University of Bucharest
Faculty of Mathematics and Computer Science
Doctoral School of Mathematics

Summary of PhD. Thesis

Contributions to the study of multigraded algebras associated with some combinatorial objects

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Overview of the thesis and outline of the original results

The core of this thesis is built on three papers, [1], [2] and [3]. The main topic consists in associating multigraded objects of commutative algebra, namely rings and ideals, with combinatorial objects such as simplicial complexes and graphs.

In the first part of this thesis, we focus on the study of some classes of toric ideals, namely convex polyomino ideals. A polyomino $\mathcal{P}$ is a finite connected set of adjacent cells in the cartesian plane $\mathbb{N}^2$, where a cell in $\mathbb{N}^2$ means a unitary square. A connection of polyominoes to commutative algebra first appeared in [45]. In that paper, to each polyomino it is assigned its ideal of 2-inner minors. There it was shown that if $\mathcal{P}$ is a convex polyomino, then the quotient ring modulo this ideal, called the coordinate ring of $\mathcal{P}$, is a normal Cohen-Macaulay domain. This was proved by viewing the coordinate ring of $\mathcal{P}$ as the edge ring of a suitable bipartite graph $G_{\mathcal{P}}$ associated with $\mathcal{P}$.

We follow this research direction and we contribute to the study of the coordinate ring of a convex polyomino. More precisely, we classify all convex polyominoes whose coordinate rings are Gorenstein. We compute the Castelnuovo-Mumford regularity of the coordinate ring of any stack polyomino in terms of the smallest interval which contains its vertices and we give a recursive formula for computing the multiplicity of the coordinate ring of a stack polyomino.

In the second part of this thesis, we study square-free monomial ideals, namely t-spread monomial ideals which have been recently introduced in [22]. In that paper, V. Ene, J. Herzog and A. Qureshi proved that every t-spread strongly stable ideal is componentwise linear. They also gave formulas for graded Betti numbers and height, and they computed the generic initial ideal of a t-spread strongly stable ideal. In this research direction, we have the last two papers. More precisely, in [3], we introduce the f_t-vector of a t-spread ideal and a new t-operator which is involved in the proof of Kruskal-Katona Theorem for t-spread strongly stable ideals. We show that any t-spread strongly stable ideal has a unique t-spread lex ideal with the same f_t-vector. The main theorem in [3] gives a complete classification of the sequences of positive integers which are the f_t-vectors of some t-spread strongly stable ideals.

In paper [2], we obtain the sequential Cohen-Macaulay and the strong persistence properties of a t-spread principal Borel ideal. We also prove that a t-spread principal Borel ideal satisfies the ℓ-exchange property with respect to a certain sorting order. That implies that a t-spread principal Borel ideal satisfies an x-condition which guarantees that all the powers of such ideals have linear quotients. Finally, we characterize the limit behavior of the depth for the powers of t-spread principal Borel ideals.

The first chapter starts with a brief description of simplicial complexes. We recall some methods to classify Cohen-Macaulay simplicial complexes and we pay attention to some properties of Stanley-Reisner ring by studying Alexander duality. Next, we recall some basic

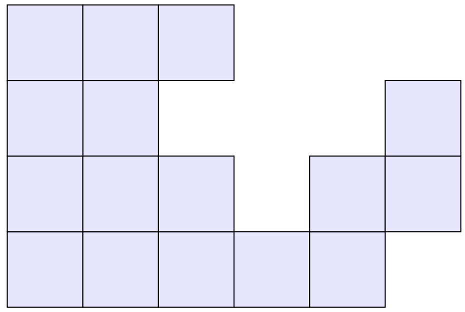


Figure 1: A polyomino

definitions and known facts about toric rings. Finally, we look at the behaviour of the set of the associated prime ideals of the powers of a monomial ideal. These properties will be needed in the following chapters of the thesis.

