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OPTIMAL INEQUALITIES FOR CHEN INVARIANTS ON SUBMANIFOLDS

PH. D. THESIS SUMMARY

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CHAPTER 1

INTRODUCTION

The scope of this doctoral research is to find simple relationships between the main intrinsic invariants and the main extrinsic invariants of submanifolds, more specifically, to find inequalities involving δ-invariants for various submanifolds, like Lagrangian submanifolds in complex space forms, or Lagrangian, slant, quaternionic CR-submanifolds or QR-submanifolds in quaternionic space forms.

As a field of differential geometry, the theory of submanifolds is as old as differential geometry itself. It has been studied since the invention of calculus. It started with the theory of plane curves and surfaces and Fermat (1601-1665) is regarded as an innovator in this field [16]. In his time, differential geometry of plane curves was dealing with curvature, circles of curvature, evolutes, envelopes, etc. It has been developed since then as a significant part of calculus and has been extended to analogous studies of space curves and surfaces, particularly of lines of curvatures, geodesics on surfaces, and rotational and ruled surfaces [17].

Euler (1707-1783) was the first major contributor to the subject. He introduced in 1736 the notion of arclength and the radius of curvature. He initiated the study of the intrinsic differential geometry of submanifolds [17].

The intrinsic geometry of a surface S in $\mathbb{E}^3$ can be determined from the Euclidean inner product as applied to tangent vectors of S, as C.F. Gauss proved in 1827 in his general theory of curved surfaces. The Gauss's Theorema Egregium claim the invariance of the Gauss curvature under isometric deformations of surfaces which live in the Euclidean world. This conclusion is remarkable and it had a tremendous impact on the development of mathematics [16]. This theorem lead to the difference between the intrinsic and the extrinsic qualities of surfaces. One has two essential quantities for a surface in Euclidean 3-dimensional space: the mean curvature and the Gauss curvature. The mean curvature is the major extrinsic invariant and it measures the tension of the surface arisen from the ambient space [16].

In his famous inaugural lecture at Göttingen (1854), Riemann debated the foundation of the geometry. He introduced the notion of n-dimensional manifolds, formulated the concept of Riemannian manifolds and defined their curvature. Later, all of this provides the mathematical foundation of Einstein's Theory of General Relativity (1915). Further generalizations came out: the positiveness of the inner product was weakened to nondegeneracy, the notion of pseudo-Riemannian manifolds developed. Inspired by the string theory and Kaluza-Klein's theory, mathematicians and physicists started to study not only submanifolds of Riemannian manifolds but also of pseudo-Riemannian manifolds as well [17].

As suggested S.S. Chern in his 1970 talk at the international congress of mathematics at Nice, the Riemannian geometry forms the theory of modern differential geometry. Riemannian invariants are the intrinsic characteristics of the Riemannian manifolds. As M. Berger said [8]: *"Curvature invariants are* $N^{\circ}1$ *Riemannian invariants and the most natural. Gauss and then Riemann saw it instantly."* This Riemannian DNA disturb the behavior in general of the Riemannian manifold [16]. They have several interesting connections to many areas of mathematics. For example, they are giving rise to new obstructions to minimal and Lagrangian isometric immersions, or associating closely to the first nonzero eigenvalue of the Laplacian on a Riemannian manifold.

These curvature invariants play also an important task in physics. For instance, the motion of a body in a gravitational field is ruled by the curvature of the space time, according to Einstein. The curvature of space time is essential for the positioning of satellites into orbit around earth. The magnitude of a force needed to move an object at constant speed, according to Newton's laws, is a constant multiple of the curvature of the trajectory. From soap bubbles, waves surface and snails shell to red blood cells, all sorts of shapes seems to be governed by various curvature [46], [16].

The differential geometry of surfaces in Euclidean 3-dimensional spaces was generalized to the differential geometry of higher dimensional submanifolds of Riemannian manifolds. One of the most significant problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a space form or, more specifically, in a Euclidean space [13]. According to acclaimed J.F. Nash embedding theorem published in 1956 [43], every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension. This gives us the opportunity to consider the Riemannian manifolds as Riemannian submanifolds of some Euclidean spaces and, more important, to use the extrinsic help for derive new intrinsic results. This idea had not been unfolded as late as 1985. According to M. Gromov the main argument is due to the absence of controls of the extrinsic properties of the submanifolds by the known intrinsic invariants.

For studying embedding problems it is natural to demand some suitable conditions on the immersions, for example, requiring the minimality condition, which leads to the following problem:

Given a Riemannian manifold M*, what are the necessary conditions for* M *to admit a minimal isometric immersion in a Euclidean* $m-$*space* $\mathbb{E}^m$*?* [17]

For many years the only known necessary conditions for a general Riemannian manifold to admit a minimal isometric immersion in a Euclidean space regardless of codimension were the conditions that the Riemannian manifold to be non-compact and the Ricci tensor satisfies $\mathrm{Ric} \leq 0$. That is why S.S. Chern asked in his 1968 monograph to search for further Riemannian conditions for M to admit an isometric minimal immersion into a Euclidean space. Another principal problem in the theory of submanifolds is to establish simple relationship between the main intrinsic invariants and the main extrinsic invariants of the submanifolds [17]. No solutions to this problem and Chern's problem were known for many years before B.-Y. Chen introduced in the early 1990s new sorts of Riemannian invariants, the so-called δ-curvature invariants. Distinct from the classical Ricci and scalar curvature, both scalar and Ricci curvatures being "total sum" of sectional curvatures on a Riemannian manifold, the non-trivial δ-invariants are obtained from the scalar curvature by giving up to certain number of sectional curvatures [45]. Chen was also able to formulate general optimal relations between his new intrinsic δ-invariants and the main extrinsic invariants for Riemannian submanifolds. As an application, he was able to determine new intrinsic spectral properties of homogeneous spaces via Nash's theorem.

Let M^n be an n-dimensional Riemannian manifold and $K(\pi)$ the sectional curvature of M^n associated with a 2-plane section $\pi \subset T_pM^n$, $p \in M^n$.

For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM^n, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

We denote by

$$(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_pM^n, \dim \pi = 2\}.$$

The *Chen first invariant* is given by $\delta_M(p) = \tau(p) - (\inf K)(p)$.

If L is a subspace of T_pM^n of dimension $r \geq 2$ and $\{e_1, ..., e_r\}$ an orthonormal basis of L, the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta).$$

For given integers $n \geq 3$ and $k \geq 1$, we denote by $S(n,k)$ the finite set of all k-tuples $(n_1, ..., n_k)$ of integers satisfying $2 \leq n_1, ..., n_k < n$, $n_1 + ... + n_k \leq n$. Let $S(n) = \bigcup_{k \geq 1} S(n,k)$.

For each $(n_1, ..., n_k) \in S(n)$ and each point $p \in M^n$, B.-Y. Chen introduced a Riemannian invariant defined by

$$\delta(n_1, ..., n_k)(p) = \tau(p) - \inf\{\tau(L_1) + ... + \tau(L_k)\},$$

where $L_1, ..., L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_j = n_j$, $j = 1, ..., k$.

Many geometers studied the δ-invariants. For example, in [14] B.-Y. Chen established sharp inequalities for submanifolds in a complex space form. A. Oiagă and I. Mihai investigated in [44] δ-invariants for slant submanifolds in complex space forms. J.S. Kim, Y.M. Song and M.M. Tripathi studied them in generalized complex space forms ([28]) and later, P. Alegre, A. Carriazo, Y.H. Kim and D.W. Yoon ([1]) studied δ-invariants in both generalized complex space forms and generalized Sasakian space forms, with arbitrary codimension. K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C. Ozgur ([3], [4]) studied δ-invariants for submanifolds in locally conformal almost cosymplectic manifolds and (k,μ)-contact space forms.

The present thesis describes contributions to basic research on inequalities involving δ-invariants for Lagrangian submanifolds in complex space forms and Lagrangian, quaternionic CR-submanifolds and QR-submanifolds of a quaternionic space form.

CHAPTER 2

COMPLEX MANIFOLDS

2.1 Complex manifolds

In this section we recall fundamental notions on complex space forms, giving some examples and important results.

We denote by $\chi(\tilde{M})$ the set of all (differentiable) vector fields and by $T_p\tilde{M}$, $T_p^{\perp}\tilde{M}$ the tangent space, respectively the normal space of $\tilde{M}$ at $p \in \tilde{M}$.

