University of Bucharest
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Summary of Ph.D. thesis

Good gradings on structural matrix algebras

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Bucharest
2018

Introduction

Group gradings on algebras play a key role in linear algebra, representation theory, ring theory, commutative algebra, combinatorics, algebraic geometry and Lie theory. Gradings are tools that can be useful to study specific properties of algebraic structures on which they are considered. There are two main sources where the concept of a graded algebra arose from. The first one is the study of polynomials; the algebra of polynomials in an arbitrary number of indeterminates has a natural grading by the additive group of integers given by the usual polynomial degree. The second one is group representation theory, where the group algebra kG of a group G over the field k has a natural G-grading.

If A is an algebra over the field k, and G is a (multiplicative) group, a G-grading on A is a decomposition $A = \oplus_{g\in G} A_g$ of A as a direct sum of k-subspaces, such that $A_g A_h \subset A_{gh}$ for any $g, h \in G$.

One of the first constructions of a graded algebra structure who has played a fundamental role in commutative algebra was done by Krull in 1938 (see [35], [12]). He considered $\mathfrak{m}$ a maximal ideal in a local Noetherian ring A and $\{\alpha_i\}_{1\leq i\leq r}$ a minimal system of generators for $\mathfrak{m}$. He defined for all nonzero elements x in A the "initial forms" of x as the set of all homogeneous polynomials $P(X_1, \ldots, X_r)$ of degree j with coefficients in the quotient field $k = A/\mathfrak{m}$ such that $x \equiv P(\alpha_1, \ldots, \alpha_r) \pmod{\mathfrak{m}^{j+1}}$, where $j = \max\{q \in \mathbb{Z} \mid x \in \mathfrak{m}^q\}$. To every ideal $\mathfrak{a}$ of A he put in correspondence the "Leitdeal", i.e. the graded ideal of $k[X_1, \ldots, X_r]$ generated by the "initial forms" of all the elements of $\mathfrak{a}$; for Krull these two notions took the place of the associated $\mathbb{Z}$-graded algebra.

If G is a group and k is a field, the group algebra $A = kG$ has the natural G-grading given by $A_g = kg$ for any $g \in G$. Moreover, if H is a normal subgroup of G, the algebra kG also has a natural grading by the factor group G/H; the homogeneous component of degree gH is $\sum_{h\in H} kgh$ for any $g \in G$. Thus the homogeneous component of trivial degree of this grading is just the group algebra kH of H, and this point of view has been used in connecting representations of G to representations of H. Note that this G/H-grading of $A = kG$ has the property that $A_{gH}A_{rH} = A_{grH}$ for any $g, r \in G$. This approach was initiated by E. Dade in [17], [18], [19], who studied strongly graded algebras, i.e. G-graded algebras A for which $A_g A_h = A_{gh}$ for any $g, h \in G$, the connection between modules over the homogeneous component A_e (where e is the neutral element of G) and (graded) modules over A, and as an application he extended Clifford theory by investigating simple modules over A_e in relation to (graded) simple modules over A. This applies to the classical case, where irreducible representations of G are connected to irreducible representations of the normal subgroup H, since kG endowed with the above mentioned G/H-grading is a strongly graded algebra.

Many concepts and constructions from Ring Theory have graded versions for graded algebras. For example, a graded left ideal of a G-graded algebra A is a left ideal I with the property that for any element $a \in I$, with decomposition

$a = \sum_{g \in G} a_g$ in the graded structure of A, all homogeneous components a_g lie in I, too. Then A is called graded left Noetherian if any ascending chain of graded left ideals of A is stationary. One relevant application of the graded theory is to investigate certain ring properties of an algebra via the grading. More precisely, it can be studied the connection between an algebra having a certain ring property and having the graded version of that property; if it happens that the property is equivalent to its graded version, then it is usually easier to check whether the graded version is satisfied. For example, it was proved in [37] that for an algebra A graded by the additive group of integers, A is left Noetherian if and only if it is graded left Noetherian. More generally, the equivalence between the left Noetherian property and the graded left Noetherian property was proved in [14] for any algebra graded by a polycyclic-by-finite group.

Graded ring theory became a direction of study of great interest in the 1970's. The first books in this direction [38], [39] were written by C. Năstăsescu and F. van Oystaeyen. One major advance in the theory was done in [16], where a duality between group actions and group gradings was explained in the finite group case; this suggested an enlightening Hopf algebra approach. More precisely, a G-grading on a k-algebra A is just a coaction of the group Hopf algebra kG on A (in other words, A is a kG-comodule algebra). If G is finite, this is the same with a action of the dual Hopf algebra $(kG)^*$ on A. If k has enough roots of unity, then the group Hopf algebra kG is selfdual, and then a G-grading, i.e. a coaction, is in fact an action of G as algebra automorphisms.

Given a graded algebra A, one can consider A-modules, but also graded A-modules, which are just A-modules endowed with a grading compatible with the grading of A. A theory of graded modules over a graded ring may seem not to differ much from ordinary module theory. Indeed, the category of graded left A-modules is a Grothendieck category, and many concepts can be defined as in the case of un-graded modules. However, graded modules come equipped with a shift because of a possible partitioning followed by a rearrangement of the partitions (see [28]). From this point of view, graded module theory demonstrates a specific complexity. For example, a local theory of graded modules is introduced by Green and Marcos in [27]; it is applied for quotients of path algebras. Furthermore, graded modules play an essential role in the study of homological aspect of rings (see [43]).

It is of interest to study group gradings on certain given algebras. Thus a general problem is: given an algebra A, determine all G-gradings on A, for all possible groups G. Describing gradings was useful in solving certain problems in Ring Theory. For example, in his solution to the Specht problem for associative algebras in characteristic zero, see [32] and [49], Kemer needed to describe all gradings on the 2×2 matrix algebra by the cyclic group of order two. In [29], there is the description of $\mathbb{Z}$-gradings on finite-dimensional complex Lie algebras. It was proved in [47] that all finite $\mathbb{Z}$-gradings on a simple associative algebra can be obtained from the Pierce decomposition of this algebra. In [24], there are classified all gradings of a Cayley-Dickson algebra. A classification of finite $\mathbb{Z}$-gradings on infinite-dimensional simple Lie algebras is described in [48].

Gradings on matrix algebras are an important object of study among graded algebras in general and have a wide range of applications. E. Zelmanov posed the following general problem (see [30], [31]): find all G-gradings of the matrix algebra $M_n(k)$, where G is a group, k is a field and n a positive integer. This seems to be a complex problem and a lot of work has been done in this direction. The first results concerning gradings on matrix algebras were obtained by Knus in 1969 (see [34]). Many results so far have given solutions to this problem depending on the structure of G and k and the value of n; the general problem is still unsolved. Steps towards the general solution have been made. For instance, in [26] and [27], a special case of gradings (such that all the matrix units e_{ij} are homogeneous elements) was considered; this type of gradings are called good gradings in [21]. All gradings of $M_2(k)$ by C_2 were described in [21] making use of computational methods and the duality between group actions and group gradings. These methods appear in [9] in the form of actions and coactions of Hopf algebras (a G-grading on an algebra A coincides with a structure of a kG-comodule algebra on A). This idea was also used in the study of gradings on matrix algebras by cyclic groups (see [9]). All the isomorphism types of C_2-gradings on $M_2(k)$ are illustrated in [9] (for $\mathrm{char}(k) \neq 2$) and in [7] (for $\mathrm{char}(k) = 2$). If k is algebraically closed, it was proved in [13] (see also [46]) that any C_m-grading on $M_n(k)$ is isomorphic to a good grading; there were further described (in cohomological terms) all gradings over cyclic groups for an arbitrary field k through descent theory. Additionally, if G is torsion-free (see [21]) or if $G = C_p$ (where p is a prime), k has a pth root of unity, and $p \nmid n$ (see [9]) again any grading on the matrix algebra is isomorphic to a good grading. This demonstrates the importance of good gradings. The study of gradings by non-cyclic groups is more difficult. In [8] the gradings of $M_2(k)$ over the Klein group $C_2 \times C_2$ (for an arbitrary field k) were classified; it was applied the Hopf algebra technique and duality. Furthermore, in [33] a classification of all group gradings of $M_2(k)$ for any field k was provided; in addition, an elementary approach was given to the results about C_2-gradings and $C_2 \times C_2$-gradings without any use of the technique mentioned above and it was also proved that any grading on $M_2(k)$ is either isomorphic to a good grading or reduces to a grading of C_2 or by $C_2 \times C_2$. In the case of k being an algebraically closed field, in [1] there were described all gradings on $M_n(k)$ by abelian groups; it was shown that any such grading is isomorphic with the tensor product between a good grading and a fine grading (where all the G-indexed direct summands are at most 1-dimensional). In [2] a similar result was proved for gradings by arbitrary finite groups in case of k being a algebraically closed of characteristic zero. For an arbitrary field k, and an arbitrary group G, in [10] there were described and classified all G-gradings on $M_3(k)$ by using Galois extensions; it was shown that any such grading is either isomorphic to a good grading or it reduces to a grading by C_3 or by $C_3 \times C_3$.

