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MATHEMATICS DOCTORAL SCHOOL

Geometric properties of locally conformally symplectic manifolds

-Summary of PhD Thesis-

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1 Subject of the thesis

This thesis is concerned with studying the geometric properties of locally conformally symplectic (LCS for short) and, to a certain extend, of locally conformally Kähler (LCK) manifolds. It is based on [OS], [S1] and [S2].

LCS manifolds are quotients of symplectic manifolds by a discrete group of homotheties. Products of contact manifolds with S^1 are LCS ([V]), but there are many other examples. Equivalently, they are even-dimensional manifolds endowed with a non-degenerate two-form ω and a closed one-form θ (called Lee form) such that

$$d\omega = \theta \wedge \omega.$$

LCK manifolds intertwine the above definition with the existence of both a complex and Riemannian structures *i.e.* we ask ω above to be the two-form given by a Riemannian metric composed with the complex structure endomorphism.

There is a growing interest in LCS manifolds, concerning both their geometry and topology (*e.g.* [Ba], [CM] and the references therein). A strong motivation for studying LCS manifolds is a recent result by Eliashberg and Murphy [EM] proving that all compact almost complex manifolds with non-trivial integer one-cohomology do have LCS structures, and hence these manifolds are much more numerous than symplectic ones.

The broad goal of the thesis is to generalize well-known procedures and theorems in symplectic and Kähler geometry and bring them into the LCS/LCK setting, leading to a better understanding of the field and, perhaps, to new examples of LCS manifolds.

2 Contents of the thesis. Summary of original results

I will now explain the structure of my thesis and, doing so, give an outline of the original results therein.

2.1 The preliminaries

Chapter 2 serves as a collection of preliminary facts needed for the description and proof of the original results in the later chapters. This includes the definitions and basic facts about symplectic, Kähler, contact and Sasaki geometry, as well as a brief overview on LCS and LCK geometry. As an interesting and rich subclass in LCK geometry, I also look at Vaisman manifolds. I give special importance to the relations between all these different objects, relations which I will exploit to prove my own results.

An especially important part of Chapter 2 is Subsection 2.1.2, concerning symplectic reduction. The Marsden-Weinstein reduction for symplectic manifolds is a classic and well known procedure (see [AM, pp. 298-299], [Br, ch. 7]) by which, given a Poisson action of a Lie group on a symplectic manifold, one can produce other symplectic manifolds as quotients of level sets of the momentum mapping. Since its introduction, the general idea has been applied to many geometric structures (contact, Kähler etc.).

The symplectic reduction consists of the following steps:

- Using the Poisson character of the action, one constructs a momentum mapping from the symplectic manifold to the dual of the Lie algebra of the group. This will be equivariant with respect to the group action and to the coadjoint action.
- Factoring the level set of a regular value of the momentum mapping through the action of the stabilizer of this value. The restriction of symplectic form to the level set will project to a symplectic form on the quotient.

S. Haller and T. Rybicki ([HR1]) adapted this symplectic reduction to LCS manifolds, and their construction was used in [MTP] to obtain universal models for several types of LCS manifolds. The path chosen in [HR1] was to retain the idea of factoring by the group action itself; however, in the general case, this means the level sets of the momentum mapping no longer satisfy the hypotheses they impose.

Of significance is also Subsection 2.2.2, discussing contact reduction. At regular value zero, this procedure has been well established, independently, by various authors (see [A], [G1], [Lo]), for some time. Lerman and Willett [LW] studied the topological structure of contact quotients. Albert [A] in 1989 and Willett [W] in 2002 developed different reduction procedures for non-zero regular values; the first depends on the choice of a contact form for the contact structure, the latter imposes more conditions than in the symplectic analogue. They were later unified by Zambon and Zhu [ZZ] using groupoid actions.