We call an n-dimensional *complex manifold* any pair $(\tilde{M}, \mathcal{A})$, where $\tilde{M} \neq \emptyset$ is a paracompact Hausdorff space (generally assumed to be connected) and $\mathcal{A}$ is a family of functions with the following properties:

i) If $\phi_\alpha \in \mathcal{A}$, $\phi_\alpha : U_\alpha \to \tilde{M}$, then U_α is an open subset of $\mathbb{C}^n$ and ϕ_α is an one-to-one function (coordinates map).

ii) If $\phi_\alpha, \phi_\beta \in \mathcal{A}$, then $\phi_\alpha^{-1}(\mathrm{Im}\ \phi_\beta)$ is an open subset of $\mathbb{C}^n$ and the function $\phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(\mathrm{Im}\ \phi_\beta) \to \phi_\beta^{-1}(\mathrm{Im}\ \phi_\alpha)$ is holomorphic.

iii) $\{U_\alpha\}$ is an open covering of $\tilde{M}$.

iv) $(\tilde{M}, \mathcal{A})$ is maximal related to i), ii) and iii).

We call $\phi_\alpha^{-1}(z) = (z_1^\alpha, z_2^\alpha, \dots, z_n^\alpha)$ *local coordinates* for the point z, ϕ_α *charts* and $\mathcal{A}$ *atlas*.

Let $\tilde{M}$ be a differentiable manifold. An endomorphism J defined on the tangent bundle $T\tilde{M}$, $J_p : T_p\tilde{M} \to T_p\tilde{M}$ linear, with the property that $J_p^2 = -1_{T_p\tilde{M}}$, $\forall p \in \tilde{M}$, is called an *almost complex structure* on $\tilde{M}$. $\tilde{M}$ endowed with the complex structure J is called an *almost complex manifold*.

The following theorems are known.

Theorem 2.1: [41]. *Any almost complex manifold has an even dimension and is orientable.*

Theorem 2.2: [41]. *Any complex manifold admits a standard almost complex structure.*

If $(\tilde{M}, J)$ is an almost complex manifold, a Hermitian metric on $\tilde{M}$ is a J-invariant Riemannian metric g, i.e.,

$$g(JX, JY) = g(X, Y),\ \forall X, Y \in \Gamma T\tilde{M}.$$

It's easy to prove that

Theorem 2.3: [41]. *Every almost complex manifold admits a Hermitian metric.*

If $(\tilde{M}, J)$ is an almost complex manifold, a Hermitian metric g on $\tilde{M}$ defines a non-degenerate 2-form ω, by $\omega(X,Y) = g(JX,Y)$, $X, Y \in \Gamma T\tilde{M}$, called the *fundamental 2-form*. It is obvious that $\omega(JX, JY) = \omega(X, Y)$.

An (almost) complex manifold with a Hermitian metric is called *(almost) Kähler manifold* if the fundamental 2-form ω is closed, i.e. $d\omega = 0$.

2.2 Lagrangian submanifolds in complex space forms

Let $\tilde{M}^m$ be a complex m-dimensional Kähler manifold endowed with the standard almost complex structure J, $J_p : T_p\tilde{M} \to T_p\tilde{M}$, $p \in M$, and a Hermitian metric $\tilde{g}$, and $f : M^n \to \tilde{M}^m$ an isometric immersion of an n-dimensional manifold M^n into $\tilde{M}^m$.

If $X \in T_p\tilde{M}$, we can write $JX = PX + FX$, where $PX \in T_pM$, $FX \in T_p^{\perp}M$, $P : TM \to TM$, $F : TM \to T^{\perp}M$.

M is called a *complex submanifold* if $J(T_pM) = T_pM$ (i.e., $F = 0$).

The submanifold M^n is called a *totally real submanifold* if $J(T_pM^n) \subset T_p^{\perp}M^n$, $\forall p \in M^n$. A totally real submanifold of maximum dimension, i.e., $\dim_{\mathbb{R}} M^n = \dim_{\mathbb{C}} \tilde{M}^n = n$, is called a *Lagrangian submanifold*.

In [11], B.-Y. Chen introduced the notion of a slant submanifold. A *slant submanifold* is a submanifold M of a Kähler manifold $\tilde{M}$ such that, for any nonzero vector $X \in T_pM$, the angle $\theta(X)$ (called the *Wirtinger angle* of X) between JX and the tangent space T_pM is a constant (independent of the choice of the point $p \in M$ and the choice of the tangent vector $X \in T_pM$). The Wirtinger angle of a slant submanifold is called the *slant angle* of the slant submanifold.

It is obvious that complex and totally real submanifolds are in fact slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.

A slant submanifold is said to be *proper* if it is neither complex nor totally real.

If the canonical endomorphism P defined above is parallel, that is, $\nabla P = 0$, a proper slant submanifold is called *Kählerian slant*.

Let $\tilde{M}$ is a Kähler manifold and $M \subset \tilde{M}$ a submanifold.

We denote $\mathcal{H}_p = T_pM \bigcap J(T_pM) \subset T_pM$ the maximal holomorphic subspace. $\mathcal{H}$ is J-invariant, i.e. $J\mathcal{H}_p = \mathcal{H}_p$. M is called *generic submanifold* if $\dim \mathcal{H}_p$ is a constant.

If $\tilde{M}$ is a Kähler manifold and $M \subset \tilde{M}$ is a generic submanifold, then M is called *CR-submanifold* if $J(\mathcal{H}_p^{\perp}) \subset T_p^{\perp}M$.

If $\tilde{M}^m$ has constant holomorphic sectional curvature $4c$, then it is called a *complex space form* and it is denoted by $\tilde{M}^m(4c)$. Its Riemannian curvature tensor is given by

$$\tilde{R}(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ],$$

for any vector fields X, Y, Z tangent to $\tilde{M}^m(4c)$.

2.3 Chen invariants

The Riemannian invariants of a Riemannian manifold are the intrinsic characteristics of the Riemannian manifold. In this section, we recall a string of Riemannian invariants on a Riemannian manifold, which are known as Chen invariants (see [17]). The theory of Chen invariants was initiated in [12].

Let M^n be an n-dimensional Riemannian manifold and $K(\pi)$ the sectional curvature of M^n associated with a 2-plane section $\pi \subset T_pM^n$, $p \in M^n$.

For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM^n, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

We denote by

$$(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_pM^n, \dim \pi = 2\}.$$

The *Chen first invariant* is given by $\delta_M(p) = \tau(p) - (\inf K)(p)$.

If L is a subspace of T_pM^n of dimension $r \geq 2$ and $\{e_1, ..., e_r\}$ an orthonormal basis of L, the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta).$$

For given integers $n \geq 3$ and $k \geq 1$, we denote by $S(n, k)$ the finite set of all k-tuples $(n_1, ..., n_k)$ of integers satisfying $2 \leq n_1, ..., n_k < n$, $n_1 + ... + n_k \leq n$. Let $S(n) = \bigcup_{k \geq 1} S(n, k)$.

For each $(n_1, ..., n_k) \in S(n)$ and each point $p \in M^n$, B.-Y. Chen introduced a Riemannian invariant defined by

$$\delta(n_1, ..., n_k)(p) = \tau(p) - \inf\{\tau(L_1) + ... + \tau(L_k)\},$$

where $L_1, ..., L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_j = n_j$, $j = 1, ..., k$.

B.-Y. Chen proved some inequalities involving δ_M and $\delta(n_1, ..., n_k)$ for submanifolds in real space forms, which are known as Chen inequalities.

Theorem 2.4: [12]. *Let M^n be an n-dimensional ($n \geq 3$) submanifold of a real space form $\widetilde{M}^m(c)$ of constant sectional curvature c. Then*

$$\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + (n+1)c \right\}. \tag{2.1}$$

The equality case holds if and only if, with respect to suitable orthonormal frame fields $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, the shape operators $A_r = A_{e_r}$, $r = n+1, \ldots, m$, of M in $\widetilde{M}^m(c)$

take the following forms

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, a+b=\mu; \tag{2.2}$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \dots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, r = n+2, \dots, m. \tag{2.3}$$

B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken [20] showed that the same inequality holds for totally real submanifolds in complex space forms.

A corresponding inequality for slant submanifolds in complex space forms was obtained by A. Oiagă and I. Mihai.

