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Extended English Abstract

Contributions to proof mining

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Extended English Abstract

General presentation of the field

The interpretative flavour of proof theory originally arose from the same motivations that drove, since David Hilbert, the very study of proofs as objects in themselves: showing, by acceptable means, the consistency of logical systems that can be thought to act as a workable foundation to modern mathematics. This goal had already been stifled in the 1930s by Kurt Gödel's second incompleteness theorem, which indicated that those means could not be a subset of the same logical system which was under investigation, and then somewhat salvaged by Gerhard Gentzen's work, which indicated that the non-finitary methods necessarily playing a role in the consistency proof of first-order Peano arithmetic could be limited to the schema of primitive recursive induction up to a large countable ordinal. Gödel himself later introduced [25] the alternative method, his functional or *Dialectica* interpretation (named after the journal it was published in), of proving the consistency of arithmetic, which works by translating the proofs of intuitionistic arithmetic into a quantifier-free, higher-typed calculus of his own devising, dubbed System T, whose consistency is *a priori* less doubtful, arguably, than that of the source theory.

Gentzen had identified in his papers three levels of the use of infinity in mathematics – arithmetic, analysis and set theory – and he had already started preliminary work on the consistency of the next level, analysis (in the form exhibited in the monograph of Hilbert and Bernays [30] and equivalent to its modern formulation as a two-sorted, first-order theory known as "second-order arithmetic"), before his untimely death in 1945. A non-trivial, interpretative solution to the problem was first proposed by another who came to meet an early death, namely Spector [78], who augmented Gödel's system with new constants denoting bar-recursive functionals (representing an extension to higher types of Brouwer's bar induction principle, which had already been suggested by Gödel as a possible way forward). However, these additional functionals were not nearly as intuitively sound as pure System T (and a second solution later conceived by Girard [24] using yet another extension called System F suffered from similar shortcomings). Georg Kreisel, who had recently introduced another proof interpretation for intuitionistic arithmetic called modified realizability [52], an extension of an earlier work of Kleene, convened a seminar on the foundations of analysis at Stanford in the summer of 1963 with the goal of finding a justification for Spector's functionals that would be acceptable on constructive grounds. Unfortunately, the seminar concluded by declaring the answer to be "negative by a wide margin" [53].

It was Kreisel, though, that foresaw an entirely different way of looking at these proof interpretations. He proposed that instead of considering them simply a destructive instrument that translates a hypothetical proof of contradiction inside a celebrated system to an almost impossible proof in a more reliable one, we should focus instead on translating existing proofs in mathematics, carrying the hope that the result of the translation will also contain more information (for example, witnesses on existentially quantified variables) than the original proof. That way, one could hope to answer the following question [51]:

> *"What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"*

which became the driving force behind a research program proposed by Kreisel under the name "unwinding of proofs". In the following decades (which would, however, see some seminal research and expository work in "pure" interpretative proof theory [66, 81]) only sporadic advances would be made, one of the most significant being H. Luckhardt's 1989 analysis [67] of the proof of Roth's theorem on diophantine approximations.

In the early 1990s, Luckhardt's student Ulrich Kohlenbach devised the "monotone" variants of both modified realizability and Gödel's *Dialectica* [38, 39], new proof interpretations that could only extract bounds instead of full witnesses, gaining instead the power of accepting more commonly used proof principles like the weak König lemma as additional axioms (or similarly bounded universal-existential sentences). This triggered a complete overhaul of Kreisel's program, which was renamed 'proof mining' (a name originally suggested by D. Scott), and which quickly led to quantitative results being obtained in the nonlinear analysis of separable spaces, particularly in approximation theory. The next major step was taken in the early 2000s, when Kohlenbach and his collaborators, including Laurenţiu Leuştean, started to analyze proofs in functional analysis in the context of abstract (metric) spaces [40, 44]. This culminated into a series of 'general logical metatheorems' (developed by Kohlenbach [41] and by Gerhardy and Kohlenbach [22, 23]) for both classical and semi-intuitionistic systems of higher-order arithmetic (appropriately modified in order to be able to tackle the abstract spaces used in nonlinear analysis), of proof-theoretic strength less than or comparable to classical analysis, which detail the circumstances in which bounds may be effectively extracted. Kohlenbach's monograph from 2008 [42] covers the major results in the field until then, while a survey of recent developments is [43].

This thesis positions itself firmly within this continuing mission of unwinding proofs in abstract nonlinear analysis, and the contributions to proof mining that give it its title are situated mainly in the fields of nonlinear functional analysis and convex optimization. The results that are analyzed typically feature the asymptotic behaviour of some sequence in a metric space.

Studying pseudocontractions

The results of this section can be found in the papers [75, 76, 60].

The class of k-strict pseudocontractions was introduced by Browder and Petryshyn in [12] for Hilbert spaces. If H is a Hilbert space, $C \subseteq H$ is a convex subset and $k \in [0, 1)$, then a mapping $T : C \to H$ is called a **k-strict pseudocontraction** if for all $x, y \in C$ we have that:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2. \qquad (1)$$

If we set $k := 0$ in the above, we obtain the condition $\|Tx - Ty\| \leq \|x - y\|$, which states that the mapping T is nonexpansive. The search of algorithms for finding fixed points of nonexpansive self-mappings of subsets of metric spaces belonging to various established classes has been a longstanding research program. In the sequel, we will restrict ourselves to classes of Banach spaces. We denote the unit sphere of a Banach space E by $S(E)$. A Banach space E is called **smooth** if for any $u \in S(E)$, its norm is Gâteaux differentiable at u, i.e. for any $v \in S(E)$, the limit

$$\lim_{h \to 0} \frac{\|u + hv\| - \|u\|}{h}$$

exists. This condition has been proven to be equivalent to the fact that the normalized duality mapping of the space, $J : E \to 2^{E^*}$, is single-valued – and we shall denote its unique section by $j : E \to E^*$. Therefore, for all $x \in E$, $j(x)(x) = \|x\|^2$ and $\|j(x)\| = \|x\|$. Moreover, E has a **Fréchet differentiable norm** if, in addition, the limit above is attained uniformly in the variable $v \in S(E)$ and it is **uniformly smooth** (or has a **uniformly Fréchet differentiable norm**) if the limit is attained uniformly in the pair of variables $(u, v) \in S(E) \times S(E)$. One can also define the **modulus of smoothness** of E to be the map $\rho_E : (0, \infty) \to \mathbb{R}$, defined, for all $\tau \in (0, \infty)$, by

$$\rho_E(\tau) := \sup \left\{ \frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 \;\middle|\; u, v \in S(E) \right\}.$$

It is known that a space E is uniformly smooth iff

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

This can happen, for example, if there are $c > 0$ and $q > 1$ such that for all τ, $\rho_E(\tau) \leq c\tau^q$. In that case, E is said to be **q-uniformly smooth**.