2.2 The original results

In **Chapter 3**, based on a joint work with A. Otiman ([OS]), I state and prove the following result, generalizing in a natural way the classical Darboux-Weinstein theorem to the LCS setting:

Theorem 2.1: ([OS]) *Let M be a manifold, θ_0 and θ_1 closed 1-forms and ω_0, ω_1 2-forms on M such that $d_{\theta_i}\omega_i = 0$. Let $Q \subset M$ be a compact submanifold such that ω_0 and ω_1 are nondegenerate and equal on T_qM for all $q \in Q$, and $\theta_{0|TQ} = \theta_{1|TQ}$.*

Then there exist $\mathcal{N}_0, \mathcal{N}_1$ neighborhoods of Q and $\varphi : \mathcal{N}_0 \longrightarrow \mathcal{N}_1$ a diffeomorphism such that

$$\varphi^*\omega_1 \sim \omega_0 \quad \text{and} \quad \varphi_{|Q} = id.$$

where by " $\sim$ " we mean conformally equivalent.

The Darboux-Weinstein theorem in symplectic geometry has several important consequences, one of which is the fundamental Darboux Theorem. Another is a theorem due to Weinstein (see [We, Theorem 6.1, pages 338-339]) characterizing, up to diffeomorphism, any symplectic form around a compact Lagrangian submanifold. To show the natural quality of our theorem, we also proved a generalization of this fact to the LCS context in:

Theorem 2.2: ([OS]) *Let (M, ω) be an LCS manifold with Lee form θ and $Q \subset M$ a compact Lagrangian submanifold.*

Then there exists a neighborhood $\mathcal{M}$ of Q, a neighborhood $\mathcal{N}$ of the zero section in T^*Q and a diffeomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ such that

$$\varphi^*\omega_\theta = \omega,$$

where $\omega_\theta = d\eta - \theta \wedge \eta$ and η is the tautological one-form on the cotangent bundle T^*Q.

In **Chapter 4**, based on [S1], I adapt the symplectic reduction to LCS manifolds in a natural way by doing, in a sense, the opposite of the reduction defined by [HR1]: I factor the level sets of the momentum mapping, as in the symplectic case, but not through the group action, but along a foliation derived by "twisting" the group action. This amounts to the known symplectic reduction if the Lee form θ is zero.

I do not ask that the group action preserve the LCS form, only the LCS structure; however, if only the latter is true, certain additional conditions have to be imposed. Specifically, the main theorem states:

Theorem 2.3: ([S1]) *Let (M, ω, θ) be a connected LCS manifold and G a connected Lie group acting twisted Hamiltonian on it.*

Let μ be the momentum mapping and $\xi \in \mathfrak{g}^*$ a regular value. Denote by $\mathcal{F} = T\mu^{-1}(\xi) \cap (T\mu^{-1}(\xi))^\omega$. Assume that one of the following conditions is met:

- *The action of G preserves the LCS form ω.*
- *G is compact, $\xi \wedge \theta_x(X.) = 0$ for all $x \in \mu^{-1}(\xi)$ and there exists a function h on $\mu^{-1}(\xi)$ such that $\theta_{|\mathcal{F}} = dh$.*

If $N_\xi := \mu^{-1}(\xi)/_{\mathcal{F}}$ is a smooth manifold and $\pi : \mu^{-1}(\xi) \longrightarrow N_\xi$ is a submersion, then N_ξ has an LCS structure such that the LCS form ω_ξ satisfies

$$\pi^*\omega_\xi = e^f \omega_{|\mu^{-1}(\xi)}$$

for some $f \in C^\infty(\mu^{-1}(\xi))$.

Moreover, one can take $f = h$; in particular, $f = 0$ if the action preserves the LCS form.

I then prove that this reduction along a foliation can be expressed as a quotient of a group action of the universal covering of the stabilizer of the regular value ξ.

The reduction described above can produce a great number of LCS manifolds, by varying either the regular value or the LCS form in its conformal class. I show that in both cases and under certain conditions, the quotients produced are cobordant.