Theorem 2.5: [44]. *Given an m-dimensional complex space form $\tilde{M}(4c)$ and a θ-slant submanifold M, $\dim M = n$, $n \geq 3$, we have*

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1+3\cos^2\theta)c\right\}. \tag{2.4}$$

The equality case holds if there exists an orthonormal basis on $\tilde{M}$ such that the shape operator of M takes the forms in (2.2) and (2.3).

However, for Lagrangian submanifolds in complex space forms the above inequality, known as Chen first inequality, was improved by J. Bolton, F. Dillen, J. Fastenakels and L. Vrancken [9]. Moreover, A. Mihai [39] improved the Chen first inequality for Kählerian slant submanifolds in complex space forms.

Theorem 2.6: [9]. *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}(c)$ of real dimension $2n$, $n \geq 3$. Then*

$$\delta_M \leq \frac{(n-2)(n+1)}{2}\frac{c}{4} + \frac{n^2}{2}\frac{2n-3}{2n+3}\|H\|^2. \tag{2.5}$$

The equality case holds if there exists a suitable orthonormal basis on $\tilde{M}$ such that the shape operator of M takes the forms in (2.2) and (2.3).

Theorem 2.7: [39]. *Let M be an n-dimensional ($n \geq 3$) purely real submanifold of an m-dimensional complex space form $\tilde{M}(4c)$, $p \in M$ and $\pi \subset T_pM$ a 2-plane section. Then*

$$\tau(p) - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + [(n+1)(n-2) + 3\|P\|^2 - 6\Phi^2(\pi)]\frac{c}{2}, \tag{2.6}$$

where $\Phi^2(\pi) = g^2(Je_1, e_2)$ and $\{e_1, e_2\}$ is an orthonormal basis of π. The equality case holds if there exists an orthonormal basis on $\tilde{M}$ such that the shape operator of M takes the forms in (2.2) and (2.3).

For each $(n_1, ..., n_k) \in S(n)$, we put:

$$d(n_1, ..., n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)},$$

$$b(n_1, ..., n_k) = \frac{1}{2}[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)].$$

The following sharp inequality involving the Chen invariants and the squared mean curvature obtained in [15] plays a very important role in this topic.

Theorem 2.8: *For each* $(n_1, ..., n_k) \in S(n)$ *and each* n*-dimensional submanifold* M^n *in a Riemannian space form* $\widetilde{M}^m(4c)$ *of constant sectional curvature* $4c$*, we have*

$$\delta(n_1, ..., n_k) \leq d(n_1, ..., n_k) \, \|H\|^2 + b(n_1, ..., n_k)c. \tag{2.7}$$

The equality case holds if and only if, with respect to suitable orthonormal frame fields $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$*, the shape operators* $A_r = A_{e_r}$*,* $r = n+1, \ldots, m$*, of* M *in* $\widetilde{M}^m(c)$ *take the following forms*

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}, \tag{2.8}$$

$$A_r = \begin{pmatrix} A_1^r & \ldots & 0 & 0 & \ldots & 0 & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \ldots & A_k^r & 0 & \ldots & 0 & \\ 0 & \ldots & 0 & \mu_r & \ldots & 0 & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & & \ldots & 0 & 0 & \ldots & \mu_r \end{pmatrix}, r = n+2, \ldots, m, \tag{2.9}$$

where $a_1, \ldots, a_n$ *satisfy the relations*

$$a_1 + \ldots + a_{n_1} = \ldots = a_{n_1+\ldots+n_{k-1}+1} + \ldots + a_{n_1+\ldots+n_k} = a_{n_1+\ldots+n_k+1} = \ldots = a_n$$

and every A_j^r *is a symmetric matrix of* $n_j \times n_j$ *type, satisying*

$$\mathrm{trace}(A_1^r) = \ldots = \mathrm{trace}(A_1^r) = \mu_r.$$

B.-Y. Chen also pointed-out that a similar inequality holds for totally real (in particular Lagrangian) submanifolds in a Kählerian space form.

2.4 An inequality for $\delta(2,2)$ for Lagrangian submanifolds in complex space forms

B.-Y. Chen at al. established the following inequalities for Chen invariants of Lagrangian submanifolds in complex space forms, which improve the inequality (2.7).

Theorem 2.9: [19]. *Let* M^n *be a Lagrangian submanifold of a complex space form* $\tilde{M}^n(4c)$. *Let* $n_1, n_2, \dots, n_k$ *be integers satisfying* $2 \le n_1 \le \dots \le n_k \le n-1$ *and* $n_1 + \dots n_k < n$. *Then, at any point of* M^n, *we have*

$$\delta(n_1, n_2, \dots, n_k) \le \frac{n^2\{n - \sum_{i=1}^k n_i + 3k - 1 - 6\sum_{i=1}^k (2+n_i)^{-1}\}}{2\{n - \sum_{i=1}^k n_i + 3k + 2 - 6\sum_{i=1}^k (2+n_i)^{-1}\}} \|H\|^2 + \tag{2.10}$$

$$+\frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_1 - 1)\right\} c.$$

Assume that equality holds at a point $p \in M^n$. *Then with the choice of basis and the notations introduced above, one has*

- $h^A_{BC} = 0$ *if* A, B, C *are mutually different and not all in the same* Δ_i, $i \in \{1, \dots, k\}$,
- $h^{\alpha_i}_{\alpha_j \alpha_j} = h^{\alpha_i}_{rr} = \sum_{\beta_i \in \Delta_i} h^{\alpha_i}_{\beta_i \beta_i} = 0$ *for* $i \ne j$,
- $h^r_{rr} = 3h^r_{ss} = (n_i + 2)h^r_{\alpha_i \alpha_i}$ *for* $r \ne s$,

where

$$\Delta_1 = \{1, \dots, n_1\},$$
$$\Delta_2 = \{n_1 + 1, \dots, n_1 + n_2\},$$
$$\vdots$$
$$\Delta_k = \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\},$$
$$\Delta_{k+1} = \{n_1 + \dots + n_k + 1, \dots, n\},$$

$A, B, C \in \{1, \dots, n\}$, $i, j \in \{1, \dots, k\}$, $\alpha_i, \beta_i \in \Delta_i$, $r, s \in \Delta_{k+1}$, $n_{k+1} = n - n_1 - \dots - n_k$.

Theorem 2.10: [19]. *Let* M^n *be a Lagrangian submanifold of a complex space form* $\tilde{M}^n(4c)$. *Let* $n_1, n_2, \dots, n_k$ *be integers satisfying* $2 \le n_1 \le \dots \le n_k \le n-1$ *and* $n_1 + \dots n_k = n$. *Then, at any point of* M^n, *we have*

$$\delta(n_1, n_2, \dots, n_k) \le \frac{n^2(k - 1 - 2\sum_{i=1}^k (2+n_i)^{-1})}{2(k - 2\sum_{i=1}^k (2+n_i)^{-1})} \|H\|^2 + \tag{2.11}$$

$$+\frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_i - 1)\right\} c.$$

Assume that equality holds at a point $p \in M^n$. *Then with the choice of basis and the notations introduced above, one has*

- $h^A_{\alpha_i\alpha_j} = 0$ *for* $i \neq j$ *and* $A \neq \alpha_i, \alpha_j$,
- *if* $n_j \neq \min\{n_1, \dots, n_k\}$:

$$h^{\beta_j}_{\alpha_i\alpha_i} = 0 \text{ if } i \neq j \text{ and } \textstyle\sum_{\alpha_j\in\Delta_j} h^{\beta_j}_{\alpha_i\alpha_i} = 0,$$

- *if* $n_j = \min\{n_1, \dots, n_k\}$:

$$\textstyle\sum_{\alpha_j\in\Delta_j} h^{\beta_j}_{\alpha_i\alpha_i} = (n_i+2)h^{\beta_j}_{\alpha_i\alpha_i} \text{ for any } i \neq j \text{ and any } \alpha_i \in \Delta_i.$$

By using the method of constrained maxima, we proved an improved B.-Y. Chen inequality for the invariant $\delta(2,2)$ of Lagrangian submanifolds in complex space forms in the next

Theorem 2.11: [37]. *Let* M^n *be a Lagrangian submanifold of a complex space form* $\tilde{M}^n(4c)$, $n \geq 4$. *Then we have*

$$\delta(2,2) \leq \frac{n^2}{2} \cdot \frac{n-2}{n+1}\|H\|^2 + \frac{1}{2}[n(n-1)-4]c. \tag{2.12}$$

The equality sign holds at a point $p \in M^n$ *if and only if there is an orthonormal basis* $\{e_1, e_2, \dots, e_n\}$ *at* p *such that with respect to this basis the second fundamental form* h *satisfies the following conditions*

$$h^C_{iA} = 0,\ A, C \in \{1, \dots, n\} \diagdown \{i\},\ A < C,\ i = \overline{1,3},$$

$$h^A_{BC} = 0,\ A = \overline{1,n},\ 4 \leq B < C \leq n,\ A \notin \{B, C\}.$$

CHAPTER 3

LAGRANGIAN SUBMANIFOLDS OF A QUATERNIONIC SPACE FORM

3.1 Lagrangian submanifolds of a quaternionic space form

In this section, we recall some basic definition of quaternionic space forms and Lagrangian submanifolds in a quaternionic space form, proving two inequalities involving the Riemannian invariants δ_M and $\delta(n_1, n_2, \ldots, n_k)$.