We define now the **modulus of convexity** of a space E to be the map $\delta_E : [0, 2] \to \mathbb{R}_+$, defined, for all $\varepsilon \in [0, 2]$, by

$$\delta_E(\varepsilon) := \inf \left\{ 1 - \frac{\|x + y\|}{2} \;\middle|\; x, y \in S(E), \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is called **uniformly convex** iff for all $\varepsilon > 0$, $\delta_E(\varepsilon) > 0$ – or, equivalently, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $u, v \in S(E)$ such that $\|u - v\| \geq \varepsilon$ we have that $\frac{\|u+v\|}{2} \leq 1 - \delta$. If $\eta : \mathbb{R} \to \mathbb{R}$ is a function such that for any $\varepsilon > 0$, $\eta(\varepsilon)$ is such a δ, we call η a **valid modulus of uniform convexity** for the space. We remark that the modulus of convexity from above is, in a sense, the "optimal" valid modulus of uniform convexity.

The algorithms mentioned above are usually iterative in nature – a typical example is the **Mann iteration** associated to such a self-mapping $T : C \to C$, an initial point $x \in C$ and a sequence $(t_n)_{n \in \mathbb{N}} \subseteq (0, 1)$, which is the sequence $(x_n)_{n \in \mathbb{N}} \subseteq C$ defined[1] by:

$$x_0 := x$$

$$x_{n+1} := t_n T x_n + (1 - t_n) x_n$$

A result on the Mann iteration for nonexpansive mappings is the following theorem of Reich, which will be needed in the sequel.

[1] We note that for the definition to make sense, C should be presupposed to be convex.

Theorem 1 (Reich (1979), [72, Theorem 2])**.** *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, $C \subseteq E$ a convex, closed set and $T : C \to C$. Suppose that T is nonexpansive with $Fix(T) \neq \emptyset$. Let $x \in C$ and $(t_n)_{n\in\mathbb{N}} \subseteq (0, 1)$ such that*

$$\sum_{n=0}^{\infty} t_n(1 - t_n) = \infty$$

Then the Mann iteration corresponding to T, x and $(t_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

The class of k-strict pseudocontractions has been less readily amenable to classical iterative schemas. Still, in 2007, Marino and Xu proved[2] the following theorem.

Theorem 2 (Marino and Xu (2007), [69, Theorem 3.1])**.** *Let H be a Hilbert space, $C \subseteq H$ a convex, closed set and $T : C \to C$. Let k be in $(0, 1)$ and suppose that T is a k-strict pseudocontraction with $Fix(T) \neq \emptyset$. Let $x \in C$ and $(t_n)_{n\in\mathbb{N}} \subseteq (0, 1 - k)$ such that*

$$\sum_{n=0}^{\infty} t_n(1 - k - t_n) = \infty$$

Then the Mann iteration corresponding to T, x and $(t_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

Marino and Xu asked in their paper whether this result can be generalized to uniformly convex Banach space with a Fréchet differentiable norm, in the same vein as Reich's. Since then, various authors have tried to solve this problem to some degree.

The first issue that arises is that of the proper generalization of k-strict pseudocontractions to the case of Banach spaces. The solution (used, for example, in [15, 84]) comes from the initial observation ([12, Theorem 1.(2)]) that condition (1) is equivalent to the following:

$$\langle (x - Tx) - (y - Ty), x - y\rangle \geq \frac{1-k}{2}\|(x - Tx) - (y - Ty)\|^2.$$

Now, if E is a smooth Banach space, as we said above, it admits a single-valued normalized duality mapping $j : E \to E^*$. Hence the natural extension of the condition above in this framework is the following one, which we will take as our official definition of k-strict pseudocontractions in Banach spaces:

$$j(x - y)((x - Tx) - (y - Ty)) \geq \frac{1-k}{2}\|(x - Tx) - (y - Ty)\|^2.$$

A significant advance in this direction was made by Zhou in 2014, who, using a variant[3] of a function previously defined by Cholamjiak and Suantai – namely, for any Banach space E with a Fréchet differentiable norm, one can define the function $\beta^*_E : E \times (0, \infty) \to \mathbb{R}$ by:

$$\beta^*_E(x, t) := \sup\left\{ \frac{\|x + tv\|^2 - \|x\|^2}{t} - 2j(x)(v) \;\middle|\; v \in S(E)\right\},$$

[2] Marino and Xu use the notation $\alpha_n := 1 - t_n \in (k, 1)$, so that the condition becomes $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$. We have chosen to present it as above in order to maintain uniformity with Reich's approach.

[3] The original definition in [15, Lemma 3.2] was:

$$\beta^*_E(x, t) := \sup\left\{ \left|\frac{\|x + tv\|^2 - \|x\|^2}{t} - 2j(x)(v)\right| \;\middle|\; v \in S(E)\right\}.$$

which would make Zhou's condition (2) unnecessarily stronger.

for any $x \in E$ and $t \in (0, \infty)$ – proved the following result, which is implicitly contained in [84, Theorem 3.1].

Theorem 3 (Zhou (2014), [84])**.** *Let E be a uniformly convex Banach space which is also uniformly smooth, $C \subseteq E$ a convex, closed set and $T : C \to C$. Let $d \in [1, \infty)$ be such that for any $x \in E$ and $t \in (0, \infty)$,*

$$\beta_E^*(x, t) \leq dt. \tag{2}$$

Let k be in $(0, 1)$ and suppose that T is a k-strict pseudocontraction with $Fix(T) \neq \emptyset$. Let $x \in C$ and $(t_n)_{n\in\mathbb{N}} \subseteq (0, \frac{1-k}{2d})$ such that

$$\sum_{n=0}^{\infty} t_n = \infty.$$

Then the Mann iteration corresponding to T, x and $(t_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

Our aim is to show that condition (2) above is actually equivalent to 2-uniform smoothness and then to give simpler and immediate proofs of both Theorem 2 and Theorem 3.

By convention, we will set for any Banach space E and any $x \in E$, $\beta_E^*(x, 0) := 0$. We first note that for any Hilbert space H, any $x \in H$ and any $t \geq 0$, $\beta_H^*(x, t) = t$. Indeed, in that case, for any $x \in H$, $t > 0$ and $v \in S(H)$, we have that:

$$\frac{\|x + tv\|^2 - \|x\|^2}{t} - 2j(x)(v) = \frac{\|x + tv\|^2 - \|x\|^2 - 2\langle x, tv\rangle}{t} = \frac{\|x + tv - x\|^2}{t} = t.$$

The characterization lemma is the following.

Lemma 4. *Let E be a smooth Banach space. The following statements are equivalent:*

1. *E is 2-uniformly smooth, i.e. there is a $c > 0$ such that for all τ, $\rho_E(\tau) \leq c\tau^2$;*
2. *there is a $d > 0$ such that for all $x, y \in E$ we have that $\|x + y\|^2 \leq \|x\|^2 + 2j(x)(y) + d\|y\|^2$;*
3. *there is a $d > 0$ such that for all $x \in E$ and $t \geq 0$ we have that $\beta_E^*(x, t) \leq dt$ (i.e., Zhou's condition (2)).*

Moreover, the constants in (ii) and (iii) may be taken to be the same (and hence we have used the same designator).