2. Properties of the coordinate ring of a convex polyomino. Main results

In this chapter, we classify all convex polyomino whose coordinate rings are Gorenstein. We also compute the Castelnuovo-Mumford regularity of the coordinate ring of any stack polyomino in terms of the smallest interval which contains its vertices. Finally, we give a recursive formula for computing the multiplicity of the coordinate ring of a stack polyomino.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. Two cells A and B of $\mathcal{P}$ are *connected*, if there is a sequence of cells of $\mathcal{P}$ given by $A = A_1, A_2, \ldots, A_{n-1}, A_n = B$ such that $A_i \cap A_{i+1}$ is an edge of A_i and A_{i+1} for each $i \in \{1, \ldots, n-1\}$. Such a sequence is called a *path* connecting the cells A and B.

Definition 1 *[45] A collection of cells $\mathcal{P}$ is called a polyomino if any two cells of $\mathcal{P}$ are connected.*

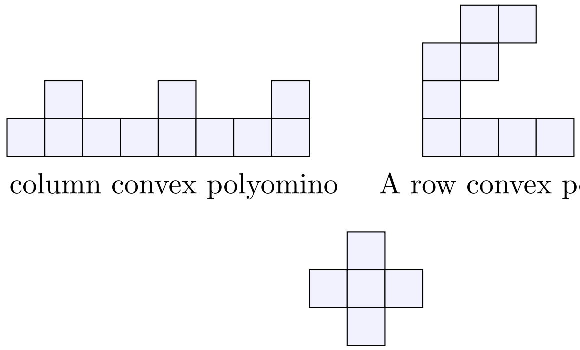
Definition 2 *[45] A polyomino $\mathcal{P}$ is called* row *(respectively* column*)* convex*, if for any two cells A and B of $\mathcal{P}$ with left lower corners $a = (i,j)$ and $b = (k,j)$ (respectively $a = (i,j)$ and $b = (i,l)$), the horizontal (respectively vertical) cell interval $[A,B]$ is contained in $\mathcal{P}$.*

If $\mathcal{P}$ is row and column convex, then $\mathcal{P}$ is called a convex polyomino.

In Figure 2, we give an example of column (row) convex polyomino which is not row (column) convex polyomino. The third polyomino of this figure is a convex polyomino.

Let $\mathcal{P}$ be a convex polyomino. After a possible translation, we consider $[(1,1),(m,n)]$ to be the smallest interval which contains the vertices of $\mathcal{P}$. In this case, we say that $\mathcal{P}$ is a convex polyomino on $[m] \times [n]$, where $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$.

Fix a field $\mathbb{K}$ and a polynomial ring $S = \mathbb{K}[x_{ij} \mid (i,j) \in V(\mathcal{P})]$. We consider the ideal $I_{\mathcal{P}} \subset S$ generated by all binomials $x_{il}x_{kj} - x_{ij}x_{kl}$ for which $[(i,j),(k,l)]$ is an interval in $\mathcal{P}$.



A column convex polyomino A row convex polyomino

A convex polyomino

Figure 2: Some polyominoes

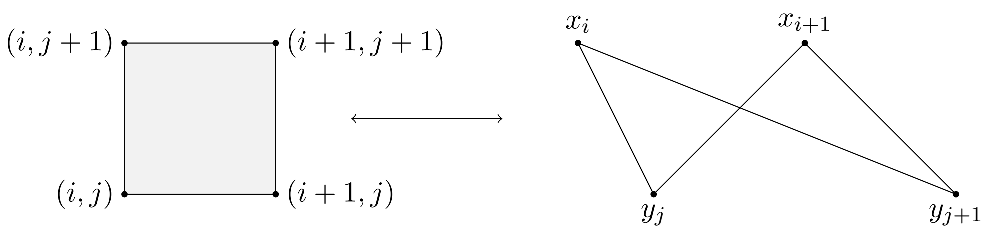


Figure 3: The bipartite graph attached to a cell in $\mathbb{N}^2$

The $\mathbb{K}$-algebra $S/I_{\mathcal{P}}$ is denoted $\mathbb{K}[\mathcal{P}]$ and is called the *coordinate ring* of $\mathcal{P}$. By [45, Theorem 2.2], $\mathbb{K}[\mathcal{P}]$ is a normal Cohen-Macaulay domain.