Let $\tilde{M}$ be a differentiable manifold and we assume that there is a rank 3-subbundle σ of $\mathrm{End}(T\tilde{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\mathrm{Id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{3.1}$$

where Id denotes the identity field of type $(1, 1)$ on $\tilde{M}$ and the indices are taken from $\{1, 2, 3\}$ modulo 3. The bundle σ is called an *almost quaternionic structure* on $\tilde{M}$ and $\{J_1, J_2, J_3\}$ is called a *canonical basis of* σ. $(\tilde{M}, \sigma)$ is said to be an *almost quaternionic manifold*. It's easy to see that any almost quaternionic manifold is of dimension $4m$, $m \geq 1$.

A Riemannian metric $\tilde{g}$ on $\tilde{M}$ is said to be *adapted to the almost quaternionic structure* σ if it satisfies

$$\tilde{g}(J_\alpha X, J_\alpha Y) = \tilde{g}(X, Y), \quad \forall \alpha \in \{1, 2, 3\}, \tag{3.2}$$

for all vector fields X, Y on $\tilde{M}$ and any canonical basis $\{J_1, J_2, J_3\}$ on σ. $(\tilde{M}, \sigma, \tilde{g})$ is said to be an *almost quaternionic Hermitian manifold*.

$(\tilde{M}, \sigma, \tilde{g})$ is said to be a *quaternionic Kähler manifold* [26] if the bundle σ is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$, i.e., there exist locally defined 1-forms ω_1, ω_2, ω_3 such that we have

$$\tilde{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}, \tag{3.3}$$

for all $\alpha \in \{1, 2, 3\}$ and for any vector field X on $\tilde{M}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Let $(\tilde{M}, \sigma, \tilde{g})$ be a quaternionic Kähler manifold and let X be a non-null vector on $\tilde{M}$. The 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a *quaternionic 4-plane* and is denoted by $Q(X)$. Any 2-plane in $Q(X)$ is called a *quaternionic plane*. The sectional curvature of a quaternionic plane is called a *quaternionic sectional curvature*. A quaternionic Kähler manifold is a *quaternionic space form* if its quaternionic sectional curvatures are equal to a constant, say c, i.e., its curvature tensor is given by

$$\tilde{R}(X,Y)Z = \frac{c}{4}\{\tilde{g}(Z,Y)X - \tilde{g}(X,Z)Y+ \tag{3.4}$$
$$+\sum_{\alpha=1}^{3}[\tilde{g}(Z,J_\alpha Y)J_\alpha X - \tilde{g}(Z,J_\alpha X)J_\alpha Y + 2\tilde{g}(X,J_\alpha y)J_\alpha Z]\Bigg\},$$

for all vector fields $X,\ Y,\ Z$ on $\tilde{M}$ and any local basis $\{J_1, J_2, J_3\}$ on σ.

Let $f : M \to \tilde{M}(c)$ an isometric immersion of an n-dimensional Riemannian manifold M in the $4n$-dimensional quaternionic space form $\tilde{M}(c)$. M is said to be a *Lagrangian submanifold* if

$$J_\alpha(T_pM) \subset T_p^\perp M, \forall p \in M, \forall \alpha \in \{1,2,3\}.$$

3.2 An improved Chen first inequality for Lagrangian submanifolds in quaternionic space forms

In this section we have proved a similar inequality to (2.5), for Lagrangian submanifolds of a quaternionic space form.

Theorem 3.1: [30]. *Let M be an n-dimensional Lagrangian submanifold of a quaternionic space form $\tilde{M}(c)$. Then, we have*

$$\delta_M \le \frac{(n-2)(n+1)}{2}\cdot\frac{c}{4} + \frac{n^2}{2}\cdot\frac{2n-3}{2n+3}\cdot ||H||^2. \tag{3.5}$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ at p such that with respect to this basis the second fundamental form h satisfies the conditions

$$h_{ij}^{\phi_r(k)} = 0\ ,\ i = \overline{1,n}\ ,\ j = \overline{3,n}\ ,\ i \neq j,\ k \in \{1,\dots,n\}\diagdown\{i,j\}.$$

3.3 Improved Chen inequalities for Lagrangian submanifolds of a quaternionic space form

Using the same method of constrained maxima, we also proved in the third section an inequality involving the Riemannian invariant $\delta(n_1, \dots, n_k)$ for Lagrangian submanifolds of a quaternionic space form, similar to the theorems 2.9 and 2.10.

Theorem 3.2: [30]. *Let M be an n-dimensional Lagrangian submanifold of a quaternionic space form $\tilde{M}(c)$. For a given k-tuple $(n_1, n_2, \dots, n_k) \in S(n)$, we put $N = n_1 + n_2 + \dots + n_k$ and $Q = \sum_{i=1}^{k}(2+n_i)^{-1}$. If $N < n$ then we have* *a) if* $Q \leq \frac{1}{3}$,

$$\delta(n_1, n_2, \ldots, n_k) \leq \frac{n^2\{n - N + 3k - 1 - 6Q\}}{2\{n - N + 3k + 2 - 6Q\}}\|H\|^2 + \tag{3.6}$$

$$+\frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_i - 1)\right\}\frac{c}{4}.$$

The equality sign holds at a point $p \in M$ *if and only if there is an orthonormal basis* $\{e_1, e_2, \ldots, e_n\}$ *at* p *such that with respect to this basis the second fundamental form* h *takes the following form*

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{r=1}^{3}\left(\sum_{\gamma_i \in \Delta_i} h_{\alpha_i\beta_i}^{\phi_r(\gamma_i)}\phi_r e_{\gamma_i} + \frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda\phi_r(e_{N+1})\right),\ \alpha_i, \beta_i \in \Delta_i,\ i = \overline{1,k},$$

$$h(e_{\alpha_i}, e_{\alpha_j}) = 0,\ \sum_{\alpha_i \in \Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(\gamma_i)} = 0,\ r = \overline{1,3},\ \alpha_i \in \Delta_i,\ \alpha_j \in \Delta_j,\ i \neq j,\ i,j \in \{1,2,\ldots,k\},$$

$$h(e_{\alpha_i}, e_{N+1}) = \frac{3\lambda}{2+n_i}\sum_{r=1}^{3}\phi_r(e_{\alpha_i}),\ h(e_{\alpha_i}, e_u) = 0,\ u \in \{N+2, \ldots, n\},$$

$$h(e_{N+1}, e_{N+1}) = 3\lambda\sum_{r=1}^{3}\phi_r(e_{N+1}),$$

$$h(e_{N+1}, e_u) = \lambda\sum_{r=1}^{3}\phi_r(e_u),\ u \in \{N+2, \ldots, n\},$$

$$h(e_u, e_v) = \lambda\delta_{uv}\sum_{r=1}^{3}\phi_r(e_{N+1}),\ u, v \in \{N+2, \ldots, n\},$$

for $\lambda = \frac{1}{3}h_{e_{N+1}e_{N+1}}^{N+1}$.

b) if $Q > \frac{1}{3}$,

$$\delta(n_1, n_2, \ldots, n_k) \leq \frac{n^2\{n - N + 3k - 3\}}{2\{n - N + 3k\}}\|H\|^2 + \tag{3.7}$$

$$+\frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_i - 1)\right\}\frac{c}{4}.$$

The equality sign holds at a point $p \in M$ *if and only if there is an orthonormal basis* $\{e_1, e_2, \ldots, e_n\}$ *at* p *such that with respect to this basis the second fundamental form* h *takes the following form*

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{r=1}^{3}\sum_{\gamma_i \in \Delta_i} h_{\alpha_i\beta_i}^{\phi_r(\gamma_i)}\phi_r(e_{\gamma_i}),$$

$$\sum_{r=1}^{3}\sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(\gamma_i)}\phi_r(e_{\gamma_i}) = 0,$$

$$h(e_A, e_B) = 0 \quad \textit{otherwise},$$

for $\alpha_i,\ \beta_i,\ \gamma_i \in \Delta_i,\ i = \overline{1,k},\ A,\ B,\ C = \overline{1,n}.$

CHAPTER 4

QUATERNIONIC $\mathcal{CR}$-SUBMANIFOLDS OF A QUATERNIONIC SPACE FORM

4.1 Quaternionic $\mathcal{CR}$-submanifolds of a quaternionic space form

In this section, we recall the notion of a CR-submanifold, the definition of quaternionic CR-submanifolds.