Most importantly, I also show that, with an additional hypothesis, our construction is compatible with the existence of a complex structure *i.e.* if the manifold M is LCK, the resulting reduced manifold is also LCK:

Theorem 2.4: ([S1]) *Let $(M, J, h, \omega, \theta)$ be a connected LCK manifold and G a connected Lie group whose action on M is twisted Hamiltonian and holomorphic.*

Let μ be the momentum mapping and $\xi \in \mathfrak{g}^*$ a regular value. Denote by $\mathcal{F} = T\mu^{-1}(\xi) \cap (T\mu^{-1}(\xi))^\omega$. Assume that one of the following conditions is met:

- *The action of G preserves the LCS form ω.*
- *G is compact, $\xi \wedge \theta_x(X.) = 0$ for all $x \in \mu^{-1}(\xi)$ and there exists a function η on $\mu^{-1}(\xi)$ such that $\theta_{|\mathcal{F}} = d\eta$.*

Assume further that $G_\xi = G$ and that the s-Lee field θ^ω is holomorphic.

If $N_\xi := {}^{\mu^{-1}(\xi)}\!/_{\mathcal{F}}$ *is a smooth manifold and* $\pi : \mu^{-1}(\xi) \longrightarrow N_\xi$ *is a submersion, then* N_ξ *has an LCK structure* $(J_\xi, h_\xi, \omega_\xi, \theta_\xi)$ *which satisfies*

$$\pi^*\omega_\xi = e^f \omega_{|\mu^{-1}(\xi)}$$

for some $f \in C^\infty(\mu^{-1}(\xi))$.

Moreover, one can take $f = \eta$*; in particular,* $f = 0$ *if the action preserves the LCK form.*

The most significant of these conditions is met if the manifold belongs to a large subclass of LCK manifolds called Vaisman manifolds. Thus I also looked at reduction for Vaisman manifolds and found that, with an additional hypothesis, the reduced space also becomes Vaisman.

Theorem 2.3 can be used in the contact context. Exploiting the relationship between contact and LCS manifolds, I derive (and get a new proof of) a contact reduction method that works for any regular value of the momentum mapping, which turns out to produce the same result as the one defined by Albert. Since we present this reduction as being naturally linked to LCS reduction, this seems to be the most natural of the existing methods for non-zero contact reduction. As a byproduct, this provides a wide class of examples for the reduction method. Specifically, the contact reduction method is described in:

Theorem 2.5: *Let* (C, α) *be a connected contact manifold and* G *a connected Lie group acting on* C *and preserving the contact form. Denote by* R *the Reeb field of* C.

Let μ_C *be the momentum mapping and* $\xi \in \mathfrak{g}^*$ *a regular value. Let* $\mathcal{F}_C$ *be the foliation*

$$(\mathcal{F}_C)_x = \{v \in T_x\mu^{-1}(\xi) \mid v = (X_a)_x - \xi(a)R_x \text{ for some } a \in \mathfrak{g}\},$$

where X_a *is the fundamental vector field corresponding to* $a \in \mathfrak{g}$.

If $C_\xi := {}^{\mu_C^{-1}(\xi)}\!/_{\mathcal{F}_C}$ *is a smooth manifold and* $\pi : \mu_C^{-1}(\xi) \longrightarrow C_\xi$ *is a submersion, then* C_ξ *has a natural contact structure such that the contact form* α_ξ *satisfies*

$$\pi^*\alpha_\xi = \alpha.$$

Moreover, $(S^1 \times C_\xi, d_\theta\alpha_\xi, \theta)$ *is the LCS reduction of* $S^1 \times C$ *with respect to the regular value* $-\xi$.

Even though this turns out to be the same constructions as Albert's, having the benefit of knowing its relationship with the LCK reduction of

Theorem 2.4, I also prove that, if the original contact manifold is Sasaki, the reduced space obtained *via* this method is also Sasaki.

The chapter then concludes with a few applications of this LCS reduction on a few classes of examples.

In **Chapter 5**, I turn to the class of LCS manifolds I must thoroughly studied the effects of LCS reduction on, namely the cotangent bundles.