We have therefore established the equivalence of Zhou's condition with 2-uniform smoothness, which was only mentioned as a special case in [84, p. 762]. We note that it is immediate that if d satisfies the two equivalent conditions, and $d \leq d'$, then d' also satisfies the condition. Hence we can always take $d \geq 1$. In addition, we have derived a way to compute the constant d – and that specific choice of d for a given c (whose value will be seen to be already greater than 1) will be denoted by d_c and is given by the following:

$$d_c := k_2^{-1} = \frac{8}{\min\left(\frac{1}{16c}, 2 - \sqrt{2}\right)}. \tag{3}$$

We note that this bound is by no means an optimal one – we saw that for a Hilbert space one can simply take $d := 1$, whereas the formula would give $d_c := 64$ (using $c := \frac{1}{2}$, taken from the usual modulus of smoothness $\rho(\tau) := \sqrt{1+\tau^2} - 1 \leq \frac{\tau^2}{2}$). Still, the above argument shows that there is a simple method one can use to readily obtain a suitable $d \geq 1$ given the original smoothness constant c.

The other ingredient of our result is the following generalization of [12, Theorem 2]. For a given self-mapping of a convex set, $T : C \to C$, and a $t \in (0,1)$, set $T_t := tT + (1-t)id_C$ – that is, for all $x \in C$, $T_t x = tTx + (1-t)x$. It is immediate that for all $t_1, t_2 \in (0,1)$, $(T_{t_1})_{t_2} = T_{t_1 \cdot t_2}$. Also note that, for any t, T and T_t have the same fixed points.

Lemma 5. *Let E be a Banach space, $C \subseteq E$ a convex subset and $d \geq 1$ such that for any $x \in E$ and $t \leq 0$, $\beta_E^*(x,t) \leq dt$. Let $k \in (0,1)$ and $T : C \to C$ a k-strict pseudocontraction.*

Let $t \in (0, \frac{1-k}{d}]$. Then T_t is nonexpansive. (In particular, $T_{\frac{1-k}{d}}$ is nonexpansive.)

Now we can prove the main convergence result.

Theorem 6. *Let E be a uniformly convex Banach space which is also 2-uniformly smooth, $C \subseteq E$ a convex, closed set and $T : C \to C$. Let, therefore, $d \geq 1$ be a constant satisfying conditions (ii) and (iii) from Lemma 4 (if, for example, $\rho_E(\tau) \leq c\tau^2$, for all τ, take $d := d_c$). Let k be in $(0,1)$ and suppose that T is a k-strict pseudocontraction with $Fix(T) \neq \emptyset$. Let $x \in C$ and $(t_n)_{n\in\mathbb{N}} \subseteq (0, \frac{1-k}{d})$ such that*

$$\sum_{n=0}^{\infty} t_n \left(\frac{1-k}{d} - t_n \right) = \infty$$

Then the Mann iteration corresponding to T, x and $(t_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of T.

Proof. By Lemma 5, we have that $T_{\frac{1-k}{d}}$ is nonexpansive. For every $n \geq 0$, set $t'_n := t_n \cdot \frac{d}{1-k}$. Denote by $(x_n)_{n\in\mathbb{N}}$ the Mann iteration corresponding to T, x and $(t_n)_{n\in\mathbb{N}}$. Let $n \geq 0$. We have that:

$$\begin{aligned} x_{n+1} &= t_n T x_n + (1-t_n) x_n \\ &= T_{t_n} x_n \\ &= T_{t'_n \cdot \frac{1-k}{d}} x_n \\ &= T_{t'_n}(T_{\frac{1-k}{d}} x_n) \\ &= t'_n T_{\frac{1-k}{d}} x_n + (1-t'_n) x_n. \end{aligned}$$

We have then, that $(x_n)_{n\in\mathbb{N}}$ is the Mann iteration corresponding to $T_{\frac{1-k}{d}}$, x and $(t'_n)_{n\in\mathbb{N}}$. We seek to apply Theorem 1. For that we do the following verification:

$$\sum_{n=0}^{\infty} t'_n(1-t'_n) = \sum_{n=0}^{\infty} t_n \cdot \frac{d}{1-k} \left(1 - t_n \cdot \frac{d}{1-k} \right) = \left(\frac{d}{1-k} \right)^2 \sum_{n=0}^{\infty} t_n \left(\frac{1-k}{d} - t_n \right) = \infty.$$

We therefore get that $(x_n)_{n\in\mathbb{N}}$ weakly converges to a fixed point of $T_{\frac{1-k}{d}}$, which is also a fixed point of T. ∎

Proof of Theorem 3. Note that the hypothesis states that $(t_n)_{n\in\mathbb{N}} \subseteq (0, \frac{1-k}{2d})$. Then, for all n, $\frac{1-k}{d} - t_n \geq \frac{1-k}{2d}$, so:

$$\sum_{n=0}^{\infty} t_n \left(\frac{1-k}{d} - t_n\right) \geq \frac{1-k}{2d} \sum_{n=0}^{\infty} t_n = \infty.$$

We are therefore in the hypothesis of Theorem 6. □

Proof of Theorem 2. We have shown in the beginning of this section that one can take for a Hilbert space $d := 1$. The conclusion immediately follows. □

An area of investigation closely related to the kind of iterative algorithms mentioned above has been the problem of finding a common fixed point of a (finite or infinite) family $(T_i)_i$ of self-mappings of a subset C. An iterative scheme that is useful in the case of a finite family $(T_i)_{1\leq i\leq N}$ is the *parallel algorithm*, defined as follows. Let x be in C and $(t_n)_{n\in\mathbb{N}} \subseteq (0,1)$. For each $i \in \{1,\ldots,N\}$, let $(\lambda_i^{(n)})_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that, for any $n \in \mathbb{N}$:

$$\sum_{i=1}^{N} \lambda_i^{(n)} = 1.$$

Write, for all $n \geq 0$:

$$A_n := \sum_{i=1}^{N} \lambda_i^{(n)} T_i.$$

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by:

$$x_0 := x,$$

$$x_{n+1} := t_n x_n + (1-t_n) A_n x_n.$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is the output of the parallel algorithm associated with the inputs T, x, (t_n) and $(\lambda_i^{(n)})$.

Two remarks are in order here. Firstly, we see that the case $N = 1$ represents the well-known Mann iteration for finding a fixed point of a self-mapping and therefore, all the results pertaining to this algorithm immediately transfer to the case of a single mapping (i.e. the one treated in [33]). Secondly, we note that there exists another (equivalent) convention when working with Mann-like algorithms, pairing the t_n with the application of the appropriate mapping, i.e. the formula above would be:

$$x_{n+1} := t_n A_n x_n + (1-t_n) x_n$$

We use the "$t_n x_n$" convention, in the description of the parallel algorithm and also further below, when formalizing the passage from nonexpansive to strictly pseudocontractive mappings, as it is the one used in [69, 65]. One should be careful to check the convention used when comparing different hypotheses and convergence results.