On 1978, A. Bejancu introduced the notion of CR-*submanifolds*, which is a generalization of holomorphic and totally real submanifolds in an almost Hermitian manifold [6].

Let $\tilde{M}$ be a Kähler manifold with complex structure J and let M be a Riemannian manifold isometrically immersed in $\tilde{M}$. One denotes by $\mathcal{D}_x$, $x \in M$ the maximal complex subspace $T_xM \cap J(T_xM)$ of the tangent space T_xM of M. If the dimension of $\mathcal{D}_x$ is constant for all $x \in M$, then $\mathcal{D} : x \to \mathcal{D}_x$ defines a *holomorphic distribution* $\mathcal{D}$ on M. A subspace ν of T_xM, $x \in M$ is called *totally real* if $J(\nu)$ is a subspace of the normal space $T_x^{\perp}M$ at x. If each tangent space of M is totally real, then M is called a *totally real submanifold* of the Kähler manifold $\tilde{M}$.

If there exists a totally real distribution $\mathcal{D}^{\perp}$ on M whose orthogonal complement is the holomorphic distribution $\mathcal{D}$, i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$, $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}M$, $x \in M$, then the submanifold M is called a CR-*submanifold*.

The totally real distribution $\mathcal{D}^{\perp}$ of every CR-submanifold of a Kähler manifold is an integrable distribution (see [10]).

4.2 An inequality for the $\mathcal{CR}$ δ-invariant

In the case of a CR-submanifold M of a Kähler manifold, Chen introduced a δ-invariant $\delta(\mathcal{D})$, called CR δ-*invariant*, defined by

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x),$$

where τ is the scalar curvature of M and $\tau(\mathcal{D})$ is the scalar curvature of the holomorphic distribution $\mathcal{D}$ of M.

In [2], Al-Solamy, Chen and Deshmukh proved an inequality involving the δ-invariant $\delta(\mathcal{D})$, in the case of an anti-holomorphic submanifold in a complex space form, in terms of squared mean curvature. In this chapter, we considered a quaternionic CR-submanifold in a quaternionic space form with minimal codimension.

Let $\tilde{M}$ be a quaternionic Kähler manifold and M be a real submanifold of $\tilde{M}$. A distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ is called a *quaternionic distribution* if $J_\alpha(\mathcal{D}) \subset \mathcal{D}$, $\forall \alpha = 1, 2, 3$, so $\mathcal{D}$ is carried into itself by the quaternionic structure.

M is called a *quaternionic CR-submanifold* if it admits a differential quaternionic distribution $\mathcal{D}$ such that its orthogonal complementary distribution $\mathcal{D}^\perp$ is totally real, i.e., $J_\alpha(\mathcal{D}_x^\perp) \subset T_x^\perp M,\ \alpha = 1, 2, 3,\ \forall x \in M$.

A submanifold M in a quaternionic manifold $\tilde{M}$ is called *quaternionic submanifold* (respectively, a *totally real submanifold*) if $\dim \mathcal{D}_x^\perp = 0$ (respectively, $\dim \mathcal{D}_x = 0$). A quaternionic CR-submanifold is called *proper* if it is neither totally real nor quaternionic.

Let $\mathcal{D}_{\alpha x} = J_\alpha(\mathcal{D}_x^\perp),\ \nu_x^\perp = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}$ a $3q$-dimensional distribution $\nu^\perp : x \to \nu_x^\perp$ globally defined on M, where $q = \dim \mathcal{D}_x^\perp$ and ν the orthogonal complementary distribution of $\nu^\perp$.

Then

$$T\tilde{M} = TM \oplus T^\perp M,\ TM = \mathcal{D} \oplus \mathcal{D}^\perp,$$

$$T^\perp M = \nu \oplus \nu^\perp,\ \nu, \nu^\perp \subset T^\perp M,\ \nu_x^\perp = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}.$$

M is called *mixed geodesic* if $h(X, Y) = 0,\ \forall X \in \Gamma(\mathcal{D}),\ Y \in \Gamma(\mathcal{D}^\perp)$.

M is called *mixed foliate* if $\mathcal{D}$ is integrable and M is mixed geodesic.

M is called *$\mathcal{D}$-geodesic* if $h(X, Y) = 0,\ \forall X, Y \in \Gamma(\mathcal{D})$.

M is called *$\mathcal{D}^\perp$-geodesic* if $h(X, Y) = 0,\ \forall X, Y \in \Gamma(\mathcal{D}^\perp)$.

If M is a quaternionic CR-submanifold of minimal codimension, i.e., $\dim \nu_x = 0$ for $x \in M$, we choose the following orthonormal basis:

$$\mathcal{D}_x = Sp\{\, e_1, \ldots, e_n\},$$

$$\mathcal{D}_x^\perp = Sp\{e_{n+1}, \ldots, e_{n+q}\},$$

and then

$$TM = Sp\{e_1, \ldots, e_n;\ e_{n+1}, \ldots, e_{n+q}\},$$

$$T^\perp M = Sp\{J_1 e_{n+1}, \ldots, J_1 e_{n+q};\ J_2 e_{n+1}, \ldots, J_2 e_{n+q};\ J_3 e_{n+1}, \ldots, J_3 e_{n+q}\},$$

which correspond to the definition of a quaternionic CR-submanifold given in [5].

In [2], the authors proved an inequality for $\delta(\mathcal{D})$ in case of an anti-holomorphic submanifold of a complex space form:

Theorem 4.1: [2]. *Let M be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \operatorname{rank}_{\mathbb{C}} \mathcal{D} \geq 1$ and $p = \operatorname{rank} \mathcal{D}^\perp \geq 2$. Then we have*

$$\delta(\mathcal{D}) \leq \frac{(2h+p)^2}{2}\|H\|^2 + \frac{p(4h+p-1)}{2}c - \frac{3p^2}{2(p+2)}\|H_{\mathcal{D}^\perp}\|^2. \tag{4.1}$$

The equality sign holds identically if and only if the following three conditions are satisfied:

(a) M is $\mathcal{D}$-minimal, i.e., $\overrightarrow{H}_{\mathcal{D}} = 0$,

(b) M is mixed totally geodesic, and

(c) there exists an orthonormal frame $\{e_{2h+1}, \ldots, e_n\}$ of $\mathcal{D}^\perp$ such that the second fundamental form σ of M satisfies

$\sigma^r_{rr} = 3\sigma^r_{ss}$, *for* $2h+1 \leq r \neq s \leq 2h+p$, *and*
$\sigma^t_{rs} = 0$ *for distinct* $r, s, t \in \{2h+1, \dots, 2h+p\}$.

In this chapter we prove a corresponding inequality for a quaternionic CR-submanifold with minimal codimension of a quaternionic space form, by using a different method, more precisely the method of constrained extrema.

Theorem 4.2: [35]. *If* M *is a quaternionic CR-submanifold of a quaternionic space form* $\tilde{M}$, *of minimal codimension, i.e.* $\dim \nu_x = 0$, *for* $x \in M$, $\dim \mathcal{D}_x = n$, $\dim \mathcal{D}^\perp_x = q$ *and* $\dim \nu^\perp_x = 3q = \dim T^\perp_x M$ *then*

$$\delta(\mathcal{D}) \leq \frac{(n+q)^2}{2} \cdot \frac{q+2}{q+5} \|H\|^2 + \frac{q(2q+n-1)}{2} \cdot \frac{c}{4}. \tag{4.2}$$

The equality sign holds at a point $x \in M$ *if and only if the following conditions are satisfied:*
a) M *is mixed totally geodesic;*
b) there is an orthonormal basis $\{e_1, e_2, \dots, e_{n+q}\}$ *at* x *such that with respect to this basis the second fundamental form* h *satisfies the following conditions:*

(i)

$$\sum_{i=1}^{n} \tilde{g}(h(e_i, e_i), J_\alpha e_r) = \tilde{g}(h(e_r, e_r), J_\alpha e_r) = 3\tilde{g}(h(e_s, e_s), J_\alpha e_r),$$

$\forall \alpha = \overline{1,3}$, $\forall r \neq s \in \{n+1, \dots, n+q\}$,

(ii)

$$\tilde{g}(h(e_r, e_s), J_\alpha e_t) = 0,$$

$\forall \alpha = \overline{1,3}$, $r, s, t = \overline{n+1, n+q}$, $r \neq s \neq t \neq r$.