Subalgebras of full matrix algebras demonstrate great significance and even more difficulty when considered the associated gradings. Algebras of upper block triangular matrices (to which we will refer as UT-algebras and we will consider them as subalgebras in a $M_n(k)$) are key examples of PI-algebras. For any group and any field, a conjecture occured in [52], namely that a graded UT-algebra is isomorphic to the tensor product between another UT-algebra which is has an elementary grading (i.e. a grading for which there exists a map $f : \{1, \dots, n\} \to G$ such that $\deg(e_{ij}) = f(i)^{-1}f(j)$ for any i, j, see [4]) and a division graded full matrix algebra (i.e. it has a unit and every non-zero homogeneous element is invertible); this was proved for any algebraically closed field of characteristic zero in [53] (for any finite abelian group) and in [44] (for any abelian group). In [3] good gradings on UT-algebras were investigated; it was shown that any such grading is isomorphic to the ring of endomorphisms of a graded flag over a field. Also, it was provided a result according to which any two good gradings are isomorphic if and only if the corresponding graded flags are isomorphic up to a shift; this has allowed to classify (in the same paper) all good gradings as orbits of a certain biaction of a Young subgroup and the group G on the set G^n, where G is the grading group. In [45] there was determined when two UT-algebras graded by the same group are isomorphic. UT-algebras are a particular case of incidence algebras. In [42] group gradings were studied on the incidence algebra $I(X, k) = \{f : X \times X \to k \mid f(x, y) = 0 \text{ if } x \not\leq y\}$ for locally finite posets $(X, \leq)$ (for operations on $I(X, k)$ see [50]). It was shown that $I(X, k)$ has a grading for which a certain homogeneous component (more specifically, the one that corresponds to the neutral element of the group) is central if and only if X is a antichain. If X is a finite antichain with n elements, then $I(X, k) \simeq k^n$; to this case was directed the study from [20] where a description of all group gradings on diagonal algebras was made for an arbitrary field k.

The original results of this thesis are the content of [5] and [6]. We study structural matrix algebras; these algebras got their name in [54], but they had already been considered in [41]. Our research on structural matrix algebras (a special case of incidence algebras) was made in terms of description (a structural matrix algebra is isomorphic to the endomorphism algebra of a generalized flag) with direct relation to good gradings and in terms of presentation of the automorphism group (as a biproduct). We classified all good gradings that arise from graded generalized flags and all good gradings on structural matrix algebras that are associated to partial order relations. The second classification led the computation of the number of types of isomorphism of good gradings in some particular cases (for finite groups). Structural matrix algebras have been useful for providing examples and counterexamples in ring theory and in the study of numerical invariants of PI- algebras. The illustrated study from this thesis brings also as mentioned a new type of flags which represents an important mathematical contribution also; flags in general are concepts of great value for algebraic geometry, representation theory, algebraic groups, combinatorics (see [36]).

In the first chapter, for a field k, a positive integer n and a preorder relation ρ on $\{1, \dots, n\}$, we introduce the structural matrix algebra $M(\rho, k)$ (as a subalgebra of the full matrix algebra $M_n(k)$) consisting of all matrices with zero entries on all positions (i, j) with the property that $(i, j) \notin \rho$. In the terminology of [50], A is the incidence algebra over k associated with ρ. We define some useful structures. We associate to ρ an equivalence relation $\sim$ and we call $\mathcal{C}$ its set of equivalence classes; we connect these cosets by a partial order $\leq$. To $(\mathcal{C}, \leq)$ we correlate an oriented graph Γ; we illustrate this combinatorial object in many examples. In addition, we show that through transformations that are generated by certain permutations we can have in $M(\rho, k)$ all the inferior blocks to be null.

Chapter 2 represents a description of the automorphisms of a structural matrix algebra providing also a preparation for a classification of good G-gradings (G being a group). We show that a structural matrix algebra $M(\rho, k)$ is isomorphic to the endomorphism algebra of a certain algebraic-combinatorial structure $\mathcal{F}$ which we call a ρ-flag. If $\mathcal{F} = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ is a ρ-flag, we show that the $\mathrm{End}(\mathcal{F})$-submodules of V are in bijective correspondence with the antichains of $\mathcal{C}$. We exemplify the lattice of such submodules in some cases. We also describe the lattice automorphisms of $\mathcal{A}(\mathcal{C})$ ($\mathcal{A}(\mathcal{C})$ being the set of antichains of $\mathcal{C}$). We find that the set of algebra isomorphisms from $\mathrm{End}(\mathcal{F})$ to $\mathrm{End}(\mathcal{F}')$ (where $\mathcal{F}$ and $\mathcal{F}'$ are ρ-flags) is in bijective correspondence with the equivalence classes of a set involving the invertible matrices of $M(\rho, k)$, the automorphisms of $\mathcal{C}$ preserving the cardinality of elements, and the transitive functions on ρ, with respect to an equivalence relation. In particular, if $\mathcal{F} = \mathcal{F}'$, the automorphism group of $\mathrm{End}(\mathcal{F})$ is described as a factor group of a double semidirect product. As a biproduct, we obtain a descriptive presentation of the automorphism group of a structural matrix algebra. This automorphism group was computed in [15], and we show how the presentation in [15] can be derived from ours.

The third chapter regards G-gradings on structural matrix algebras; we classify those which arise from graded flags. If $\mathcal{F}$ is a G-graded ρ-flag, we find that its endomorphism algebra $\mathrm{End}(\mathcal{F})$ gets an induced G-graded algebra structure; we denote by $\mathrm{END}(\mathcal{F})$ the obtained G-graded algebra. This grading transfers to a G-grading on $M(\rho, k)$ via the isomorphism mentioned in the previous paragraph. The gradings produced in this way are good gradings on $M(\rho, k)$. It is an interesting question whether all good gradings are obtained like this. This is a problem of independent interest, and it can be formulated in simple terms related to the graph Γ associated with ρ: if G is a group, and on each arrow of Γ we write an element of G as a label, such that for any two paths starting from and terminating at the same points the product of the labels of the arrows is the same for both paths, does the set of labels arise from a set of weights on the vertices of Γ, in the sense that an arrow starting from v_1 and terminating at v_2 has label $g_1 g_2^{-1}$, where g_1 and g_2 are the weights of v_1 and v_2? This problem was considered in [41] in the case where G is abelian, and it was showed that the answer is positive if and only if the cohomology group $H^1(\Delta, G) = 0$, where Δ is a certain simplicial complex associated with ρ. Also, for a given ρ, the answer to the above question is positive for any abelian group G if and only if the homology group $H_1(\Delta) = 0$. We show that the answer is positive for any arbitrary group G if and only if the normal closure of two certain subgroups $A(\Gamma) \subseteq B(\Gamma)$ of the free group generated by the arrows of Γ coincide; $A(\Gamma)$ and $B(\Gamma)$ are defined in terms of cycles of the un-directed graph obtained from Γ.