When considering algorithms for finite families such as the one above, the intermediate result obtained during the proof of the convergence theorem will be still one of "asymptotic regularity", though one pertaining to the map(s) constructed as a byproduct of the algorithm (here, the A_n's), i.e. that:

$$\lim_{n\to\infty} \|x_n - A_n x_n\| = 0.$$

Given that A_n varies with n, such a result does not mean *a priori* that $(x_n)_n$ is an approximate fixed point sequence for any mapping – certainly not one given by the problem data. Therefore, what is actually relevant to the proof mining program is an asymptotic regularity related to the relevant mappings of the problem – that is, the T_i's. One might look for an associated rate of convergence for the statements:

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0,$$

for each $i \in \{1, ..., N\}$. A concrete extraction of such a rate can be found, for example, in [37], for the Kuhfittig iteration.

López-Acedo and Xu, in 2007, have found sufficient conditions so that the parallel algorithm weakly converges to a fixed point of a finite family of strictly pseudocontractive mappings. Their result, using the notations introduced above, is expressed as follows.

Theorem 7 (López-Acedo & Xu (2007), [65, Theorem 3.3])**.** *Let H be a Hilbert space and $C \subseteq H$ a closed, convex set. Let $N \geq 1$, $(T_i : C \to C)_{1\leq i\leq N}$ a family of mappings and $(k_i)_{1\leq i\leq N} \subseteq (0, 1)$ such that each T_i is a k_i-strict pseudocontraction. Suppose that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Set $k := \max_{1\leq i\leq N} k_i$. Let x be in C, $(t_n)_{n\in\mathbb{N}} \subseteq [k, 1]$ be such that*

$$\sum_{n=0}^{\infty}(t_n - k)(1 - t_n) = \infty.$$

Impose the conditions

$$\inf_{i,n} \lambda_i^{(n)} > 0$$

and

$$\sum_{j=0}^{\infty}\sqrt{\sum_{i=1}^{N}|\lambda_i^{(j+1)} - \lambda_i^{(j)}|} < \infty$$

on $(\lambda_i^{(n)})$. Then the the parallel algorithm associated with the inputs T, x, (t_n) and $(\lambda_i^{(n)})$ weakly converges to a common fixed point of the family $(T_i : C \to C)_{1\leq i\leq N}$.

Our main result here will be to obtain rates of asymptotic regularity for this instance of the parallel algorithm – that is, a rate of convergence for each sequence $(\|x_n - T_i x_n\|)_{n\in\mathbb{N}}$, with the sequence $(x_n)_{n\in\mathbb{N}}$ being defined as before.

We will take $\theta : \mathbb{N} \to \mathbb{N}$ be a rate of divergence for the series

$$\sum_{n=0}^{\infty}(t_n - k)(1 - t_n),$$

i.e., for all $N \in \mathbb{N}$,

$$\sum_{n=0}^{\theta(N)}(t_n - k)(1 - t_n) \geq N.$$

and $\gamma : (0, \infty) \to \mathbb{N}$ be a Cauchy modulus for the series

$$\sum_{j=0}^{\infty}\sum_{i=1}^{N}|\lambda_i^{(j+1)} - \lambda_i^{(j)}|,$$

i.e., for all $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\sum_{j=\gamma(\varepsilon)+1}^{\gamma(\varepsilon)+n} \sum_{i=1}^{N} |\lambda_i^{(j+1)} - \lambda_i^{(j)}| \leq \varepsilon.$$

We will assume that there is a $b > 0$ and a $p \in \bigcap_{i=1}^{N} Fix(T_i)$ such that $\|x\| \leq b$ and $\|x - p\| \leq b$. Let $a > 0$ be such that

$$a \leq \inf_{i,n} \lambda_i^{(n)}.$$

Note that if $0 \leq k \leq k' < 1$ and T is a k-strict pseudocontraction, then it is also a k'-strict pseudocontraction. Hence, instead of considering each T_i to be a k_i-strict pseudocontraction, we can take k to be the maximum of the N k_i's and work, without loss of generality, with a finite family of k-strict pseudocontractions.

The result is then expressed as follows:

Theorem 8 ("general rate of T_i-asymptotic regularity")**.** *Set:*

$$\begin{aligned}
\Delta_{b,\theta}(\varepsilon, m) :=& \theta\left(m + \left\lceil \frac{b^2}{\varepsilon^2} \right\rceil\right). \\
\Phi_{b,\theta,\gamma}(\varepsilon) :=& \ \Delta_{b,\theta}\left(\frac{\varepsilon}{2}, \gamma\left(\frac{\varepsilon}{6b}\right) + 1\right) \\
=& \ \theta\left(\gamma\left(\frac{\varepsilon}{6b}\right) + \left\lceil \frac{4b^2}{\varepsilon^2} \right\rceil + 1\right). \\
P_{a,b}(\varepsilon) :=& \min\left\{\frac{\varepsilon}{2}, \sqrt{\frac{a\varepsilon^2}{4(1-a)} + b^2} - b\right\} \\
\Phi'_{a,b,\theta,\gamma}(\varepsilon) :=& \ \Phi_{b,\theta,\gamma}(P_{a,b}(\varepsilon)). \\
\Phi''_{a,b,k,\theta,\gamma}(\varepsilon) :=& \ \Phi'_{a,b,\theta,\gamma}((1-k)\varepsilon).
\end{aligned}$$

Then for all $\varepsilon > 0$ we have that for all $n \geq \Phi''_{a,b,k,\theta,\gamma}(\varepsilon)$ and all i, $\|x_n - T_i x_n\| \leq \varepsilon$.

Efforts to extend the Mann iteration to Lipschitzian pseudo-contractions were not successful. Later, Chidume and Mutangadura [14] would exhibit an example of a Lipschitzian pseudo-contractive map with a unique fixed point for which no Mann sequence converges.

We recall that T is said to be *L-Lipschitzian* (for an $L > 0$) if for all $x, y \in C$ we have that $\|Tx - Ty\| \leq L\|x - y\|$. Examples of Lipschitzian pseudo-contractions are the strict pseudo-contractions defined above, hence, in particular, nonexpansive mappings.

Meanwhile, some alternate algorithms were proposed, the first of which being the one of Ishikawa [32], who deployed it successfully in the case of Lipschitzian pseudo-contractions acting on a compact convex subset of a Hilbert space. It is defined as follows.

If $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ are sequences in $[0, 1]$, then the *Ishikawa iteration* starting with an $x \in C$ using the two sequences as weights is defined by:

$$x_0 := x, \quad x_{n+1} := \alpha_n T(\beta_n T x_n + (1 - \beta_n)x_n) + (1 - \alpha_n)x_n. \tag{4}$$

We recognize the Mann iteration in the special case where $\beta_n := 0$ for all $n \in \mathbb{N}$.