CHAPTER 5

QR-SUBMANIFOLDS OF A QUATERNIONIC SPACE FORM

5.1 QR-submanifolds of a quaternionic space form

In this section, we recall the definitions of the CR δ-invariant $\delta(\mathcal{D}^{\perp})(x)$ and the QR-submanifolds.

In 1986, A. Bejancu [7] introduced the notion of QR-submanifolds as a generalization of real hypersurfaces of a quaternionic Kähler manifold (see also [48]).

Let $\tilde{M}$ be a quaternionic Kähler manifold and M be a real submanifold of $\tilde{M}$. M is called a *QR-submanifold* if there exists a vector subbundle ν of the normal bundle such that we have

$$J_\alpha(\nu_x) = \nu_x \text{ and } J_\alpha(\nu_x^{\perp}) \subset T_xM,\ x \in M,\ \alpha = \overline{1,3},$$

where $\nu^{\perp}$ is the complementary orthogonal bundle.

Taking into account the research done until now ([35],[36]), we remark that quaternionic CR-submanifolds and QR-submanifolds have very little in common. The differences between the quaternionic CR-submanifolds and QR-submanifolds in quaternionic space forms can be represented as follows.

Fig. 1: Quaternion CR-submanifolds

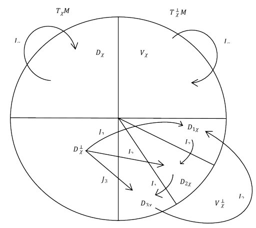
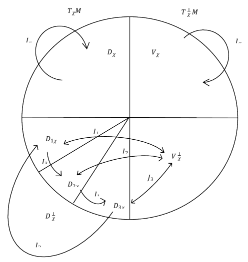


Fig. 2: QR-submanifolds



5.2 A new δ-invariant for a QR-submanifold of a quaternionic space form

In the case of a CR-submanifold M of a Kähler manifold, Chen also introduced the δ-invariant $\delta(\mathcal{D}^\perp)$, called CR δ-*invariant*, defined by Chen in [18]:

$$\delta(\mathcal{D}^\perp)(x) = \tau(x) - \tau(\mathcal{D}^\perp_x),$$

where τ is the scalar curvature of M and $\tau(\mathcal{D}^\perp)$ is the scalar curvature of the totally real distribution $\mathcal{D}^\perp$ of M.

In this section, we give a corresponding inequality to the inequality (4.1), for $\delta(\mathcal{D}^\perp)$ in the case of a QR-submanifold of a quaternionic space form with minimal codimension, i.e., $\dim \nu_x = 0$.

Let $\mathcal{D}_{\alpha x} = J_\alpha(\nu^\perp_x)$, $\mathcal{D}^\perp_x = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}$ a 3q-dimensional distribution $\mathcal{D}^\perp : x \to \mathcal{D}^\perp_x$ globally defined on M, where $q = \dim \nu^\perp_x$. One has

$$J_\alpha(\mathcal{D}_{\alpha x}) = \nu^\perp_x,\ J_\alpha(\mathcal{D}_{\beta x}) = \mathcal{D}_{\gamma x},\ \forall x \in M,$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

$\mathcal{D}$ is the orthogonal complementary distribution of $\mathcal{D}^\perp$ in TM and $J_\alpha(\mathcal{D}_x) = \mathcal{D}_x$.

$\mathcal{D}$ is called the *quaternionic distribution*.

So

$$T\tilde{M} = TM \oplus T^\perp M,\ TM = \mathcal{D} \oplus \mathcal{D}^\perp,$$

$$T^\perp M = \nu \oplus \nu^\perp,\ \nu, \nu^\perp \subset T^\perp M,\ \mathcal{D}^\perp_x = \mathcal{D}_{1x} \oplus \mathcal{D}_{2x} \oplus \mathcal{D}_{3x}.$$

For $Y \in \Gamma(TM)$ we consider the decomposition $J_\alpha Y = \Phi_\alpha Y + F_\alpha Y$, $\alpha = \overline{1,3}$; $\Phi_\alpha Y$, $F_\alpha Y$ are the tangential and normal components of $J_\alpha Y$, respectively.

For $V \in \Gamma(T^\perp M)$ we consider the decomposition $J_\alpha V = t_\alpha V + f_\alpha V$, $\alpha = \overline{1,3}$; $t_\alpha V$, $f_\alpha V$ are the tangential and normal components of $J_\alpha V$, respectively.

If $M \subset \tilde{M}$ is a QR-submanifold of minimal codimension, i.e., $\dim \nu_x = 0$ for $x \in M$, we consider the following orthonormal bases:

$$\{e_1, \ldots, e_n\} \subset \mathcal{D}_x;$$

$$\{J_1 e_{n+1}, \ldots, J_1 e_{n+q};\ J_2 e_{n+1}, \ldots, J_2 e_{n+q};\ J_3 e_{n+1}, \ldots, J_3 e_{n+q}\} \subset \mathcal{D}_x^\perp;$$

$$\{\ e_{n+1}, \ldots, e_{n+q}\} \subset T_x^\perp M.$$

The main result of this chapter is the following inequality involving $\delta(\mathcal{D}^\perp)$ in the case of a QR-submanifold of a quaternionic space form.

Theorem 5.1: [36]. *Let M be a QR-submanifold of minimal codimension of a quaternionic space form $\tilde{M}(c)$, $\dim \mathcal{D}_x = n$, $\dim \mathcal{D}_x^\perp = 3q$, $\dim \nu_x = 0$, $\dim \nu_x^\perp = q$, $x \in M$. Then we have:*

$$\delta(\mathcal{D}^\perp) \leq \frac{n(n+3q)^2}{2(n+1)} \cdot \|H\|^2 + \frac{n(n+6q+8)}{2} \cdot \frac{c}{4}. \tag{5.1}$$

The equality sign holds identically if and only if the following three conditions are satisfied:
(a) M is mixed totally geodesic,
(b) the distribution $\mathcal{D}$ is totally umbilical, and
(c) there exists an orthonormal frame

$$\{J_1 e_{n+1}, \ldots, J_1 e_{n+q};\ J_2 e_{n+1}, \ldots, J_2 e_{n+q};\ J_3 e_{n+1}, \ldots, J_3 e_{n+q}\}$$

of $\mathcal{D}_x^\perp$ such that the second fundamental form σ of M satisfies

$$h_{ij}^r = 0,\ i,j = \overline{1,n},\ i \neq j,\ r = \overline{n+1, n+q}.$$

CHAPTER 6

WINTGEN INEQUALITY

6.1 Wintgen inequality

In 1979, P. Wintgen ([49]) proved that the Gauss curvature K, the squared mean curvature $\|H\|^2$ and the normal curvature $K^{\perp}$ of any surface M^2 in E^4 satisfy the inequality

$$K \leq \|H\|^2 - K^{\perp}.$$

The equality holds if and only if the ellipse of curvature of M^2 in E^4 is a circle.

An extension of the Wintgen inequality was given later by B. Rouxel ([47]) and by I.V.Guadalupe and L.Rodriguez ([25]) independently for surfaces M^2 of arbitrary codimension m in real space forms $\tilde{M}^{2+m}(c)$

$$K \leq \|H\|^2 - K^{\perp} + c.$$

In 2004, A. Mihai ([40]) found a corresponding inequality for totally real surfaces in n-dimensional complex space forms. Also, the equality case was studied and the author gived a non-trivial example of a totally real surface satisfying the equality case.

The conjecture on Wintgen inequality, which is also known as the *DDVV conjecture*, was formulated in 1999 by P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken ([22]).

Conjecture. *Let $f : M^n \to \tilde{M}^{n+m}$ be an isometric immersion, where $\tilde{M}^{n+m}$ is a real space form of constant sectional curvature c. Then*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c,$$

where ρ is the normalized scalar curvature and $\rho^{\perp}$ is the normalized normal scalar curvature.