This parallels the result in the abelian case, where $H_1(\Delta) = B/A$ for similar subgroups A and B in a free abelian group associated with Γ. In fact we use slightly different A and B, by working with a different graph. For classifying G-gradings arising from graded flags, we consider two G-graded ρ-flags $\mathcal{F}$ and $\mathcal{F}'$, and we look at the isomorphisms between the graded algebras $\mathrm{END}(\mathcal{F})$ and $\mathrm{END}(\mathcal{F}')$. Using the structure of isomorphisms between $\mathrm{End}(\mathcal{F})$ and $\mathrm{End}(\mathcal{F}')$, which we already know by now, and adding the additional information about gradings, we obtain that $\mathrm{END}(\mathcal{F}) \simeq \mathrm{END}(\mathcal{F}')$ if and only if the connected components of $\mathcal{F}$ and $\mathcal{F}'$ are pairwise isomorphic up to a permutation, some graded shifts and an automorphism of $\mathcal{C}$. Using this result, we show that the isomorphism types of graded algebras of the form $\mathrm{END}(\mathcal{F})$ are classified by the orbits of the action of a certain group, which is a double semidirect product of a Young subgroup of S_n, a certain subgroup of automorphisms of $\mathcal{C}$, and G^q, where q is the number of connected components of $\mathcal{C}$, on the set G^n.

In the last chapter we consider structural matrix algebras $M(\rho, k)$ in the case of ρ being a partial order. We describe in an explicit way (and illustrate in some examples) the automorphisms of $M(\rho, k)$, by following the approach from chapter 2. The explicit description is used to show that the isomorphism classes of good G-gradings on $M(\rho, k)$ are in a bijective correspondence to the orbits of a certain action of the automorphism group of the poset $(\{1, \ldots, n\}, \rho)$ on the set of G-valued transitive functions on ρ. An alternative version in terms of the graph associated with ρ is given. Furthermore, we compute explicitly the number of isomorphism types of good gradings for certain partial order relations.

Graded flags allowed the description of all good gradings on UT-algebras. Unfortunately, it is not the case for structural matrix algebras, i.e. not any good grading comes from a graded generalized flag. This is a problem to be solved by future research.

1 Preliminaries

Let k be a field, n a positive integer. Let ρ a preorder relation on the set $\{1, \ldots, n\}$.

We define a **structural matrix algebra associated with** ρ **and** k a subalgebra in $M_n(k)$ by

$$M(\rho, k) = \{(a_{ij})_{1 \leq i,j \leq n} \in M_n(k) \mid a_{ij} = 0 \text{ if } (i,j) \notin \rho\}.$$

Let $\sim$ be the equivalence relation on $\{1, \ldots, n\}$:

$$i \sim j \Leftrightarrow i\rho j \text{ and } j\rho i.$$

Let $\mathcal{C}$ be the set of equivalence classes. On $\mathcal{C}$ we define a partial order:

$$\hat{i} \leq \hat{j} \Leftrightarrow i\rho j.$$

The structure of the partially ordered set $(\mathcal{C}, \leq)$ can be illustrated via an associated oriented graph Γ where:
- the vertices are the elements of $\mathcal{C}$
- if $\alpha, \beta \in \mathcal{C}$, then we draw an arrow a from α to β (we write $s(a) = \alpha$ and $t(a) = \beta$) if and only if
  $\alpha < \beta$
  and
  there is no $\gamma \in \mathcal{C}$ such that $\alpha < \gamma < \beta$.

Let $\sigma \in S_n$. We define a bijection

$$\begin{aligned} \varphi_\sigma : \{1, \ldots, n\} \times \{1, \ldots, n\} &\rightarrow \{1, \ldots, n\} \times \{1, \ldots, n\} \\ (i,j) &\mapsto (\sigma(i), \sigma(j)). \end{aligned}$$

If ρ is a preoder relation, then $\varphi_\sigma(\rho)$ is a preorder relation too; we denote $\rho_\sigma = \varphi_\sigma(\rho)$. Thus,

$$\begin{aligned} M(\rho, k) &\simeq M(\rho_\sigma, k) \\ A = (a_{ij})_{i,j} &\mapsto A_\sigma = (a_{\sigma(i)\sigma(j)})_{i,j}. \end{aligned}$$

Proposition 1.1 *Let $M(\rho, k)$ be a structural matrix algebra. Then there exists $\sigma \in S_n$ for which $M(\rho_\sigma, k)$ is a block matrix algebra with the property that all the blocks below the main diagonal blocks are null.*

2 The automorphisms of structural matrix algebra

• **Structural matrix algebras as endomorphism algebras**

A ρ**-flag** is an n-dimensional vector space V with a family $(V_\alpha)_{\alpha\in\mathcal{C}}$ of subspaces such that there is a basis B of V and a partition $B = \bigcup\limits_{\alpha\in\mathcal{C}} B_\alpha$ with the property that $|B_\alpha| = m_\alpha$ and $\bigcup\limits_{\beta\leq\alpha} B_\beta$ is a basis of V_α for any $\alpha \in \mathcal{C}$.

If $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ and $\mathcal{F}' = (V', (V'_\alpha)_{\alpha\in\mathcal{C}})$ are ρ-flags, then a **morphism of** ρ**-flags** from $\mathcal{F}$ to $\mathcal{F}'$ is a linear map $f : V \to V'$ such that $f(V_\alpha) \subset V'_\alpha$ for any $\alpha \in \mathcal{C}$.

Proposition 2.1 *Let* $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ *be a* ρ*-flag. Then the algebra* $\mathrm{End}(\mathcal{F})$ *of endomorphisms of* $\mathcal{F}$ *(with the map composition as multiplication) is isomorphic to* $M(\rho, k)$*.*

We note that if there is $B = \{v_i \mid 1 \leq i \leq n\}$ a basis on V and $B_\alpha = \{v_i \mid i \in \alpha\}$ for any $\alpha \in \mathcal{C}$ as in the definition of a ρ-flag, then

$$E_{ij}(v_t) = \delta_{jt}v_i \text{ for any } i\rho j,\, t$$
$$E_{ij}E_{pq} = \delta_{jp}E_{iq} \text{ for any } i, j, p, q.$$

Thus

$$\mathrm{End}(\mathcal{F}) \simeq M(\rho, k)$$
$$E_{ij} \leftrightarrow e_{ij}$$

for any $i\rho j$.

• **The lattice of** $\mathrm{End}(\mathcal{F})$**-submodules of** V

Let $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ a ρ-flag. We consider the action of $\mathrm{End}(\mathcal{F})$ on V the restriction of the usual $\mathrm{End}(V)$-action on V.

If $\mathcal{D}$ is a subset in $\mathcal{C}$, we denote

$$V_\mathcal{D} = \sum_{\alpha\in\mathcal{D}} V_\alpha.$$

By convention $V_\varnothing = 0$.

Proposition 2.2 *The* $\mathrm{End}(\mathcal{F})$*-submodules of* V *are the subspaces of the form* $V_\mathcal{D}$*, where* $\mathcal{D}$ *is a subset of* $\mathcal{C}$*.*

If $\mathcal{D} \subseteq \mathcal{C}$, then $\mathcal{D}_{max}$ is the set of maximal elements of $\mathcal{D}$. The $\mathrm{End}(\mathcal{F})$-submodules of V are $V_\mathcal{D}$ with $\mathcal{D}$ an antichain in $\mathcal{C}$. We denote by $\mathcal{A}(\mathcal{C})$ the set of all antichains of $\mathcal{C}$.