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. The ring $R = \mathbb{K}[x_iy_j \mid (i,j) \in V(\mathcal{P})] \subset \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ can be viewed as the edge ring of the bipartite graph $G_{\mathcal{P}}$ with vertex set $V(G_{\mathcal{P}}) = X \cup Y$, where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ and edge set $E(G_{\mathcal{P}}) = \{\{x_i, y_j\} \mid (i,j) \in V(\mathcal{P})\}$. In Figure 3, we displayed the bipartite graph attached to a cell in $\mathbb{N}^2$. According to [45], $\mathbb{K}[\mathcal{P}]$ can be identified with $\mathbb{K}[G_{\mathcal{P}}]$.

We set $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ and, if needed, we identify the point (x_i, y_j) in the plane with the vertex $(i,j) \in V(\mathcal{P})$.

Proposition 3 *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then the bipartite graph $G_{\mathcal{P}}$ is 2-connected.*

Definition 4 *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$ and $T \subset X$. The set $N_Y(T) = \{y \in Y \mid (x,y) \in V(\mathcal{P})$ for some $x \in T\}$ is called a neighbor vertical interval if $N_Y(T) = \{y_a, y_{a+1}, \ldots, y_b\}$ with $a < b$ and for every $i \in \{a, a+1, \ldots, b-1\}$ there exists $x \in T$ such that $[(x, y_i), (x, y_{i+1})]$ is an edge in $\mathcal{P}$.*

In the polyomino of Figure 4, if $T_1 = \{x_1, x_4\}$ and $T_2 = \{x_1, x_2\}$, then $N_Y(T_1) = \{y_1, y_2, y_3, y_4\} = N_Y(T_2)$. We notice that $N_Y(T_2)$ is a neighbor vertical interval, while $N_Y(T_1)$ is not.

Definition 5 *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$ and $U \subset Y$. The set $N_X(U) = \{x \in X \mid (x,y) \in V(\mathcal{P})$ for some $y \in U\}$ is called a neighbor horizontal interval if $N_X(U) = \{x_a, x_{a+1}, \ldots, x_b\}$ with $a < b$ and for every $i \in \{a, a+1, \ldots, b-1\}$ there exists $y \in U$ such that $[(x_i, y), (x_{i+1}, y)]$ is an edge in $\mathcal{P}$.*

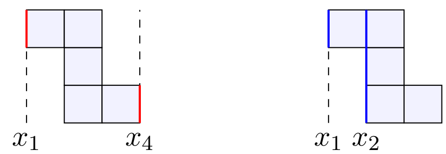


Figure 4: (Non-)Neighbor vertical interval

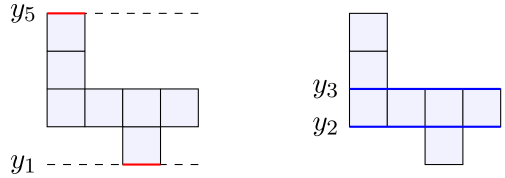


Figure 5: (Non-)Neighbor horizontal interval

In the polyomino of Figure 5, let $U_1 = \{y_2, y_3\}$ and $U_2 = \{y_1, y_5\}$. We notice that $N_X(U_1) = \{x_1, x_2, x_3, x_4, x_5\}$ is a neighbor horizontal interval, while $N_X(U_2) = \{x_1, x_2, x_3, x_4\}$ is not.

