in U \cap K \setminus ((n-2) - skeleton)$.

Consider now the process

$$\tilde{H}^{f\circ\varphi(p)} \circ C_{t\wedge\tau_U} \overset{denote}{=} \tilde{R}_{t\wedge\tau_U}^{f\circ\varphi(p)}.$$

$\tilde{R}_{t\wedge\tau_U}^{f\circ\varphi(p)}$ is obviously a continuous local martingale and it is also almost surely equal (using (17) and (19)) to the process

$$R_{t\wedge\tau_U}^{f\circ\varphi(p)} := f(X^{\varphi(p)} \circ C_{t\wedge\tau_U}) - f(\varphi(p)) - \frac{1}{2}\int_0^{t\wedge\tau_U} \Delta_N f(X^{\varphi(p)} \circ C_s)ds.$$

So $R^{f\circ\varphi(p)}_{t\wedge\tau_U}$ is also a continuous local martingale for any $p \in U \cap K \setminus ((n-2)-skeleton)$, which means, by definition, that $X^{\varphi(p)} \circ C_{s\wedge\tau_U}$ is a Brownian motion on N.

" $\Leftarrow$ " Conversely, suppose that for any $p \in K \setminus ((n-2)-skeleton)$, $(\varphi(B^p_t))_{t\geq 0}$ is a Brownian motion on N up to a change of time.

Let V be an open set of N such that $\varphi^{-1}(V) \subset U$, where U is taken as in the hypothesis of the theorem. Let $f : V \to \mathbb{R}$ be a local harmonic function on N.

Fix $p_0 \in U \setminus ((n-2)-skeleton)$ with $\varphi(p_0) \in V$ and τ denote the first exit time of $(B^{p_0}_t)_{t\geq 0}$ from $\varphi^{-1}(V)$.

By hypothesis, the process

$$(f(\varphi(B^{p_0}_{t\wedge\tau})))_{t\geq 0} = (f \circ \varphi(B^{p_0}_{t\wedge\tau}))_{t\geq 0}$$

is equal to $f({}^N B^{\varphi(p)} \circ A_{t\wedge\tau})$.

The latter process is a continuous local martingale (because, by definition the process $(f({}^N B^{\varphi(p)}_{t\wedge\tau}))_{t\geq 0}$ is a continuous local martingale and the martingale property is stable under change of time).

So we have shown that for every $p_0 \in U \setminus ((n-2)-skeleton)$ and for every (local) harmonic function on N, $(f \circ \varphi(B^{p_0}_{t\wedge\tau}))_{t\geq 0}$ is a continuous local martingale. By Corollary 3.10 this means that $f \circ \varphi$ is harmonic.

In other words, we have shown that φ pulls-back (local) harmonic function on N to (local) harmonic function on $K \setminus ((n-2)-skeleton)$. But we have already proved in the proof of the Corollary 3.10 that (local) harmonic function on $K \setminus ((n-2)-skeleton)$ are (local) harmonic on K.

We conclude that φ is a harmonic morphism. □

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