The lattice of End($\mathcal{F}$)-submodules of V is isomorphic to the lattice $\mathcal{A}(\mathcal{C})$ where the infimum and the supremum are given by

$$\mathcal{D} \wedge \mathcal{E} = \{\alpha \in \mathcal{C} \mid \text{there exist } \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{E} \text{ such that } \alpha \leq \beta_1 \text{ and } \alpha \leq \beta_2\}_{max},$$

$$\mathcal{D} \vee \mathcal{E} = (\mathcal{D} \cup \mathcal{E})_{max}.$$

The partial order on $\mathcal{A}(\mathcal{C})$ is

$$\mathcal{D} \leq \mathcal{E} \iff V_{\mathcal{D}} \subset V_{\mathcal{E}} \iff \text{ for any } \alpha \in \mathcal{D} \text{ there exists } \beta \in \mathcal{E} \text{ such that } \alpha \leq \beta.$$

Proposition 2.3 *If g is an automorphism of the poset $(\mathcal{C}, \leq)$, then the map $f_g : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(\mathcal{C})$, $f_g(\mathcal{D}) = g(\mathcal{D}) = \{g(\alpha) \mid \alpha \in \mathcal{D}\}$ is an automorphism of the lattice $\mathcal{A}(\mathcal{C})$. Moreover, for any lattice automorphism f of $\mathcal{A}(\mathcal{C})$ there exists an automorphism g of the poset $(\mathcal{C}, \leq)$ such that $f = f_g$.*

• Isomorphisms between endomorphism algebras of flags

Let $\mathcal{F} = (V, V_\alpha)_{\alpha \in \mathcal{C}}$ be a ρ-flag.

Let $\mathrm{Aut}_0(\mathcal{C}, \leq) = \{g \in \mathrm{Aut}(\mathcal{C}, \leq) \mid |\alpha| = |g(\alpha)| \text{ for any } \alpha \in \mathcal{C}\}$. It is a subgroup in $\mathrm{Aut}(\mathcal{C}, \leq)$.

For any $g \in \mathrm{Aut}_0(\mathcal{C})$ we define a bijection $\tilde{g} : \{1, \ldots, n\} \to \{1, \ldots, n\}$:
if $\alpha = \{i_1, \ldots, i_r\}$ with $i_1 < \ldots < i_r$ and $g(\alpha) = \{j_1, \ldots, j_r\}$ with $j_1 < \ldots < j_r$, then

$$\tilde{g}(i_1) = j_1, \ldots, \tilde{g}(i_r) = j_r.$$

Let $\mathcal{T}(\rho, k^*) = \{(a_{ij})_{i\rho j} \subset k^* \mid a_{ij}a_{jr} = a_{ir} \text{ for any } i, j, r \text{ with } i\rho j, j\rho r\}$ (i.e. the set of transitive k^*-valued functions on ρ.).
Multiplication on positions (i.e. pointwise multiplication of functions) makes $\mathcal{T}(\rho, k^*)$ a group.

Let $\mathcal{F}' = (V', V'_\alpha)_{\alpha \in \mathcal{C}}$ be another ρ-flag.

Define

$$F : U(M(\rho, k)) \times \mathrm{Aut}_0(\mathcal{C}) \times \mathcal{T}(\rho, k^*) \to \mathrm{Iso}_{alg}(\mathrm{End}(\mathcal{F}), \mathrm{End}(\mathcal{F}'))$$

$$F(A, g, (a_{ij})_{i\rho j}) = \varphi$$

for any $A = (\lambda_{ij})_{1 \leq i,j \leq n} \in U(M(\rho, k))$, $g \in \mathrm{Aut}_0(\mathcal{C})$, $(a_{ij})_{i\rho j} \in \mathcal{T}(\rho, k^*)$, where

$$\varphi(E_{ij}) = a_{ij} \sum_{\substack{s\rho\tilde{g}(i) \\ \tilde{g}(j)\rho t}} \lambda_{s\tilde{g}(i)} \overline{\lambda}_{\tilde{g}(j)t} E'_{st}$$

for any $i\rho j$
($(E'_{ij})_{i\rho j}$ is the basis on End($\mathcal{F}'$) and $A^{-1} = (\overline{\lambda}_{ij})_{1 \leq i,j \leq n}$).

Proposition 2.4 *F is surjective.*

Proof plan:
Let $\varphi : \mathrm{End}(\mathcal{F}) \to \mathrm{End}(\mathcal{F}')$ be an algebra isomorphism. We have that:

- the set of all $\mathrm{End}(\mathcal{F})$-submodules of V is in a bijective correspondence with $\mathcal{A}(\mathcal{C})$;
- the algebra isomorphism φ induces a linear isomorphism $\gamma : V \to V'$ that is a φ'-isomorphism for a certain deformation of φ;
- the new algebra isomorphism φ' is obtained from φ using a k^*-valued transitive function on ρ;
- since φ' is an algebra isomorphism, γ induces an isomorphism between the lattice of $\mathrm{End}(\mathcal{F})$-submodules of V and the lattice of $\mathrm{End}(\mathcal{F}')$-submodules of V', and this lattice isomorphism reduces in fact to an automorphism of the lattice $\mathcal{A}(\mathcal{C})$;
- such an automorphism is completely determined by an automorphism g of the poset $\mathcal{C}$;
- φ can be recovered from g, the deformation constants producing φ' from φ and a matrix of γ in a fixed pair of bases.

For any $A = (\lambda_{ij})_{1\le i,j\le n} \in U(M(\rho, k))$, $g \in \mathrm{Aut}_0(\mathcal{C})$ define

$$A^g = (\lambda_{i\tilde{g}(j)})_{1\le i,j\le n} \text{ și } {}^gA = (\lambda_{\tilde{g}(i)j})_{1\le i,j\le n}.$$

We note that ${}^g(A^h) = ({}^gA)^h$ for any A, g, h.

We consider the relation $\approx$ on $U(M(\rho, k)) \times \mathrm{Aut}_0(\mathcal{C}) \times \mathcal{T}(\rho, k^*)$:

$$(A, g, (a_{ij})_{i\rho j}) \approx (B, h, (b_{ij})_{i\rho j}) \text{ if and only if}$$

$$\begin{cases} g = h \\ \text{there exist } d_1,\ \ldots,\ d_n \in k^* \text{ such that } a_{ij}b_{ij}^{-1} = d_i d_j^{-1} \text{ for any } i\rho j \\ B^g = A^g \,\mathrm{diag}(d_1, \ldots,\ d_n). \end{cases}$$

$\approx$ is an equivalence relation.

We have that $F(A, g, (a_{ij})_{i\rho j}) = F(B, h, (b_{ij})_{i\rho j})$ if and only if $(A, g, (a_{ij})_{i\rho j}) \approx (B, h, (b_{ij})_{i\rho j})$.