Theorem 6 *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$ and $G := G_{\mathcal{P}}$ its associated bipartite graph.*

Then $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if the following conditions are fulfilled:

1. *$|U| \le |N_X(U)|$ for every $U \subset Y$ and $|T| \le |N_Y(T)|$ for every $T \subset X$;*
2. *For every $\emptyset \neq T \subsetneq X$ with the properties*
   (a) $N_Y(T)$ is a neighbor vertical interval,
   (b) $N_X(Y \setminus N_Y(T)) = X \setminus T$ is a neighbor horizontal interval,

   one has $|N_Y(T)| = |T| + 1$.

We consider $\mathcal{P}$ to be a polyomino and we may assume that $[(1, 1), (m, n)]$ is the smallest interval containing the vertices of $\mathcal{P}$. Then $\mathcal{P}$ is called a *stack polyomino*, if it is a convex polyomino and for $i \in [m-1]$, the cell $[(i, 1), (i + 1, 2)]$ belongs to $\mathcal{P}$.

Theorem 7 *If $\mathcal{P}$ is a stack polyomino on $[m] \times [n]$, then the a-invariant of $\mathbb{K}[\mathcal{P}]$ is $-\max\{m, n\}$.*

Corollary 8 *If $\mathcal{P}$ is a stack polyomino on $[m] \times [n]$, then the Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$ is $\min\{m, n\} - 1$.*

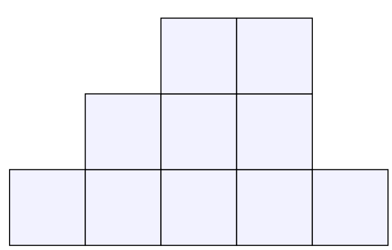


Figure 6: A stack polyomino

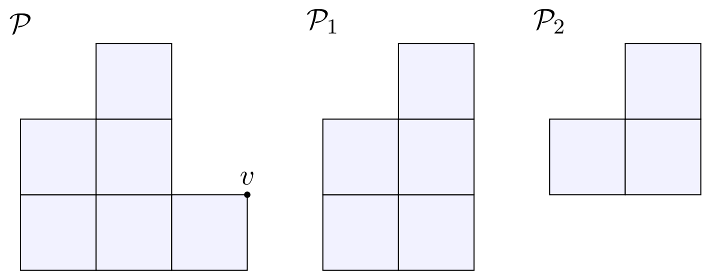


Figure 7: The multiplicity of $\mathbb{K}[\mathcal{P}]$

Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$. For every $i \in [m]$, we define the *height* of i as

$$\text{height}(i) = \max\{j \in [n] \mid (i,j) \in V(\mathcal{P})\}.$$

Following the proof of [44, Theorem], we give a total order on the variables x_{ij}, with $(i,j) \in V(\mathcal{P})$, as follows:

$$x_{ij} > x_{kl} \text{ if and only if} \tag{1}$$

$$(\text{height}(i) > \text{height}(k)) \text{ or } (\text{height}(i) = \text{height}(k) \text{ and } i > k) \text{ or } (i = k \text{ and } j > l).$$

Let $<$ be the reverse lexicographical order induced by this order of variables.

Theorem 9 *Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$ and $v = (i,j) \in V(\mathcal{P})$ with the properties:*

1. *x_{i1} is the smallest variable in S and*
2. *$j = \text{height}(i)$.*

We consider $\mathcal{P}_1$ and $\mathcal{P}_2$ to be the following polyominoes:

1. *$\mathcal{P}_1$ is the polyomino obtained from $\mathcal{P}$ by deleting the cell which contains the vertex v if $i = 1$. Otherwise, $\mathcal{P}_1$ is given by deleting the cell of $\mathcal{P}$ which contains the vertex $(m, \text{height}(m))$.*
2. *$\mathcal{P}_2$ is the polyomino obtained from $\mathcal{P}$ be deleting all the cells of $\mathcal{P}$ which lie below the horizontal edge interval containing the vertex v.*

Then the multiplicity of $\mathbb{K}[\mathcal{P}]$ has the following recursive formula

$$e(\mathbb{K}[\mathcal{P}]) = e(\mathbb{K}[\mathcal{P}_1]) + e(\mathbb{K}[\mathcal{P}_2]).$$

In Figure 7, we present the first step which is applied for computing the multiplicity of $\mathbb{K}[\mathcal{P}]$.