Denoting by K and $R^{\perp}$ the sectional curvature function and the normal curvature tensor on M^n, respectively, the normalized scalar curvature and the normalized normal scalar curvature are given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\le i<j\le n} \sum_{1\le\alpha<\beta\le n} \left(R^{\perp}(e_i, e_j, \xi_\alpha, \xi_\beta)\right)^2},$$

where τ is the scalar curvature.

This conjecture was proven by the authors for submanifolds M^n of arbitrary dimension $n \geq 2$ and codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of constant sectional curvature c and a detailed characterization of the equality case in terms of the shape operators of M^n in $\tilde{M}^{n+2}(c)$ was given.

T.Choi and Z.Lu ([21]) proved that this conjecture is true for all 3-dimensional submanifolds M^3 of arbitrary codimension $m \geq 2$ in $\tilde{M}^{3+m}(c)$ and give also a characterization for the equality case.

Other extensions of the Wintgen inequality for invariant submanifolds in Kähler, nearly Kähler and Sasakian spaces have been studied by P.J. De Smet, F. Dillen, J. Fastenakels, A. Mihai, J.Van der Veken, L. Verstraelen and L. Vrancken.

Recently, Z. Lu and independently J. Ge and Z. Tang finally settled the general case of the DDVV-conjecture.

Theorem 6.1: [29]. *The Wintgen inequality*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c,$$

holds for every submanifold M^n *in any real space form* $\tilde{M}^{n+m}(c)$, $n \geq 2$, $m \geq 2$.

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i\}$ *and* $\{\xi_\alpha\}$, *the shape operators of* M^n *in* $\tilde{M}^{n+m}(c)$ *take the forms*

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \dots & 0 \\ \mu & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 - \mu & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix},$$

$A_{\xi_4} = \cdots = A_{\xi_m} = 0$, *where* $\lambda_1, \lambda_2, \lambda_3$ *and* μ *are real functions on* M^n.

6.2 Generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms

I. Mihai proved the following generalized Wintgen inequality for Lagrangian submanifolds in complex space forms.

Theorem 6.2: [42]. *Let M^n a Lagrangian submanifold in a complex space form $\tilde{M}^m(4c)$. Then*

$$(\rho^{\perp})^2 \leq (\|H\|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - c)c + \frac{2c^2}{n(n-1)}.$$

In this section, we give a similar result for Lagrangian submanifolds of quaternionic space forms.

Let M^n be an n-dimensional totally real submanifold of a $4m$-dimensional quaternionic space form $\tilde{M}^{4m}(4c)$ and $\{e_1, \dots, e_n\}$ an orthonormal frame on M^n and $\{\xi_{n+1}, \dots, \xi_{4m}\}$ an orthonormal frame in the normal bundle $T^{\perp}M^n$, respectively.

The *scalar normal curvature* of M^n is defined by

$$K_N = \frac{1}{4}\sum_{r,s=n+1}^{4m} \mathrm{Trace}[A_r, A_s]^2. \tag{6.1}$$

Then the normalized scalar normal curvature is given by $\rho_N = \dfrac{2\sqrt{K_N}}{n(n-1)}$.

From (6.1) we get

$$K_N = \frac{1}{2}\sum_{1\leq r<s\leq 4m-n} \mathrm{Trace}[A_r, A_s]^2 = \sum_{1\leq r<s\leq 4m-n}\sum_{1\leq i<j\leq n}\big(g([A_r, A_s]e_i, e_j)\big)^2 \tag{6.2}$$

Denoting by $h_{ij}^r = g(h(e_i, e_j), \xi_r)$, $i, j = \overline{1, n}$, $r = \overline{1, 4m-n}$, we have

$$K_N = \sum_{1\leq r<s\leq 4m-n}\sum_{1\leq i<j\leq n}\left(\sum_{k=1}^{n}(h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s)\right)^2. \tag{6.3}$$

One of the main results of this chapter is the following

Theorem 6.3: [31]. *Let M^n be a Lagrangian submanifold of a quaternionic space form $\tilde{M}^{4n}(4c)$. Then*

$$\left(\rho^{\perp}\right)^2 \leq \left(\|H\|^2 - \rho + c\right)^2 + \frac{6}{n(n-1)}c^2 + \frac{4}{n(n-1)}c(\rho - c). \tag{6.4}$$

We prove this theorem using the following lemma, also proved here

Lemma 6.1: [31]. *Let M^n be a totally real submanifold of an $4m$-dimensional quaternionic space form $\tilde{M}^{4m}(4c)$. Then we have*

$$\|H\|^2 - \rho_N \geq \rho - c. \tag{6.5}$$

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i\}$ *and* $\{\xi_\alpha\}$*, the shape operators of* M^n *in* $\tilde{M}^{4m}(4c)$ *take the forms*

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \dots & 0 \\ \mu & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 - \mu & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix},$$

$A_{\xi_4} = \dots = A_{\xi_{4m-n}} = 0$, *where* $\lambda_1, \lambda_2, \lambda_3$ *and* μ *are real functions on* M^n.

6.3 Slant submanifolds in quaternionic space forms

We also give a similar result concerning the θ-slant submanifolds of a quaternionic space form.

We recall the definition of a slant submanifold in a quaternionic space form.

Definition 6.1: A submanifold M of a quaternionic Kähler manifold $(\tilde{M}, \sigma, \tilde{g})$ is said to be a *slant submanifold* if for each non-zero vector X tangent to M at p, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1, 2, 3\}$ is constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_pM$.

Theorem 6.4: [31]. *Let* M^n *be an* n*-dimensional* θ*-slant submanifold of an* $4m$*-dimensional quaternionic space form* $\tilde{M}^{4m}(4c)$*. Then, we have*

$$\|H\|^2 \geq \rho + \rho_N - c - \frac{9c}{n-1}\cos^2\theta. \tag{6.6}$$

The results mentioned in this thesis and others on related subjects were already published or presented in mathematical journals or conferences.

1. Macsim, G., *Improved Chen's inequalities for Lagrangian submanifolds in quaternionic space forms*, Romanian J. Math. Comp. Sci. **6** (2016), 61-84.

2. Macsim, G. and Mihai, A., *An inequality on quaternionic CR-submanifolds*, Ann. Univ. Ovidius Constanţa **26**(3) (2018), 181-196.

3. Macsim, G. and Mihai, A., *A $\delta-$invariant for QR-submanifolds in quaternion space forms*, Int. Electron. J. Geom. **11**(2) (2018), 8-17.

4. Macsim, G., Mihai, A. and Olteanu A., *On rectifying-type curves in a Myller configuration*, Bull. Korean Math. Soc. **56**(2) (2019), 383-390.

5. Macsim, G. and Ghişoiu, V., *Generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms*, va apărea în Math. Inequal. Appl.

6. Macsim, G. and Mihai, A., *A constrained maxima method for improving certain Chen inequalities*, submitted.

7. Macsim, G. and Mihai, A., *An optimization method for improved Chen inequalities*, Proceedings of Mathematics and Educational Symposium of Department of Mathematics and Computer Sciences, Technical University of Civil Engineering Bucharest, The 2nd edition, May 28, 2016, 75-80.

8. Macsim, G. and Mihai, A., *A CR $\delta-$ invariant for quaternionic CR-submanifolds in quaternionic space forms*, The 14th Workshop of Scientific Communications, Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, May 27, 2017, 66-71.

9. Macsim, G. and Ghişoiu, V., *Generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms*, The 14-th International Workshop on Differential Geometry and its Applications, Petroleum-Gas University of Ploiesti (UPG), Romania, July, 9th - 11th, 2019.

10. Mihai, A., Macsim, G. and Olteanu, A., *Curves in a Myller configuration*, International Conference on Applied and Pure Mathematics (ICAPM 2017), Iasi, November 2-5 2017.

11. Macsim, G. and Ghişoiu, V., *Optimal inequalities involving Casorati curvature for Lagrangian submanifolds in quaternionic space forms*, submitted.

12. Ghişoiu, G., Ghişoiu, V. and Macsim, G., *Generalized Wintgen inequality for Lagrangian submanifolds in generalized complex space forms*, submitted.

13. Ghişoiu, V. and Macsim, G., *Generalized Wintgen inequality for submanifolds in generalized Sasaki space forms*, submitted.