We introduce the following conditions that sequences (α_n), (β_n) in $[0,1]$ may satisfy:

$$\begin{aligned}
&\text{(A1)} \qquad \lim_{n\to\infty} \beta_n = 0;\\
&\text{(A2)} \qquad \sum_{n=0}^{\infty} \alpha_n\beta_n = \infty;\\
&\text{(A3)} \quad \alpha_n \le \beta_n, \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

As pointed out in [32], an example of a pair of sequences satisfying all three conditions is $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}}$.

We can now state the exact form of Ishikawa's 1974 strong convergence result for the above iteration.

Theorem 9. *Let H be a Hilbert space, $C \subseteq H$ a nonempty convex compact subset, $T : C \to C$ a Lipschitzian pseudo-contraction and (α_n), (β_n) sequences in $[0,1]$ that satisfy (A1)-(A3). Then, for all $x \in C$, the Ishikawa iteration starting with x, using (α_n) and (β_n) as weights, converges strongly to a fixed point of T.*

When analysing from the viewpoint of proof mining Ishikawa's above result, whose conclusion states that a sequence converges, a quantitative version would be a rate of convergence that computes the corresponding N_ε given the ε and perhaps some additional parameters. However, the high logical complexity of the definition of convergence makes it intractable for proofs that involve some notion of excluded middle, as it is the case here. Therefore, an equivalent formulation (identifiable in logic as its Herbrand normal form) introduced in this case by Tao [79, 80] under the name of *metastability*, is used in its stead. The following sentence expresses the metastability of a given sequence (x_n) in a normed space:

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i,j \in [N, N+g(N)] \quad \left(\|x_i - x_j\| \le \frac{1}{k+1} \right).$$

One can immediately glimpse the reduced complexity of this statement: no unbounded universal quantifier occurs after the existential one (as it clearly does in the usual formulations of convergence or Cauchyness). It is a simple exercise, however, to check that the sentence is equivalent to the assertion that (x_n) is Cauchy – and one should note that an appeal to *reductio ad absurdum* is inevitable in the process. We shall now exhibit an effective rate of metastability – that is, a bound $\Omega(k,g)$ on the N in the above formulation – for the Ishikawa iteration.

Theorem 10. *Let H be a Hilbert space, $C \subseteq H$ a nonempty totally bounded convex subset, $T : C \to C$ an L-Lipschitzian pseudo-contraction with $F := Fix(T) \neq \emptyset$, (α_n), (β_n) sequences in $[0,1]$ satisfying (A1)-(A3) and (x_n) be the Ishikawa iteration starting with $x \in C$. Assume, furthermore, that γ is a modulus of total boundedness for C, $b \in \mathbb{N}$ is an upper bound on the diameter of C, β is a rate of convergence of (β_n) and θ is a rate of divergence of $\sum_{n=0}^{\infty} \alpha_n\beta_n$.*

Let $\Sigma_{b,\theta,\gamma,\beta,L}$ and $\Omega_{b,\theta,\gamma,\beta,L} : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be defined as in Table 1. Then

1. *$\Sigma_{b,\theta,\gamma,\beta,L}$ is a rate of metastability for (x_n).*

2. *There exists $N \leq \Omega_{b,\theta,\gamma,\beta,L}(k,g)$ such that*

$$\forall i,j \in [N, N+g(N)] \ \left(\|x_i - x_j\| \leq \frac{1}{k+1} \text{ and } \|x_i - Tx_i\| \leq \frac{1}{k+1} \right).$$

$$\begin{aligned}
&\Sigma_{b,\theta,\gamma,\beta,L}(k,g) := K + \tilde{\Sigma}_{b,\theta,\gamma}(k,h),\\
&\quad \tilde{\Sigma}_{b,\theta,\gamma} : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \quad \tilde{\Sigma}_{b,\theta,\gamma}(k,g) := (\tilde{\Sigma}_0)_{b,\theta}(P,k,g),\\
&\quad (\tilde{\Sigma}_0)_{b,\theta} : \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \quad (\tilde{\Sigma}_0)_{b,\theta}(0,k,g) := 0,\\
&\quad (\tilde{\Sigma}_0)_{b,\theta}(n+1,k,g) := \theta^M\left(2(b^2+1)\big(Zg^M\left((\tilde{\Sigma}_0)_{b,\theta}(n,k,g)\right)+1\big)^2\right),\\
&\Omega_{b,\theta,\gamma,\beta,L}(k,g) := K + \tilde{\Omega}_{b,\theta,\gamma,L}(k,h),\\
&\quad \tilde{\Omega}_{b,\theta,\gamma,L} : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \quad \tilde{\Omega}_{b,\theta,\gamma,L}(k,g) := (\tilde{\Omega}_0)_{b,\theta,L}(P_0,k,g),\\
&\quad (\tilde{\Omega}_0)_{b,\theta,L} : \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \quad (\tilde{\Omega}_0)_{b,\theta,L}(0,k,g) := 0,\\
&\quad (\tilde{\Omega}_0)_{b,\theta,L}(n+1,k,g) := \theta^M\left(2(b^2+1)\big(\max\{2k+1, Z_0 g^M((\tilde{\Omega}_0)_{b,\theta,L}(n,k,g))\}+1\big)^2\right),\\
&K := \beta\left(\lceil 1+\sqrt{2L^2+4}\rceil\right), \qquad h(n) := g(K+n), \qquad Z := 8b(8k^2+16k+10),\\
&P := \gamma\left(\left\lceil \sqrt{8k^2+16k+9}\right\rceil\right), \qquad k_0 := \left\lceil \frac{\lceil L\rceil(4k+4)-1}{2}\right\rceil,\\
&P_0 := \gamma\left(\left\lceil \sqrt{8k_0^2+16k_0+9}\right\rceil\right), \qquad Z_0 := 8b(8k_0^2+16k_0+10).
\end{aligned}$$

Table 1: Functionals and constants.

The proximal point algorithm

The results of this section can be found in the papers [59, 61].

The proximal point algorithm is a fundamental tool of convex optimization, going back to Martinet [68], Rockafellar [73] and Brézis and Lions [10]. Since its inception, the schema turned out to be highly versatile, covering in its various developments, *inter alia*, the problems of finding zeros of monotone operators, minima of convex functions and fixed points of nonexpansive mappings. For a general introduction to the field in the context of Hilbert spaces, see the book of Bauschke and Combettes [7].

We shall fix a complete CAT(0) space X in this section.

We first state the most general, albeit unnatural, conditions which imply convergence for the algorithm.