3. T-spread monomial ideals. Main results

In this chapter, we study t-spread strongly stable ideals with $t \geq 1$. They have been recently introduced in [22] and they represent a special class of square-free monomial ideals.

In the first part, we prove that any t-spread strongly stable ideal has a unique t-spread lex ideal with the same f_t-vector. This result is the main step to characterize the possible

f_t-vectors of t-spread strongly stable ideals in the "t-spread" analogue of Kruskal-Katona theorem.

In the last part, we study t-spread principal Borel ideals. In fact, we give explicitly all the generators of the ideal of the Alexander dual $\Delta^\vee$, where Δ is the simplicial complex associated to a t-spread principal Borel ideal. We then derive that the Stanley-Reisner ideal of $\Delta^\vee$ has linear quotients, which yields the sequential Cohen-Macaulay property of a t-spread principal Borel ideal, by Alexander duality.

Fix a field $\mathbb{K}$ and a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. Let t be a positive integer. A monomial $x_{i_1}x_{i_2}\cdots x_{i_d} \in S$ with $i_1 \leq i_2 \leq \ldots \leq i_d$ is called *t-spread*, if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$. A monomial ideal in S is called a *t-spread monomial ideal*, if it is generated by t-spread monomials.

Let I be a t-spread monomial ideal of S. We denote by I_j, the j-th graded component of I and call the set of t-spread monomials in I_j, the t-spread part of I_j and denote it by $[I_j]_t$. Furthermore, we set

$$f_{t,j-1}(I) = |[S_j]_t| - |[I_j]_t|.$$

Then the vector

$$\mathbf{f}_t(I) = (f_{t,-1}(I), f_{t,0}(I) \ldots, f_{t,j}(I), \ldots)$$

is called the *f_t-vector* of the t-spread monomial ideal I. By convention, we set $f_{t,-1} = 1$. Note that if $t = 1$, then I is the Stanley-Reisner ideal of a uniquely determined simplicial complex Δ and $\mathbf{f}_1(I)$ is the classical f-vector of Δ.

We denote by $M_{n,d,t}$ the set of the t-spread monomials of degree d in the polynomial ring S. For a monomial $u \in S$, we set $\operatorname{supp}(u) = \{i : x_i \mid u\}$.

Definition 10 *(a) A subset $L \subset M_{n,d,t}$ is called a* t-spread strongly stable set*, if for all t-spread monomials $u \in L$, all $j \in \operatorname{supp}(u)$ and all $1 \leq i < j$ such that $x_i(u/x_j)$ is a t-spread monomial, it follows that $x_i(u/x_j) \in L$.*

(b) Let I be a t-spread monomial ideal. Then I is called a t-spread strongly stable ideal*, if $[I_j]_t$ is a t-spread strongly stable set for all j.*

A special class of t-spread strongly stable ideals consists of t-spread lex ideals which are defined as follows.

Definition 11 *(a) A subset $L \subset M_{n,d,t}$ is called a* t-spread lex set*, if for all $u \in L$ and for all $v \in M_{n,d,t}$ with $v >_{\text{lex}} u$, it follows that $v \in L$.*

(b) Let I be a t-spread monomial ideal. Then I is called a t-spread lex ideal*, if $[I_j]_t$ is a t-spread lex set for all j.*

Let $I \subset S$ be a t-spread strongly stable monomial ideal. Then a t-spread lex ideal $J \subset S$ with $\mathbf{f}_t(I) = \mathbf{f}_t(J)$, if exists, is uniquely determined. We then denote this ideal J by $I^{\text{t-lex}}$.

Theorem 12 *For any t-spread strongly stable ideal I, the t-spread lex ideal $I^{\text{t-lex}}$ exists.*

In general, a t-spread monomial ideal may not have a t-spread lex ideal with the same f_t-vector.