Theorem 2.5 *F induces a bijection*

$$\overline{F} : \frac{U(M(\rho, k)) \times \mathrm{Aut}_0(\mathcal{C}) \times \mathcal{T}(\rho, k^*)}{\approx} \longrightarrow \mathrm{Iso}_{alg}(\mathrm{End}(\mathcal{F}), \mathrm{End}(\mathcal{F}')).$$

The group $\mathrm{Aut}_0(\mathcal{C})$ acts to the right on $\mathcal{T}(\rho, k^*)$ by

$$(a_{ij})_{i\rho j} \cdot g = (a_{\tilde{g}(i)\tilde{g}(j)})_{i\rho j}.$$

Thus we can form a right crossed product $\mathrm{Aut}_0(\mathcal{C}) \ltimes \mathcal{T}(\rho, k^*)$, where the multiplication is given by

$$(g \ltimes (a_{ij})_{i\rho j})(h \ltimes (b_{ij})_{i\rho j}) = gh \ltimes ((a_{\tilde{h}(i)\tilde{h}(j)} b_{ij})_{i\rho j}.$$

The group $\mathcal{T}(\rho, k^*)$ acts to the left on the group $U(M(\rho, k))$:

if $(a_{ij})_{i\rho j} \in \mathcal{T}(\rho, k^*)$ and $A = (\alpha_{ij})_{1 \le i,j \le n} \in U(M(\rho, k))$,
then $(a_{ij})_{i\rho j} \cdot A$ is the matrix $(m_{ij})_{1 \le i,j \le n}$ where

$$m_{ij} = \begin{cases} a_{ij}\alpha_{ij}, & \text{if} \quad (i,j) \in \rho \\ 0, & \text{if} \quad (i,j) \notin \rho \end{cases} .$$

The group $\mathrm{Aut}_0(\mathcal{C})$ acts to the left on $U(M(\rho, k))$ by

$$g \cdot A = {}^{g^{-1}}A^{g^{-1}}.$$

We obtain that $\mathrm{Aut}_0(\mathcal{C}) \ltimes \mathcal{T}(\rho, k^*)$ acts to the left on $U(M(\rho, k))$ by

$$(g \ltimes (a_{ij})_{i\rho j}) \cdot A = {}^{g^{-1}}((a_{ij})_{i\rho j} \cdot A)^{g^{-1}},$$

can form the left crossed product

$$U(M(\rho, k)) \rtimes (\mathrm{Aut}_0(\mathcal{C}) \ltimes \mathcal{T}(\rho, k^*)).$$

The multiplication on this group is defined as

$$(A \rtimes (g \ltimes (a_{ij})_{i\rho j})) \cdot (B \rtimes (h \ltimes (b_{ij})_{i\rho j})) =$$

$$= (A \ {}^{g^{-1}}((a_{ij})_{i\rho j} \cdot B)^{g^{-1}}) \rtimes (gh \ltimes (a_{\tilde{h}(i)\tilde{h}(j)} b_{ij})_{i\rho j}).$$

Theorem 2.6 *F is a morphism of groups, and it induces a group isomorphism*

$$\frac{U(M(\rho, k)) \rtimes (\mathrm{Aut}_0(\mathcal{C}) \ltimes \mathcal{T}(\rho, k^*))}{D} \simeq \mathrm{Aut}(\mathrm{End}(\mathcal{F})),$$

where $D = \{diag(d_1, \ldots, d_n) \rtimes (Id \ltimes (d_i^{-1} d_j)_{i\rho j}) \mid d_1, \ldots, d_n \in k^*\}$.

We showed that the description of the automorphism group of $M(\rho, k)$ given in [15] can be deduced from this theorem.

3 Good gradings on structural matrix algebras

If G is a (multiplicative) group and A is a k-algebra, a G-**grading** on A is a decomposition

$$A = \bigoplus_{g \in G} A_g$$

(as a direct sum of linear spaces) such that $A_g A_h \subseteq A_{gh}$ for any $g, h \in G$.

Let $A \subseteq M_n(k)$ a subalgebra. A **good** G-grading on A is a G-grading such that all matrices

$$e_{ij} = (\delta_{(i,j),(p,q)})_{p,q} \in A$$

are homogeneous elements.

A G-**graded** ρ-**flag** is a ρ-flag $(V, (V_\alpha)_{\alpha \in \mathcal{C}})$ such that V is a G-graded vector space, and the basis B from ρ-flag's definition consists of homogeneous elements.

If $\mathcal{F} = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ and $\mathcal{F}' = (V', (V'_\alpha)_{\alpha \in \mathcal{C}})$ are G-graded ρ-flags, then a **morphism of graded flags** from $\mathcal{F}$ to $\mathcal{F}'$ is a morphism $f : V \to V'$ of ρ-flags, which is also a morphism of graded vector spaces.

If $\mathcal{F} = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ is a G-graded ρ-flag and $\sigma \in G$, define

$$\mathrm{End}(\mathcal{F})_\sigma = \{f \in \mathrm{End}(\mathcal{F}) \mid f(V_g) \subseteq V_{\sigma g} \text{ for any } g \in G\}.$$

Proposition 3.1 $\mathrm{End}(\mathcal{F}) = \bigoplus_{\sigma \in G} \mathrm{End}(\mathcal{F})_\sigma$, *and this decomposition makes* $\mathrm{End}(\mathcal{F})$ *a* G*-graded algebra.*

We consider

$$\mathrm{END}(\mathcal{F}) = \mathrm{End}(\mathcal{F}) \ + \ \text{the grading from the proposition.}$$

The isomorphism

$$\mathrm{End}(\mathcal{F}) \simeq M(\rho, k)$$

$$E_{ij} \leftrightarrow e_{ij}$$

is a graded algebra isomorphism. Thus

$$\mathrm{END}(\mathcal{F}) \simeq M(\rho, k)$$

and via this isomorphism, $M(\rho, k)$ becomes a G-graded algebra.

Moreover,

any grading on $M(\rho, k)$ that arises from a graded ρ-flag is a good grading.

If $M(\rho, k)$ is an upper blocked triangular matrix algebra, it was shown in [3] that any good grading on $M(\rho, k)$ is of type $\mathrm{END}(\mathcal{F})$.

This doesn't happen for any structural matrix algebra, i.e. not any good grading on a structural matrix algebra is of type $\mathrm{END}(\mathcal{F})$.

We denote

Γ_0 = the set of all vertices of Γ,

Γ_1 = the set of all arrows of Γ,

Γ^u = the un-oriented graph obtained from Γ by omitting the directions of arrows,

$\tilde{\Gamma}$ = the oriented graph obtained from Γ by doubling the arrows

$$\text{if } \alpha \xrightarrow{a} \beta \text{ in } \Gamma, \text{ then } \alpha \underset{a}{\overset{\tilde{a}}{\rightleftarrows}} \beta \text{ in } \tilde{\Gamma}$$

$\mathcal{T}(\Gamma, G)$ = the set of all functions $v : \Gamma_1 \to G$ such that $v(a_1)\dots v(a_r) = v(b_1)\dots v(b_p)$ for any paths $a_1 \dots a_r$ and $b_1 \dots b_p$ in Γ with $s(a_1) = s(b_1)$ and $t(a_r) = t(b_p)$.

If $v \in \mathcal{T}(\Gamma, G)$, we consider $\tilde{v} : \Gamma_1 \cup \{\tilde{a} \mid a \in \Gamma_1\} \to G$

$$\tilde{v}|_{\Gamma_1} = v \text{ and } \tilde{v}(\tilde{a}) = v(a)^{-1} \text{ for any } a \in \Gamma_1.$$

Define

$F(\Gamma)$ = the free group generated by the set Γ_1,

$A(\Gamma)$ = the subgroup of $F(\Gamma)$ generated by all elements of the form $a_1 \dots a_r b_p^{-1} \dots b_1^{-1}$, where $a_1 \dots a_r$ and $b_1 \dots b_p$ are two paths (in Γ) starting from the same vertex and terminating at the same vertex,

$B(\Gamma)$ = the subgroup $B(\Gamma)$ of $F(\Gamma)$ generated by all elements of the form $a_1 a_2^{\varepsilon_2} \dots a_m^{\varepsilon_m}$, where $a_1, \dots, a_m$ are arrows forming in this order a cycle in Γ^u, and $\varepsilon_i = 1$ if a_i is in the direction of the directed cycle given by a_1, and $\varepsilon_i = -1$ otherwise.

We recall: if X is a group and $Y \subseteq X$ is a subgroup,

$$Y^N = < xyx^{-1} \mid x \in X \text{ and } y \in Y > .$$

Y^N is the normal closure of Y, i.e. the smallest normal subgroup of X containing Y.