The canonical symplectic structure of the cotangent bundle of any differentiable manifold has a kind of universality property with respect to reduction. Namely, if a group G acts on a manifold Q such that the quotient Q/G is a manifold, the action can be naturally lifted to a Hamiltonian action on the cotangent bundle T^*Q and the reduction at 0 of T^*Q is symplectomorphic with the cotangent bundle $T^*(Q/G)$. When performing reduction at a non-zero regular value of the momentum map, the symplectomorphism becomes a symplectic embedding (see [MMOPR]).

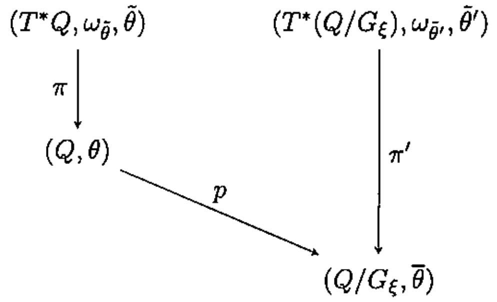
On the other hand, the cotangent bundle of a manifold has many LCS structures, given by choices of a closed 1-form on the manifold (this was first noticed by [HR2]). It is then natural to try to adapt the symplectic cotangent reduction theorems to the LCS setting. A useful observation is that if a Lie group acts on a manifold, determining a principal bundle, then the corresponding action on the cotangent bundle has the same momentum map considered with the symplectic structure, as well as with any LCS structure.

The main result of this chapter is that the cotangent bundle reduction theorem is true, in the same conditions, for the LCS structures of a cotangent bundle, reduced as in Theorem 2.3:

Theorem 2.6: ([S2]) *Let Q be a manifold with a free and proper action of a Lie group G and take $\overline{\theta} \in \Omega^1(Q/G)$ a closed 1-form.*

Denote by $\mu : T^*Q \longrightarrow \mathfrak{g}^*$ the corresponding momentum mapping and take $\xi \in \mathfrak{g}^*$.

We have the following diagram:



where $\theta = p^*\overline{\theta}$, $\tilde{\theta} = \pi^*\theta$, $\tilde{\theta}' = \pi'^*\overline{\theta}$ *and* $\omega_{\tilde{\theta}}$ *and* $\omega_{\tilde{\theta}'}$ *are the LCS forms on the respective cotangent bundles.*

Assume there exists $\alpha_\xi \in \Omega^1(Q)$ *such that* $(\alpha_\xi)_q \in \mu'^{-1}(\xi')$ *for all* $q \in Q$ *and which satisfies*

$$\mathcal{L}_{X_a}\alpha_\xi = \xi(a)\theta, \ \forall a \in \mathfrak{g}_\xi. \tag{2.1}$$

Then:

i) *There exists a* $d_{\overline{\theta}}$*-closed 2-form* β_ξ *on* Q/G_ξ *such that* $p^*\beta_\xi = d_\theta\alpha_\xi$*.*
Let $B_\xi = \pi'^*\beta_\xi$*.*

ii) *There is a canonical embedding of LCS manifolds*

$$\varphi : ((T^*Q)_\xi, \omega_\xi, \tilde{\theta}_\xi) \longrightarrow (T^*(Q/G_\xi), \omega_{\tilde{\theta}'} + B_\xi, \tilde{\theta}'),$$

(where $(T^*Q)_\xi$ *is the LCS reduction of* T^*Q *performed as in Theorem 2.3) whose image covers* Q/G_ξ*.*

This is an isomorphism if and only if $\mathfrak{g}_\xi = \mathfrak{g}$*.*

iii) *If, additionally,* $(\alpha_\xi)_q \in \mu^{-1}(\xi)$ *for all* $q \in Q$*, then*

$$\operatorname{Im}\varphi = \operatorname{Ann}(p_*\mathcal{O}),$$

where $\mathcal{O}$ *is the subbundle of* TQ *tangent to the orbits of* G*.*

In addition to the result itself, this shows the naturality of the reduction scheme introduced in Chapter 4.

The chapter then ends with a discussion on the hypotheses imposed and a concrete and computed example.

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