For example, if $I = (x_2x_8, x_2x_6, x_2x_4) \subset \mathbb{K}[x_1, \ldots x_8]$, then we have

$$B_2 = L_2 = \{x_1x_3, x_1x_4, x_1x_5\} \text{ and } |\operatorname{Shad}_2(B_2)| = 9 > 5 = |[I_3]_2|,$$

which leads to the impossibility to construct the 2-spread lex ideal with the same f_2-vector with I.

Remark 13 *Since every square-free monomial ideal has a square-free lexsegment ideal with the same f-vector, the proof of Kruskal-Katona Theorem given in [30] works for all square-free monomial ideals. In our case, a t-spread ideal may not have an associated t-spread lex ideal with the same f_t-vector. Therefore, we will restrict to t-spread strongly stable ideals.*

Kruskal-Katona Theorem for t-spread strongly stable ideals gives a complete answer to the following question: When is a given sequence of positive integers

$$f_t = (f_{t,-1}, f_{t,0}, f_{t,1}, \ldots, f_{t,d}, \ldots)$$

the f_t-vector of a t-spread strongly stable ideal?

To answer this question, we proceeded like in the proof of Kruskal-Katona Theorem given in [30, Chapter 6]. To this aim, we need to define a *"t-operator"* analog to the operator $a \to a^{(d)}$ which is involved in the proof of Kruskal-Katona Theorem.

Definition 14 *Let n, d, t and a be positive integers with $a \leq \binom{n-(d-1)(t-1)}{d}$. If*

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_r}{r}$$

is the binomial expansion of a with respect to d, then we set

$$a_{r-1} = r-2,\ a_{d+1} = n-(d-1)(t-1) \text{ and } a_{d+2} = a_{d+1} + (t+1)$$

and we define

$$a^{[d]_t} := a^{[d]_t^k},$$

where k is the largest integer of the interval $[-1, d-r+1]$ with the property that $a_{d-k+1} - a_{d-k} \geq t+1$ and

$$a^{[d]_t^k} := \sum_{j=d+1-k}^{d} \binom{a_j-(t-1)}{j+1} + \binom{a_{d-k}-(2t-1)}{d-k+1} + \sum_{j=r}^{d-k} \binom{a_j}{j}$$

for all $k \geq 0$ and

$$a^{[d]_t^{-1}} := \binom{n-d(t-1)}{d+1}.$$

Again for convenience, we set $0^{[d]_t} = 0$ for positive integers d and t.

Theorem 15 *Let $f = (f(0), f(1), \ldots, f(d), \ldots)$ be a sequence of non-negative integers and $t \geq 1$ be an integer. The following conditions are equivalent:*

(1) *there exists an integer $n \geq 0$ and a t-spread strongly stable ideal*

$$I \subset \mathbb{K}[x_1, \ldots, x_n]$$

such that $f(d) = f_{t,d-1}(I)$ for all d.

(2) *$f(0) = 1$ and $f(d+1) \leq f(d)^{[d]_t}$ for all $d \geq 1$.*

Let t be a positive integer. A monomial ideal $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ is called *t-spread principal Borel* if there exists a monomial $u \in G(I)$ such that I is the smallest t-spread strongly stable ideal which contains u. According to [22], we denote $I = B_t(u)$.

Let Δ be the simplicial complex such that $I_\Delta = B_t(u)$. We consider $I^\vee$ to be the Stanley-Reisner ideal of the Alexander dual of Δ.