Proposition 3.2 *Let G be a group. The following are equivalent.*
(1) *Any good G-grading on $M(\rho, k)$ is of type $END(\mathcal{F})$.*
(2) *For any $v \in \mathcal{T}(\Gamma, G)$ there exists a function $f : \Gamma_0 \to G$ such that $v(a) = f(s(a))f(t(a))^{-1}$ for any $a \in \Gamma_1$.*
(3) *For any $v \in \mathcal{T}(\Gamma, G)$ and for any cycle $z_1 ... z_m$ in $\tilde{\Gamma}$, with $z_1, ..., z_m \in \Gamma_1 \cup \{\tilde{a} \mid a \in \Gamma_1\}$ (this corresponds to a cycle in Γ^u), we have that $\tilde{v}(z_1) \dots \tilde{v}(z_m) = 1$.*
(4) *$A(\Gamma)^N = B(\Gamma)^N$.*
(5) *Any generator b of $B(\Gamma)$ can be written in the form $b = g_1 x_1 g_1^{-1} \dots g_m x_m g_m^{-1}$ for some positive integer m, some $g_1, \dots, g_m \in F(\Gamma)$ and some $x_1, \dots, x_m$ among the generators in the construction of $A(\Gamma)$.*

• **Isomorphisms between graded flag endomorphism algebras**

If V and W are G-graded vector spaces and $\sigma \in G$, we say that a linear map $f : V \to W$ is a **morphism of right degree** σ if $f(V_g) \subseteq W_{g\sigma}$ for any $g \in G$. This means that f is a morphism of graded vector spaces when regarded as $f : V \to W(\sigma)$; if $W = \bigoplus_{g\in G} W_g$, then $W(\sigma) = \bigoplus_{g\in G} W_{g\sigma}$ for any $\sigma \in G$.

If $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ is a G-graded ρ-flag and $\sigma \in G$, then the **right suspension of** $\mathcal{F}$ is $\mathcal{F}(\sigma) = (V(\sigma), (V_\alpha)_{\alpha\in\mathcal{C}})$.

Let $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ be a G-graded ρ-flag, with a homogeneous basis $B = \bigcup\limits_{\alpha\in\mathcal{C}} B_\alpha$ of V providing the flag structure.

Let $\mathcal{C} = \mathcal{C}^1 \cup \ldots \cup \mathcal{C}^q$ be the decomposition of $\mathcal{C}$ in disjoint connected components; these correspond to the connected components of the undirected graph Γ^u. For each $1 \le t \le q$, let ρ_t be the preorder relation on the set $\bigcup\limits_{\alpha\in\mathcal{C}^t} \alpha$, by restricting ρ.

If $V^t = \sum\limits_{\alpha\in\mathcal{C}^t} V_\alpha$, then $\mathcal{F}^t = (V^t, (V_\alpha)_{\alpha\in\mathcal{C}^t})$ is a G-graded ρ_t-flag with basis $\bigcup\limits_{\alpha\in\mathcal{C}^t} B_\alpha$.

Obviously, $V = \bigoplus\limits_{1\le t\le q} V^t$. In a formal way we can write $\mathcal{F} = \mathcal{F}^1 \oplus \ldots \oplus \mathcal{F}^q$, where $\mathcal{F}$ is a G-graded ρ-flag, and $\mathcal{F}^t$ is a G-graded ρ_t-flag for each $1 \le t \le q$.

Let ρ and μ be isomorphic preorder relations (i.e. the preordered sets on which ρ and μ are defined are isomorphic). Let $\mathcal{C}$ and $\mathcal{D}$ be the posets associated with ρ and μ, and let $g : \mathcal{C} \to \mathcal{D}$ be an isomorphism of posets.

We say that a ρ-flag $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}}))$ is **g-isomorphic** to a μ-flag $\mathcal{G} = (W, (W_\beta)_{\beta\in\mathcal{D}}))$ if there is a linear isomorphism $u : V \to W$ such that $u(V_\alpha) = W_{g(\alpha)}$ for any $\alpha \in \mathcal{C}$.

If $\mathcal{F}$ and $\mathcal{G}$ are G-graded flags, we say that they are **g-isomorphic as graded flags** if there is such an u which is a morphism of graded vector spaces.

We consider another G-graded ρ-flag $\mathcal{F}' = (V', (V'_\alpha)_{\alpha\in\mathcal{C}})$. As we did for $\mathcal{F}$, we also have $V' = \bigoplus\limits_{1\le t\le q} V'^t$ and $\mathcal{F}' = \mathcal{F}'^1 \oplus \ldots \oplus \mathcal{F}'^q$, where $\mathcal{F}'^t$ is a G-graded ρ_t-flag for each $1 \le t \le q$.

Theorem 3.3 *Let $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ and $\mathcal{F}' = (V', (V'_\alpha)_{\alpha\in\mathcal{C}})$ be G-graded ρ-flags. Then the following assertions are equivalent:*
(1) $\mathrm{END}(\mathcal{F})$ *and* $\mathrm{END}(\mathcal{F}')$ *are isomorphic as G-graded algebras.*
(2) *There exist $g \in \mathrm{Aut}_0(\mathcal{C})$, $\sigma_1, \ldots, \sigma_q \in G$ and a g-isomorphism $\gamma : V \to V'$ between the (ungraded) ρ-flags $\mathcal{F}$ and $\mathcal{F}'$, such that $\gamma_{|V^t}^{|V'^{\overline{g}(t)}} : V^t \to V'^{\overline{g}(t)}$ is a linear isomorphism of right degree σ_t for any $1 \le t \le q$, where $\overline{g} \in S_q$ is the permutation induced by g, i.e. $g(\mathcal{C}^t) = \mathcal{C}^{\overline{g}(t)}$.*
(3) *There exists a permutation $\tau \in S_q$, an isomorphism $g_t : \mathcal{C}^t \to \mathcal{C}^{\tau(t)}$ for each*

$1 \le t \le q$, *and* $\sigma_1, \ldots, \sigma_q \in G$, *such that* $\mathcal{F}^t$ *is* g_t*-isomorphic to* $\mathcal{F}'^{\tau(t)}(\sigma_t)$ *for any* $1 \le t \le q$.

- **Classification of gradings arising from graded flags**

We consider three group actions on the set G^n:

$\mathrm{Aut}_0(\mathcal{C})$ acts to the right on G^n by

$$(h_i)_{1\le i\le n} \leftharpoonup g = (h_{\tilde{g}(i)})_{1\le i\le n}$$

for any $(h_i)_{1\le i\le n} \in G^n$ and $g \in \mathrm{Aut}_0(\mathcal{C})$.

G^q acts to the right on G^n by

$$(h_i)_{1\le i\le n} \leftharpoonup (\sigma_t)_{1\le t\le q} = (h'_i)_{1\le i\le n}$$

where for each i we define $h'_i = h_i \sigma_{p(i)}$, where $p : \{1, \ldots, n\} \to \{1, \ldots, q\}$, $p(i) = j$ such that $\hat{i} \in \mathcal{C}^j$.

For each $\alpha \in \mathcal{C}$ let $S(\alpha)$ be the symmetric group of α (regarded as a subset of $\{1, \ldots, n\}$).

We consider the group $\prod_{\alpha\in\mathcal{C}} S(\alpha)$, which is a Young subgroup of S_n (isomorphic to $\prod_{\alpha\in\mathcal{C}} S_{m_\alpha}$).

Then $\prod_{\alpha\in\mathcal{C}} S(\alpha)$ acts to the right on G^n by

$$(h_i)_{1\le i\le n} \leftharpoonup (\psi_\alpha)_{\alpha\in\mathcal{C}} = (h'_i)_{1\le i\le n}$$

with h'_i defined by $h'_i = h_{\psi_\alpha(i)}$, where $\alpha = \hat{i}$, for each i.

Theorem 3.4 *The isomorphism types of* G*-gradings of the type* $\mathrm{END}(\mathcal{F})$*, where* $\mathcal{F}$ *is a* G*-graded* ρ*-flag, are classified by the orbits of the right action of the group* $\prod_{\alpha\in\mathcal{C}} S(\alpha) \rtimes (\mathrm{Aut}_0(\mathcal{C}) \ltimes G^q)$ *on the set* G^n.