Theorem 11 (General Proximal Point Algorithm)**.** *Let $(T_n : X \to X)_{n\in\mathbb{N}}$ be a family of self-mappings such that $F := \bigcap_{n\in\mathbb{N}} Fix(T_n) \neq \emptyset$. Let $(\gamma_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ be such that $\sum_{n=0}^{\infty} \gamma_n^2 = \infty$. Let $x \in X$. Set $x_0 := x$ and for all $n \in \mathbb{N}$, $x_{n+1} := T_n x_n$. Assume that:*

(i) the T_n's all satisfy the (P_2) property (in particular, they may all be firmly nonexpansive);

(ii) *for all $n, m \in \mathbb{N}$ and $w \in X$, $d(T_n w, T_m w) \leq \frac{|\gamma_n - \gamma_m|}{\gamma_n} d(w, T_n w)$;*

(iii) *the sequence $\left(\frac{d(x_n, x_{n+1})}{\gamma_n}\right)_{n\in\mathbb{N}}$ is nonincreasing.*

Then the sequence $(x_n)_{n\in\mathbb{N}}$ Δ-converges to an element of F.

The following set of definitions represent, in our opinion, the most natural general conditions under which we can talk meaningfully about the proximal point algorithm. In addition, all known variants of the algorithm (that we shall see in the next subsection) fall under the strongest of the definitions here (the "jointly firmly nonexpansive" one), while the weakest of them (the "jointly (P_2)" one) implies the even weaker conditions considered above.

Definition 12. *Let $(T_n : X \to X)_{n\in\mathbb{N}}$ be a family of mappings and $(\gamma_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$. We say that the family $(T_n)_{n\in\mathbb{N}}$ is* **jointly firmly nonexpansive with respect to** *the sequence $(\gamma_n)_{n\in\mathbb{N}}$ if for all $n, m \in \mathbb{N}$, $x, y \in X$ and all $\alpha, \beta \in [0, 1]$ such that $(1-\alpha)\gamma_n = (1-\beta)\gamma_m$ we have that:*

$$d(T_n x, T_m y) \leq d((1-\alpha)x + \alpha T_n x, (1-\beta)y + \beta T_m y).$$

Definition 13. *Let $(T_n : X \to X)_{n\in\mathbb{N}}$ be a family of mappings and $(\gamma_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$. We say that the family $(T_n)_{n\in\mathbb{N}}$ is* **jointly (P_2) with respect to** *the sequence $(\gamma_n)_{n\in\mathbb{N}}$ if for all $n, m \in \mathbb{N}$ and all $x, y \in X$ we have that:*

$$\frac{1}{\gamma_n}(d^2(x, T_m y) - d^2(x, T_n x) - d^2(T_n x, T_m y)) \geq \frac{1}{\gamma_m}(d^2(T_n x, T_m y) + d^2(y, T_m y) - d^2(y, T_n x)).$$

The link between them is the following.

Proposition 14. *Let $(T_n : X \to X)_{n\in\mathbb{N}}$ be a family of mappings and $(\gamma_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$. Suppose that $(T_n)_{n\in\mathbb{N}}$ is jointly firmly nonexpansive with respect to the sequence $(\gamma_n)_{n\in\mathbb{N}}$. Then $(T_n)_{n\in\mathbb{N}}$ is jointly (P_2) with respect to the sequence $(\gamma_n)_{n\in\mathbb{N}}$.*

In order to better understand and apply the jointly (P_2) condition, we shall use the *quasi-linearization function*, $\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}$, introduced by Berg and Nikolaev in [9], which is defined, for any $a, b, u, v \in X$, by the following (where we have denoted a pair $(w, w') \in X^2$ by $\overrightarrow{ww'}$):

$$\langle \overrightarrow{ab}, \overrightarrow{uv} \rangle := \frac{1}{2}(d^2(a, v) + d^2(b, u) - d^2(a, u) - d^2(b, v)).$$

Berg and Nikolaev gave the following characterization of this mapping.

Proposition 15 ([9, Proposition 14])**.** *In an arbitrary metric space X, the mapping $\langle \cdot, \cdot \rangle$ is the unique one that satisfies, for any $a, b, c, d, f \in X$, that:*

(i) $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$;

(ii) $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$;

(iii) $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$;

(iv) $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle+\langle\overrightarrow{ab},\overrightarrow{df}\rangle=\langle\overrightarrow{ab},\overrightarrow{cf}\rangle$.

The main result of [9] is that the "Cauchy-Schwarz" inequality for this "inner product" – i.e., that for all $a,b,c,d\in X$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\le d(a,b)d(c,d)$ – is actually equivalent to the CAT(0) property.

Using this mapping, we may express the jointly (P_2) condition as:

$$\frac{1}{\gamma_n}\langle\overrightarrow{T_nxT_my},\overrightarrow{xT_nx}\rangle\ge\frac{1}{\gamma_m}\langle\overrightarrow{T_nxT_my},\overrightarrow{yT_my}\rangle. \tag{5}$$

We may show that the above definitions imply the general conditions from before and hence we have the folllowing result.

Theorem 16 (Abstract Proximal Point Algorithm)**.** *Let $(T_n:X\to X)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}\subseteq(0,\infty)$ be such that $(T_n)_{n\in\mathbb{N}}$ is jointly (P_2) with respect to the sequence $(\gamma_n)_{n\in\mathbb{N}}$ (in particular, they may be jointly firmly nonexpansive). Suppose that the common fixed point set of the T_n's is nonempty and that $\sum_{n=0}^{\infty}\gamma_n^2=\infty$. Let $x\in X$. Set $x_0:=x$ and for all $n\in\mathbb{N}$, $x_{n+1}:=T_nx_n$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ Δ-converges to an element of F.*

We shall consider now the problem of finding minimizers of convex, lower semicontinuous (lsc) functions $f:X\to(-\infty,+\infty]$. For any such f, define its *Moreau-Yosida resolvent* or its *proximal point mapping*, $J_f:X\to X$, for any $x\in X$, as:

$$J_f(x):=\mathrm{argmin}_{y\in X}\left[f(y)+\frac{1}{2}d^2(x,y)\right].$$

We usually consider the resolvent of f of order $\gamma>0$, which is simply the resolvent of γf, namely:

$$J_{\gamma f}(x)=\mathrm{argmin}_{y\in X}\left[\gamma f(y)+\frac{1}{2}d^2(x,y)\right]=\mathrm{argmin}_{y\in X}\left[f(y)+\frac{1}{2\gamma}d^2(x,y)\right].$$

This definition (albeit without that factor of 2) first appeared in [35]. The argmin is unique for convex lsc functions, and so the operator is indeed well-defined, as witnessed by [35, Lemma 2].

We have the following results.