Theorem 16 *Let $t \geq 1$ be an integer and $I = B_t(u)$, where $u = x_{i_1}x_{i_2}\cdots x_{i_d} \in S$ is a t-spread monomial. We assume that $i_d = n$. Then $I^\vee$ is generated by the monomials of the following forms*

$$\prod_{k=1}^{n} x_k/(v_{j_1}\cdots v_{j_{d-1}}) \tag{2}$$

with $j_l \leq i_l$ for $1 \leq l \leq d-1$ and $j_l - j_{l-1} \geq t$ for $2 \leq l \leq d-1$, where $v_{j_k} = x_{j_k}\cdots x_{j_k+(t-1)}$ for $1 \leq k \leq d-1$.

$$\prod_{k=1}^{i_1} x_k. \tag{3}$$

$$\prod_{k=1}^{i_s} x_k/(v_{j_1}\cdots v_{j_{s-1}}) \tag{4}$$

with $2 \leq s \leq d-1$, $j_l \leq i_l$ for $1 \leq l \leq s-1$, $j_l - j_{l-1} \geq t$ for $2 \leq l \leq s-1$, where $v_{j_k} = x_{j_k}\cdots x_{j_k+(t-1)}$ for $1 \leq k \leq s-1$.

Theorem 17 *Let $t \geq 1$ be an integer and $u = x_{i_1}x_{i_2}\cdots x_{i_d} \in S$ be a t-spread monomial. Suppose that $i_d = n$. Then the t-spread principal Borel ideal $I = B_t(u)$ is sequentially Cohen-Macaulay.*

4. Powers of t-spread principal Borel ideals. Main results

The first part of this chapter is devoted to the study of the Gröbner basis of presentation ideals of Rees algebras of t-spread principal Borel ideals. The form of the binomials in this Gröbner basis shows that all the powers of a t-spread principal Borel ideal have linear quotients and the Rees algebra of a t-spread principal Borel ideal is a normal Cohen-Macaulay domain which implies that a t-spread principal Borel ideal possesses the strong persistence property.

In the last part, we study the limit behavior of the depth for the powers of t-spread principal Borel ideals.

Fix a field $\mathbb{K}$ and a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. For two monomials $v, w \in S$ of degree d, we write $vw = x_{i_1}x_{i_2}\cdots x_{i_{2d}}$ with $i_1 \leq i_2 \leq \cdots \leq i_{2d}$. Then the *sorting* of the pair (v, w) is the pair of monomials (v', w') where $v' = x_{i_1}x_{i_3}\cdots x_{i_{2d-1}}$ and $w' = x_{i_2}x_{i_4}\cdots x_{i_{2d}}$. The map

$$\mathrm{sort} : S_d \times S_d \to S_d \times S_d$$

with $\mathrm{sort}(v, w) = (v', w')$ is called *sorting operator*.

A subset $B \subset S_d$ is called *sortable* if $\mathrm{sort}(B \times B) \subset B \times B$.

Proposition 18 *[22, Proposition 3.1] Let $t \geq 1$ be an integer and $I = B_t(u)$, where $u = x_{i_1}\cdots x_{i_d}$ is a t-spread monomial. The minimal set of monomial generators of I is sortable.*

Theorem 19 *[48, Theorem 14.2] [20, Theorem 6.15] Let B be a sortable subset of monomials of S of the same degree and*

$$\mathcal{F} = \{t_u t_v - t_{u'}t_{v'} : u, v \in B, (u, v) \text{ unsorted, } (u', v') = \mathrm{sort}(u, v)\}.$$

Then there exists a monomial order $<$ on R which is called sorting order *such that for every $g = t_u t_v - t_{u'} t_{v'} \in \mathcal{F}$, $\mathrm{in}_<(g) = t_u t_v$.*

Let $I \subset S$ be a monomial ideal generated in a single degree and $\mathbb{K}[\{t_u : u \in G(I)\}]$ be the polynomial ring in $|G(I)|$ variables endowed with a monomial order $<$. Let P be the kernel of the K-algebra homomorphism

$$\mathbb{K}[\{t_u : u \in G(I)\}] \to \mathbb{K}[G(I)], t_u \mapsto u, u \in G(I).$$

A monomial $t_{u_1} \cdots t_{u_N}$ is called *standard with respect to* $<$, if it does not belong to $\mathrm{in}_<(P)$.