4 Classifying good gradings on $M(\rho, k)$ when ρ is a partial order

Let ρ be a **partial order**. In this case, $\hat{i} = \{i\}$, for any i, so we identify $\hat{i}$ with i. In addition, $\mathrm{Aut}_0(\mathcal{C}) = \mathrm{Aut}(\mathcal{C})$.

- **Automorphisms of structural matrix algebras**

We define $\overline{\rho}$ a transitive relation on $\{1, \ldots, n\}$: $i\overline{\rho}j$ if $i\rho j$ and $i \ne j$. If $i\overline{\rho}j$, the length $\ell([i,j])$ of the interval $[i,j]$ is defined by

$$\ell([i,j]) = \max\{p \mid \text{there exist } i = r_1\overline{\rho}r_2\overline{\rho}\ldots\overline{\rho}r_p = j\} - 1.$$

Proposition 4.1 *Let $L = (\lambda_{ij})_{i\rho j} \in M(\rho, k)$, and assume that L is invertible. Then $\lambda_{ii} \neq 0$ for any i, $L^{-1} \in M(\rho, k)$ and $L^{-1} = (\overline{\lambda}_{ij})_{i\rho j}$, where $\overline{\lambda}_{ii} = \lambda_{ii}^{-1}$ for any i, and*

$$\overline{\lambda}_{ij} = \sum_{p=2}^{\ell([i,j])+1} \sum_{i=r_1 \overline{\rho} r_2 \overline{\rho} \ldots \overline{\rho} r_p = j} (-1)^{p-1} \lambda_{r_1 r_1}^{-1} \ldots \lambda_{r_p r_p}^{-1} \lambda_{r_1 r_2} \ldots \lambda_{r_{p-1} r_p}$$

for any i, j with $i \overline{\rho} j$.

Any algebra automorphism Φ of $M(\rho, k)$ is of the form

$$\Phi(E_{ij}) = a_{ij} \sum_{s \rho \varphi(i), \varphi(j) \rho t} \lambda_{s \varphi(i)} \overline{\lambda}_{\varphi(j) t} E_{st} \tag{1}$$

for any $i \rho j$, where $L = (\lambda_{ij})_{i\rho j} \in U(M(\rho, k))$, $\varphi \in \mathrm{Aut}(\mathcal{C})$ and $(a_{ij})_{i\rho j} \in \mathcal{T}(\rho, k^*)$; here $(\overline{\lambda}_{ij})_{i\rho j} = L^{-1}$, the inverse of L. Using proposition 4.1, the relation (1) can be written in a more detailed manner in terms of L.

• The number of isomorphism types of good gradings for a particular case

Let $\mathcal{T}(\rho, G)$ the set of all transitive G-valued functions on ρ.

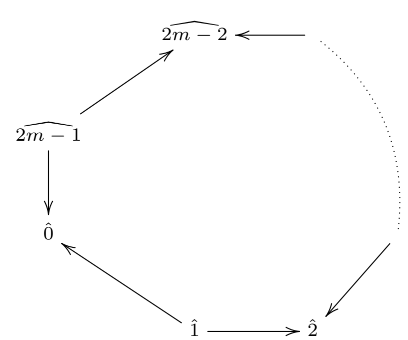
If G is a group, then $\mathrm{Aut}(\mathcal{C})$ acts to the right on $\mathcal{T}(\rho, G)$ by

$$(u_{ij})_{i\rho j} \cdot \varphi = (u_{\varphi(i)\varphi(j)})_{i\rho j}.$$

Proposition 4.2 *Let G be a group. Then the isomorphism types of good G-gradings on $M(\rho, k)$ are in bijection to the orbits of the right action of* $\mathrm{Aut}(\mathcal{C})$ *on $\mathcal{T}(\rho, G)$.*

Example

Let $m \geq 2$ be an integer. For computational reasons the elements of the poset $\mathcal{C}$ are the classes of integers modulo $2m$. We consider the partial order ρ on the set $\mathbb{Z}_{2m}$, such that the associated graph Γ is



Here we have that for any even i both adjacent arrows terminate at $\hat{i}$, while for any odd i both adjacent arrows start from $\hat{i}$.

Let $r, s : \mathbb{Z}_{2m} \to \mathbb{Z}_{2m}$ defined by

$$r(\hat{i}) = \widehat{i+2} \text{ and } s(\hat{i}) = -\hat{i}$$

for any $\hat{i} \in \mathbb{Z}_{2m}$.

$\mathrm{Aut}(\mathcal{C})$ is the subgroup of the symmetric group $S(\mathbb{Z}_{2m})$ of $\mathbb{Z}_{2m}$ generated by r and s.

Since $s^2 = 1, r^m = 1$ and $sr = r^{m-1}s$, we have that:

$$\mathrm{Aut}(\mathcal{C}) =< r, s >= \begin{cases} D_m, \text{ the dihedral group of order } 2m, & \text{if } m \geq 3 \\ \text{the Klein group }, & \text{if } m = 2 \end{cases}.$$

Because there are no distinct paths in Γ that start from the same vertex and finish in the same vertex,

$$\mathcal{T}(\Gamma, G) = \{v : \Gamma_1 \to G\}.$$

Thus the good G-gradings on $M(\rho, k)$ are in bijection with G^{2m}. In other words, we identify

$$v \in \mathcal{T}(\Gamma, G) \longleftrightarrow (g_{\hat{0}}, \dots, g_{\widehat{2m-1}}) \in G^{2m}$$

where $g_{\hat{0}}, \dots, g_{\widehat{2m-1}}$ are the values of v on the arrows of Γ, starting with the one joining $\hat{1}$ and $\hat{0}$, and continuing counterclockwise.

The right action of $\mathrm{Aut}(\rho)$ on $\mathcal{T}(\Gamma, G) \simeq G^{2m}$ is induced by:

$$\begin{aligned} (g_{\hat{0}}, g_{\hat{1}}, \dots, g_{\widehat{2m-1}}) \cdot s &= (g_{\widehat{2m-1}}, g_{\widehat{2m-2}} \dots, g_{\hat{0}}) \\ (g_{\hat{0}}, g_{\hat{1}}, \dots, g_{\widehat{2m-1}}) \cdot r &= (g_{\hat{2}}, g_{\hat{3}}, \dots, g_{\widehat{2m-2}}, g_{\widehat{2m-1}}, g_{\hat{0}}, g_{\hat{1}}). \end{aligned}$$

We assume that G is finite.

Lemma 4.3 *(Burnside)*
Let G be a finite group that acts on a set X. The number of orbits is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

Using proposition 4.2 and Burnside's lemma, the number of types of isomorphisms of good G-gradings on $M(\rho, k)$ (denoted by $N(\rho, G)$) is

$$\begin{aligned} N(\rho, G) &= \frac{1}{|\mathrm{Aut}(\mathcal{C})|} \sum_{\theta \in \mathrm{Aut}(\mathcal{C})} |\mathrm{Fix}(\theta)| \\ &= \frac{1}{2m}(|G|^{2m} + m|G|^m + \sum_{1 \leq i \leq m-1} |G|^{2(i,m)}). \end{aligned}$$

We can rewrite this as

$$N(\rho, G) = \frac{1}{2m}(m|G|^m + \sum_{d|m} \varphi(\frac{m}{d})|G|^{2d}).$$

Bibliography

[1] Yu. A. Bahturin, S. K. Sehgal, M. V. Zaicev, Group gradings on associative algebras, J. Algebra **241** (2001), 677–698.

[2] Yu. A. Bahturin, M. V. Zaicev, Group gradings on matrix algebras, Canad. Math. Bull. **45** (2002), 499–508.

[3] M. Bărăscu, S. Dăscălescu, Good gradings on upper block triangular matrix algebras, Comm. Algebra **41** (2013), 4290-4298.