BIBLIOGRAPHY

[1] Alegre, P., Carriazo, A. Kim, Y.H. and Yoon, D.W., *B.-Y. Chen's inequality for submanifolds of generalized space forms*, Indian J. Pure Appl. Math. **38** (2007), 185-201.

[2] Al-Solamy, F., Chen, B.Y. and Deshmukh, S., *Erratum to: Two optimal inequalities for anti-holomorphic submanifolds and their applications*, Taiwanese J. Math. **22(3)** (2018), 615-616.

[3] Arslan, K., Ezentas, R., Mihai, I., Murathan, C. and Ozgur, C., *B.Y. Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds*, Bull. Institute Math. Acad. Sinica **29** (2001), 231-242.

[4] Arslan, K., Ezentas, R., Mihai, I., Murathan, C. and Ozgur, C., *Certain inequalities for submanifolds in $(k, \mu)-$contact space forms*, Bull. Austral. Math. Soc. **64** (2001), 201-212.

[5] Barros, M., Chen, B.-Y. and Urbano, F., *Quaternion CR-submanifolds of quaternion manifolds*, Kodai Math. J. **4(3)** (1981), 399-417.

[6] Bejancu, A., *CR-submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc. **69** (1978), 134-142.

[7] Bejancu, A., *QR-submanifolds of quaternion Kaehler manifolds*, Chinese J. Math. **14** (1986), no. 2, 81-94.

[8] Berger, M., *La geometrie metrique des varietes Riemaniennes, Elie Cartan et les Mathematiques d'Aujourd'Hui*, Asterisque, (1985), 9-66.

[9] Bolton, J., Dillen, F., Fastenakels, J., Vrancken, L., *A best possible inequality for curvature-like tensor fields*, Math. Inequal. Appl. **12** (2009), 663-681.

[10] Chen, B.-Y., *CR-submanifolds of a Kaehler manifold. I, II*, J. Differential Geom. **16** (1981), 305-323; J. Differential Geom. **16** (1981), 493-509.

[11] Chen, B.-Y., *Slant immersions*, Bull. Austral. Math. Soc. **41** (1990), 135-147.

[12] Chen, B.-Y., *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60** (1993), 568-578.

[13] Chen, B.-Y., *A Riemannian invariant and its applications to submanifold theory*, Results Math. **27** (1995), 17-28.

[14] Chen, B.-Y., *A general inequality for submanifolds in complex space forms and its applications*, Arch. Math. J. **67** (1996), 519-528.

[15] Chen, B.-Y., *Some new obstructions to minimal and Lagrangian isometric immersions*, Japan. J. Math. (N.S.) **26** (2000), 105-127.

[16] Chen, B.-Y., *Riemannian DNA, Inequalities and Their Applications*, Tamkang Journal of Science and Engineering **Vol. 3, No. 3** (2000), 123-130.

[17] Chen, B.-Y., *δ-Invariants, Inequalities of Submanifolds and their Applications*, in *Topics in Differential Geometry*, A. Mihai, I. Mihai, R. Miron (Eds.), Editura Academiei Române, Bucharest, 2008.

[18] Chen, B.-Y., *An optimal inequality for CR-warped products in complex space forms involving CR δ-invariant*, Internat. J. Math. **23(3)** (2012), 1250045 (17 pages).

[19] Chen, B.-Y., Dillen, F., Van der Veken, J., Vrancken, L., *Curvature inequalities for Lagrangian submanifolds: the final solution*, Differential Geom. Appl. **31**(6) (2013), 808-819.

[20] Chen, B.-Y., Dillen, F., Verstraelen L. and Vrancken, L., *Totally real submanifolds of* CP^n *satisfying a basic equality*, Arch. Math. **63** (1994), 553-564.

[21] Choi, T. and Lu, Z. *On the DDVV conjecture and the comass in calibrated geometry I*, Math. Z. **260** (2008), 409-429.

[22] De Smet, P.J., Dillen, F., Verstraelen, L. and Vrancken, L., *A pointwise inequality in submanifold theory*, Arch. Math. (Brno) **35** (1999), 115-128.

[23] **Ghişoiu, G., Ghişoiu, V. and Macsim, G.**, *Generalized Wintgen inequality for Lagrangian submanifolds in generalized complex space forms*, trimis spre publicare.

[24] **Ghişoiu, V. and Macsim, G.**, *Generalized Wintgen inequality for submanifolds in generalized Sasaki space forms*, trimis spre publicare.

[25] Guadalupe, I.V. and Rodriguez, L. *Normal curvature of surfaces in space forms*, Pacific J. Math. **106** (1983), 95-103.

[26] Ishihara, S., *Quaternion Kaehlerian manifolds*, J. Differ. Geom. **9** (1974), 483-500.

[27] Kim, J.S., Song, Y.M. and Tripathi, M.M., *B.-Y. Chen's inequalities for submanifolds in generalized complex space forms*, Far East J. Math. Sci. **3**(4) (2001), 645-655.

[28] Kim, J.K., Song, Y.M. and Tripathi, M.M., *B.-Y. Chen inequalities for submanifolds in generalized complex space forms*, Bull. Korean Math. Soc. **40** (2003), 411-423.

[29] Lu, Z., *Normal scalar curvature conjecture and its applications*, J. Funct. Anal. **261** (2011), 1284-1308.

[30] **Macsim, G.**, *Improved Chen's inequalities for Lagrangian submanifolds in quaternionic space forms*, Romanian J. Math. Comp. Sci. **6** (2016), 61-84.

[31] **Macsim, G. and Ghişoiu, V.**, *Generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms*, to appear in Math. Inequal. Appl.

[32] **Macsim, G. and Ghişoiu, V.**, *Optimal inequalities involving Casorati curvature for Lagrangian submanifolds in quaternionic space forms*, trimis spre publicare.

[33] **Macsim, G. and Mihai, A.**, *An optimization method for improved Chen inequalities*, Proceedings of Mathematics and Educational Symposium of Department of Mathematics and Computer Sciences, Technical University of Civil Engineering Bucharest, The 2nd edition, May 28, 2016, 75-80.

[34] **Macsim, G. and Mihai, A.**, *A CR $\delta-$ invariant for quaternionic CR-submanifolds in quaternionic space forms*, The 14th Workshop of Scientific Communications, Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, May 27, 2017, 66-71.

[35] **Macsim, G. and Mihai, A.**, *An inequality on quaternionic CR-submanifolds*, Ann. Univ. Ovidius Constanţa **26**(3) (2018), 181-196.

[36] **Macsim, G. and Mihai, A.**, *A $\delta-$invariant for QR-submanifolds in quaternion space forms*, Int. Electron. J. Geom. **11**(2) (2018), 8-17.

[37] **Macsim, G. and Mihai, A.**, *A constrained maxima method for improving certain Chen inequalities*, submitted.

[38] **Macsim, G., Mihai, A. and Olteanu A.**, *On rectifying-type curves in a Myller configuration*, Bull. Korean Math. Soc. **56**(2) (2019), 383-390.

[39] Mihai, A., *Geometric inequalities for purely real submanifolds in complex space forms*, Results Math. **55** (2009), no. 3-4, 457-468.

[40] Mihai, A., *An inequality for totally real surfaces in complex space forms*, Kragujevac J. Math. **26** (2004), 83-88.

[41] Mihai, I., *Geometria Subvarietăţilor în Varietăţi Complexe*, Editura Universităţii Bucureşti, 2001.

[42] Mihai, I., *On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms*, Nonlinear Anal. **95** (2014), 714-720.

[43] Nash, J.F., *The imbedding problem for Riemannian manifolds*, Ann. Math. **63** (1956), 20-63.

[44] Oiagă, A. and Mihai, I., *B. Y. Chen inequalities for slant submanifolds in complex space forms*, Demonstratio Math. **32** (1999), 835-846.

[45] Olteanu, A., *Recent results in the geometry of warped product submanifolds*, Matrix Rom, Bucureşti, (2011).

[46] Osserman, R., *Curvature in the eighties*, Amer. Math. Monthly, Vol. 97 (1990), 731-754.

[47] Rouxel, B., *Sur une famille des A$-$surfaces d'un espace Euclidien $\mathbb{E}^4$*, Ősterreischer Mathematiker Kongress, Insbruck, 1981, p.185.

[48] Şahin, B., *On QR-submanifolds of a quaternionic space forms*, Turkish J. Math. **25** (2001), 413-425.

[49] Wintgen, P., *Sur l'inégalité de Chen-Willmore*, C.R.Acad.Sci., Paris Sér.A-B **288** (1979), A993-A995.