Definition 20 *[33, Definition 4.1] The monomial ideal $I \subset S$ satisfies the ℓ-exchange property with respect to $<$ if the following condition holds: for every $t_{u_1} \cdots t_{u_N}, t_{v_1} \cdots t_{v_N}$ standard monomials with respect to $<$ of the same degree N satisfying*

- *(i)* $\deg_{x_i} u_1 \cdots u_N = \deg_{x_i} v_1 \cdots v_N$ *for $1 \le i \le q-1$ with $q \le n-1$ and*
- *(ii)* $\deg_{x_q} u_1 \cdots u_N < \deg_{x_q} v_1 \cdots v_N$,

there exists integers δ, j with $q < j \le n$ and $j \in \mathrm{supp}(u_\delta)$ such that $x_q u_\delta / x_j \in I$.

Proposition 21 *Let $u = x_{i_1} \cdots x_{i_d}$ be a t-spread monomial in S. Then the t-spread principal Borel ideal $B_t(u)$ satisfies the ℓ-exchange property with respect to the sorting order $<_{\mathrm{sort}}$.*

Let $I = B_t(u) \subset S$, where $u \in S$ is a t-spread monomial.

We consider $\mathcal{R}(I) = \bigoplus_{j \ge 0} I^j t^j$ to be the Rees algebra of the ideal I. Since the minimal generators of I have the same degree, the fiber $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ of the Rees ring $\mathcal{R}(I)$ is isomorphic to $K[G(I)]$.

We fix the sorting order $<_{\mathrm{sort}}$ on the ring $T = K[\{t_v : v \in G(I)\}]$ and the lexicographic order $<_{\mathrm{lex}}$ on the ring S. Let $<$ be the monomial order on $R = S[\{t_v : v \in G(I)\}]$ defined as follows: if m_1, m_2 are monomials in S and v_1, v_2 are monomials in T, then

$$m_1 v_1 > m_2 v_2 \text{ if } m_1 >_{\mathrm{lex}} m_2 \text{ or } m_1 = m_2 \text{ and } v_1 >_{\mathrm{sort}} v_2.$$

Theorem 22 *The reduced Gröbner basis of the presentation ideal J of $\mathcal{R}(I)$ with respect to $<$ consists of the set of binomials $t_v t_w - t_{v'} t_{w'}$ where (v, w) is unsorted and $(v', w') = \mathrm{sort}(v, w)$, together with the binomials of the form $x_i t_v - x_j t_w$ where $i < j$, $x_i v = x_j w$ and j is the largest integer for which $x_i v / x_j \in G(I)$.*

Proposition 23 *All the powers of $B_t(u)$ have linear quotients. In particular, all the powers of $B_t(u)$ have a linear resolution.*

Corollary 24 *The Rees algebra $\mathcal{R}(B_t(u))$ is Koszul.*

Corollary 25 *The Rees algebra $\mathcal{R}(B_t(u))$ is a normal Cohen-Macaulay domain. In particular, $B_t(u)$ satisfies the strong persistence property. Therefore, $B_t(u)$ satisfies the persistence property.*

Theorem 26 *Let $t \ge 1$ be an integer and $I = B_t(u) \subset S$ the t-spread principal Borel ideal generated by $u = x_{i_1} \cdots x_{i_d}$ where $t+1 \le i_1 < i_2 < \cdots < i_{d-1} < i_d = n$. Then*

$$\operatorname{depth} \frac{S}{I^k} = 0, \text{ for } k \ge d.$$

In particular, the analytic spread of I is $\ell(I) = n$.

Corollary 27 *Let $t \geq 1$ be an integer and $B_t(u) \subset S$ the t-spread principal Borel ideal generated by $u = x_{i_1} \cdots x_{i_d}$ where $t+1 \leq i_1 < i_2 < \cdots < i_{d-1} < i_d = n$. Then $\dim K[G(B_t(u))] = n$.*

In the last chapter, we present a summary of the main results of this thesis. We also provide some interesting questions which have the starting point in these results.

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