[4] L. Bemm, E. Z. Fornaroli, E. A. Santulo Jr., A cohomological point of view on gradings on algebras with multiplicative basis, Journal of Pure and Applied Algebra (2018), DOI: 10.1016/j.jpaa.2018.04.019.

[5] F. Beşleagă, S. Dăscălescu, Structural matrix algebras, generalized flags and gradings, arxiv: RA/1802.03427.

[6] F. Beşleagă, S. Dăscălescu, L. Van Wyk, Classifying good gradings on structural matrix algebras, Linear and Multilinear Algebra (2018), DOI: 10.1080/03081087.2018.1476447.

[7] C. Boboc, Superalgebra structures on $M_2(k)$ in characteristic 2, Comm. Algebra **30** (2002), 255-260.

[8] C. Boboc, Gradings of $M_2(k)$ by the Klein group, Comm. Algebra **31** (2003), 2311-2326.

[9] C. Boboc, S. Dăscălescu, Gradings of matrix algebras by cyclic groups, Comm. Algebra **29** (2001), 5013-5021.

[10] C. Boboc, S. Dăscălescu, Group gradings on $M_3(k)$, Comm. Algebra **35** (2007), 2654-2670.

[11] A. Borovik, Mathematics under the microscope, American Mathematical Society Publication, Providence, 2010.

[12] N. Bourbaki, Elements of the history of mathematics, (1994) Springer Verlag.

[13] S. Caenepeel, S. Dăscălescu, C. Năstăsescu, On Gradings of Matrix Algebras and Descent Theory, Comm. Algebra **30** (2002), 5901–5920.

[14] W. Chin, D. Quinn, Rings graded by polycyclic-by-finite groups, Proc. Amer. Math. Soc. **102** (1988), 235-241.

[15] S. P. Coelho, The automorphism group of a structural matrix algebra, Linear Alg. Appl. **195** (1993), 35-58.

[16] M. Cohen, S. Montgomery, Group-graded rings, smash products and group actions, Trans. Amer. Math. Soc. **282** (1984), 237-258.

[17] E. C. Dade, Group-graded rings and modules, Math. Z. **174** (1980), 241-262.

[18] E. C. Dade, Clifford theory for group-graded rings, J. Reine Angew. Math. **369** (1986), 40-86.

[19] E. C. Dade, Clifford theory for group-graded rings II, J. Reine Angew. Math. **387** (1988), 148-181.

[20] S. Dăscălescu, Group gradings on diagonal algebras, Arch. Math. **91** (2008), 212-217.

[21] S. Dăscălescu, B. Ion, C. Năstăsescu and J. Rios Montes, Group gradings on full matrix rings, J. Algebra **220** (1999), 709-728.

[22] S. Dăscălescu, L. van Wyk, Do isomorphic structural matrix rings have isomorphic graphs?, Proc. Amer. Math. Soc. **124** (1996), 1385-1391.

[23] O. M. Di Vincenzo, P. Koshlukov, A. Valenti, Gradings on the algebra of upper triangular matrices and their graded identities, J. Algebra **275** (2004), 550-566.

[24] A. Elduque, Gradings on octonions, J. Algebra **207** (1998), 342-354.

[25] S. Foldes, J. Szigeti, L. van Wyk, Invertibility and Dedekind finiteness in structural matrix rings, Linear Multilinear Algebra **59** (2011), 221-227.

[26] E. L. Green, Graphs with relations, coverings and group-graded algebras, Thins. Amer. Math. Soc. **279** (1983), 297-310.

[27] E. L. Green, E. M. Marcos, Graded quotients of path algebras: A local theory, Journal of Pure and Applied Algebra **93** (1994), 195-226.

[28] R. Hazrat, Graded Rings and Graded Grothendieck Groups, University of Western Sydney Australia, 2015.

[29] I. L. Kantor, Some generalizations of Jordan algebras, Trudy Sem. Vect. Tenzorn. Anal. **16** (1972), 407-499.

[30] A. V. Kelarev, Applications of epigroups to graded ring theory, Semigroup Forum **50** (1995), 327-350.

[31] A. V. Kelarev, Directed graphs and Lie superalgebras of matrices. J. Algebra **285** (2005) 1-10.

[32] A. Kemer, Solution of the problem as to whether associative algebras have a finite basis of identities, Dokl. Akad. Nauk SSSR **298** (1988), 273-277; translation in Soviet Math. Dokl. **37** (1988), 60-64.

[33] R. Khazal, C. Boboc, S. Dăscălescu, Group gradings of $M_2(k)$, Bull. Austral. Math. Soc. **68** (2003), 285-293.

[34] M. A. Knus, Algebras graded by a group. Category Theory, Homology Theory Appl., Proc. Conf. Seattle, Res. Center Baltelle Mem. Inst. **2** (1969), 117-133.

[35] W. Krull, Dimensionstheorie in Stellenringe, J. de Grelle, **v. CLXXIX** (1938), pp. 204-226.

[36] V. Lakshmibai, J. Brown, Flag varieties. An interplay of geometry, combinatorics and representation theory, Hindustan Book Agency, 2009.

[37] C. Năstăsescu, Anneaux et modules gradues, Rev. Roum. Math. Pures Appl. **21** (1976), 21-41.

[38] C. Năstăsescu, F. van Oystaeyen, Graded and filtered rings and modules, Lecture Notes in Mathematics **758** (1979), Springer verlag, Berlin.

[39] C. Năstăsescu, F. van Oystaeyen, Graded ring theory, North Holland, Mathematical Library **28** (1982).

[40] C. Năstăsescu, F. van Oystaeyen, Methods of graded rings, Lecture Notes in Math., vol. 1836 (2004), Springer Verlag.

[41] A. Nowicki, Derivations of special subrings of matrix rings and regular graphs, Tsukuba J. Math. **7** (1983), 281-297.

[42] M. M. Parmenter, E. Spiegel, Central group gradings of incidence algebras, Algebra Colloquium **11** (2004), 421-426.

[43] Pratibha, R. K. Mishra, R. Mohan, A report on graded rings and graded modules, Global Journal of Pure and Applied Mathematics **9** (2017), 6827-6853.

[44] A. Ramos, D. Diniz, Graded identities and isomorphisms on algebras of upper block-triangular matrices, arXiv:1803.06949v1.

[45] A. Ramos, C. Fidelis, D. Diniz, Graded isomorphisms on upper block triangular matrix algebras, Linear Algebra and its Applications **543** (2018), 92-105.

[46] S. K. Sehgal, M. V. Zaicev, Finite gradings on simple artinian rings, Vestn. Mosk. Univ., Matem., Mechan. **3** (2001), 21-24.

[47] O. N. Smirnov, Simple associative algebras with finite $\mathbb{Z}$-grading, J. Algebra **196** (1997), 171-184.

[48] O. N. Smirnov, Finite $\mathbb{Z}$-grading on Lie algebras and symplectic involution, J. Algebra **218** (1999), 246-275.

[49] W. Specht, Gesetze in Ringen, I. Math. Z. **52** (1950), 557-589.

[50] E. Spiegel, C. J. O'Donnell, Incidence algebras, Pure and Appl. Math. **206** (1997), Marcel Dekker, New York.

[51] J. Szigeti, L. van Wyk, Subrings which are closed with respect to taking the inverse, J. Algebra **318** (2007), 1068-1076.

[52] A. Valenti, M. V. Zaicev, Group gradings on upper triangular matrices. Arch. Math. **89** (2007), 33-40.

[53] A. Valenti, M. Zaicev, Abelian gradings on upper block triangular matrices, Canad. Math. Bull. **55** (2012), 208-213.

[54] L. Van Wyk, Maximal left ideals in structural matrix rings, Comm. Algebra **16** (1988), 399-419.

[55] L. Van Wyk, Special radicals in structural matrix rings, Comm. Algebra **16** (1988), 421-435.