Proposition 17. *Let $f:X\to(-\infty,+\infty]$ be convex lsc and $(\gamma_n)_{n\in\mathbb{N}}\subseteq(0,\infty)$. Then the family $(J_{\gamma_nf})_{n\in\mathbb{N}}$ is jointly firmly nonexpansive with respect to $(\gamma_n)_{n\in\mathbb{N}}$.*

Theorem 18 (Proximal Point Algorithm for Convex Lsc Functions)**.** *Let $f:X\to(-\infty,+\infty]$ be a convex lsc function that has at least one minimizer and let $(\gamma_n)_{n\in\mathbb{N}}\subseteq(0,\infty)$ be such that $\sum_{n=0}^{\infty}\gamma_n^2=\infty$. Let $x\in X$. Set $x_0:=x$ and for all $n\in\mathbb{N}$, $x_{n+1}:=J_{\gamma_nf}x_n$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ Δ-converges to a minimizer of f.*

Similar results are obtained for other instances of the proximal point algorithm.

We now discuss results regarding the "uniform case" of the algorithm.

Definition 19. *Let $T:X\to X$ and $C\subseteq X$ such that $T(C)\subseteq C$. Let $\phi:[0,\infty)\to[0,\infty)$ be an increasing function which vanishes only at 0. We say that T is* **uniformly firmly nonexpansive** *on C with modulus ϕ if for any $x,y\in C$ and all $t\in[0,1]$, we have that:*

$$d^2(Tx,Ty)\le d^2((1-t)x+tTx,(1-t)y+tTy)-2(1-t)\phi(d(Tx,Ty)).$$

Definition 20. *Let $T : X \to X$ and $C \subseteq X$ such that $T(C) \subseteq C$. Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function which vanishes only at 0. We say that T is* **uniformly** *(P_2) on C with modulus ϕ if for any $x, y \in C$, we have that:*

$$2d^2(Tx, Ty) \leq d^2(x, Ty) + d^2(y, Tx) - d^2(x, Tx) - d^2(y, Ty) - 2\phi(d(Tx, Ty)).$$

The uniformly (P_2) condition may be expressed using the Berg-Nikolaev quasi-linearization function as follows:

$$\langle \overrightarrow{TxTy}, \overrightarrow{xTx} \rangle \geq \langle \overrightarrow{TxTy}, \overrightarrow{yTy} \rangle + \phi(d(Tx, Ty)).$$

Proposition 21. *Let $T : X \to X$ and $C \subseteq X$ such that $T(C) \subseteq C$. Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function which vanishes only at 0. Suppose that T is uniformly firmly nonexpansive on C with modulus ϕ. Then T is uniformly (P_2) on C with the same modulus ϕ.*

We now return to the framework established in Theorem 11 that we have fixed above. Remember that we have taken $b \in \mathbb{N}$ such that $d(x, z) \leq b$. Set C to be the closed ball of center z and radius b. By the results obtained in the previous sections, we have that for all n, $T_n(C) \subseteq C$. Whereas C may have been just fixed, it is understood that the definitions in the next subsection, like the ones above, apply to a general C.

We shall impose in the sequel the condition that there is a $\phi : [0, \infty) \to [0, \infty)$, an increasing function which vanishes only at 0, such that for all n, T_n is uniformly (P_2) on C with modulus $\gamma_n \phi$. It is clear than in this case z is the unique fixed point of the T_n's in C and that it is the point to which the sequence $(x_n)_{n \in \mathbb{N}}$ weakly converges. In addition, we have the following result.

Suppose that $\sum_{n=0}^{\infty} \gamma_n^2 = \infty$ with rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$, i.e. for all $K \in \mathbb{N}$ we have that

$$\sum_{n=0}^{\theta(K)} \gamma_n^2 \geq K.$$

Theorem 22. *In these circumstances, the convergence is strong. Moreover, set, for any $k \in \mathbb{N}$,*

$$\Psi_{b,\theta,\phi}(k) := \Sigma_{b,\theta} \left(\left\lceil \frac{2b}{\phi\left(\frac{1}{k+1}\right)} \right\rceil \right) + 1,$$

where $\Sigma_{b,\theta}(k) := \theta(b^2(k+1)^2)$.

Then $\Psi_{b,\theta,\phi}$ is a rate of convergence for $(x_n)_{n \in \mathbb{N}}$.

The result above is very surprising – not because it exhibits a full rate of convergence, since the technique is not fundamentally new – but because of the way the sequence of weights $(\gamma_n)_{n \in \mathbb{N}}$ disappears in the middle of the proof, even though it is not taken from a compact interval like the weight sequences studies along with pseudocontractions. It would be interesting to find out if there is a logical explanation behind this.

All the cases of the proximal point algorithm have particular "uniform cases" to which the above theorem may be applied. Take for example the case of convex functions discussed above.

Definition 23. *Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function which vanishes only at 0. A function $f : X \to (-\infty,\infty]$ is called* **uniformly convex** *on C with modulus ψ if for all $x, y \in C$ and all $t \in [0,1]$ we have that:*

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) - t(1-t)\psi(d(x,y)).$$

We have the following relationship.

Theorem 24. *Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function which vanishes only at 0. Let $f : X \to (-\infty,\infty]$ be a lsc function which is uniformly convex on C with modulus ψ. Let $\gamma > 0$. We assume that $J_{\lambda f}(C) \subseteq C$ for all $\lambda > 0$. Then $J_{\gamma f}$ is uniformly firmly nonexpansive on C with modulus $2\gamma\psi$.*

Proof mining in L^p spaces

The results of this section can be found in the paper [77].

The formalization of analysis used in proof mining is not the only way, and not even the only useful way of interfacing logic with analysis. One of the first methods to represent real numbers in a logic in a built-in matter was attempted in the 1960s – see, e.g., the book on continuous model theory by Chang and Keisler [13]. Later, Ben Yaacov and others realized that the lack of fruitful lines of research out of that logic was due to an unfortunate choice of parameters – specifically, the truth values could vary wildly along an arbitrary compact Hausdorff space (instead of just the interval $[0,1]$), while equality itself was tightly restricted to binary values. Their efforts led to what has been called "continuous first-order logic", a system in which many celebrated and relatively advanced results of 20th century model theory could be reasonably translated – see [8] for an introduction. Another strand of developments came from Henson's positive-bounded logic, introduced in [27] and later shown to be largely equivalent to continuous first-order logic. Despite this fact, due to its later exhaustive treatment by Henson and Iovino focusing on the model-theoretic ultraproduct construction [28], this logic was subject to an investigation from which it resulted that, in combination with the aforementioned ultraproducts, it could be used to prove uniformity results in nonlinear analysis and ergodic theory – see the recent paper of Avigad and Iovino [3].

We can now ask the question of whether the methods of proof mining are sufficiently powerful to provide us with all uniformity results given to us by the model-theoretic properties of positive-bounded formulas. (Proof theory already can be considered to have some upper hand in the matter of being able to deal with weak forms of extensionality.) The answer, as presented in the 2016 paper of Günzel and Kohlenbach [26], is in the affirmative. To give a rough sketch, the positive-bounded formulas are there translated into a special class of higher-order formulas denoted by $\mathcal{PBL}$, which are then turned into Δ-formulas, a class of formulas which can be freely added as additional axioms, with no negative consequences to the bound extraction procedure, as per the classical metatheorems of proof mining. A new metatheorem is then obtained for the classes of spaces which could be axiomatized by positive-bounded formulas. In addition, the treatment of a "uniform boundedness principle" tries to clarify just what exactly is the role played by the ultraproduct construction. Examples are given of such classes of spaces, and the translations for each set of axioms into the higher-order language are given explicitly, together with their metatheorems. Notable among these are the L^p and BL^pL^q Banach lattices, which are usually defined by a construction, but for which axiomatic characterizations into positive-bounded logic have been found, for the last one by Henson and Raynaud [29]. The continual addition of new classes of spaces to the list of targets of logical metatheorems has been long-pursued within proof mining – see, e.g. Leuştean's metatheorem on $\mathbb{R}$-trees [55] or Kohlenbach and Nicolae's on CAT(κ) spaces [48].

The goal here is to find an appropriate treatment of the class of L^p Banach spaces in themselves, as defined in the first chapter. It turns out – see [54, 64] for detailed expositions – that these spaces can be given an implicit characterization, which resembles a bit the axiomatization of BL^pL^q lattices which was analysed by Günzel and Kohlenbach. Notably, and in contrast to that, this characterization does not use at all the natural lattice structure. We have that a Banach space X is isometrically isomorphic to a L^p space iff it satisfies the following condition:

> *for all x_1,..., x_n in X of norm less than 1 and for all $N \in \mathbb{N}_{\geq 1}$, there is a subspace $C \subseteq X$ and y_1,..., y_n in C of norm less than 1 such that C is of dimension at most $(4nN+1)^n$, it is isometric to $\mathbb{R}_p^{\dim_{\mathbb{R}} C}$ and for all i, $\|x_i - y_i\| \leq \frac{1}{N}$.*

The advantage of the condition above is that it is both intrinsic and quantitative, therefore amenable to a logical axiomatization.

$$\psi_m(\underline{z}) := \forall \underline{\lambda} \left(\|\textstyle\sum_{i=1}^m \lambda_i z_i\| = (\sum_{i=1}^m |\lambda_i|^p)^{\frac{1}{p}} \right)$$
$$\psi'_{m,n}(\underline{y}, \underline{z}) := \textstyle\bigwedge_{k=1}^n \left(\exists \underline{\lambda} \left(y_k = \sum_{i=1}^m \lambda_i z_i\right)\right)$$
$$\psi''_{n,N}(\underline{x}, \underline{y}) := \textstyle\bigwedge_{k=1}^n \left(\|x_k - y_k\| \leq \frac{1}{N+1} \wedge \|y_k\| \leq 1 \right)$$
$$\varphi_{n,m,N}(\underline{x}) := \exists \underline{y} \exists \underline{z} \left(\psi_m(\underline{z}) \wedge \psi'_{m,n}(\underline{y}, \underline{z}) \wedge \psi''_{n,N}(\underline{x}, \underline{y}) \right)$$
$$\phi_{n,N}(\underline{x}) := \textstyle\bigvee_{0 \leq m \leq (4nN+1)^n} \varphi_{n,m,N}(\underline{x})$$
$$A_{n,N} := \forall \underline{x} \left(\left(\textstyle\bigwedge_{k=1}^n \|x_k\| \leq 1\right) \to \phi_{n,N}(\underline{x}) \right)$$

Table 2: A first axiomatization.

Table 2 shows one such axiomatization (into a crude first-order-like language), i.e. the characterization of the space is expressed by the simultaneous validity of all $A_{n,N}$ sentences. With that in mind, by closely examining the formulas, one can easily see that they represent a straightforward translation of the condition from before.

Table 3, where we have used some of the notations from [26, Definitions 7.9 and 7.10], shows how one may translate the infinite family of axioms $A_{n,N}$ into the one axiom B which is, like the one in [26], representable as a Δ-sentence. Let us see some details of the translation. Firstly, we remark that the operation $\widetilde{v} := \frac{v}{\max\{\|v\|,1\}}$ that we used excused us from writing the antecedent from $A_{n,N}$. Then we see that by substituting into $\psi_m(\underline{z})$ all λ_i's with 0, except for one which we set to 1, we obtain the fact that all z_i's are of norm one. We have also postulated that all y_k's are of norm less

$$\psi(m,z) := \forall \lambda^{1(0)(0)} \left(\|\textstyle\sum_{i=1}^{m} |\lambda(i)|_{\mathbb{R}} \cdot_X z(i)\| =_{\mathbb{R}} \left(\sum_{i=1}^{m} |\lambda(i)|_{\mathbb{R}}^{p}\right)^{1/p} \right)$$
$$\psi'(m,n,y,z,\lambda) := \forall k \preceq_0 (n-1)\, \left(y(k+1) =_X \textstyle\sum_{i=1}^{m} \lambda(i) \cdot_C z(i)\right)$$
$$\psi''(n,N,x,y) := \forall k \preceq_0 (n-1) \left(\left\| \widetilde{x(k+1)} - y(k+1) \right\| \leq_{\mathbb{R}} \tfrac{1}{N} \wedge \|y(k+1)\| \leq_{\mathbb{R}} 1 \right)$$
$$\varphi(n,m,N,x,y,z,\lambda) := \psi(m,z) \wedge \psi'(m,n,y,z,\lambda) \wedge \psi''(n,N,x,y)$$
$$B := \forall n^0, N^0 \geq 1 \forall x^{X(0)} \exists y, z \preceq_{X(0)(0)} 1_{X(0)(0)} \exists \lambda^{1(0)(0)(0)} \in [-2,2] \exists m \preceq_0 (4nN+1)^n$$
$$\varphi(n,m,N,x,y,z,\lambda)$$

Table 3: The Δ-axiomatization.

than 1. Thus, if we have, as in $\psi'_{m,n}(\underline{y},\underline{z})$, that for a given k:

$$y_k = \sum_{i=1}^{m} \lambda_i z_i,$$

the formula $\psi_m(\underline{z})$ tells us further that:

$$1 \geq \|y_k\| \geq \left\| \sum_{i=1}^{m} \lambda_i z_i \right\| = \left(\sum_{i=1}^{m} |\lambda_i|^p \right)^{\frac{1}{p}},$$

from which we get that each such λ_i is in the interval $[-1,1]$. These results allow us to correspondingly bound the y, the z and the λ (which are now properly functionals) in the axiom B. Another such bounding comes from the $(4nN+1)^n$ established before (i.e. here it matters that the characterization is quantitative), which helped us eliminate the potentially infinite disjunction in Table 1 (where such constraints were not yet relevant) and the unbounded existential quantifier in Table 3 (which would have hindered us in presenting the axiom B as a Δ-sentence). As a curiosity, we note that choosing to present B as a single axiom and not as an infinite schema like in Table 1, i.e. taking advantage of the arithmetic already present in the framework, adds a bit of strength to the system, given the fact that we do not work here with any sort of ω-rule.

We denote by $\mathcal{A}^\omega[X, \|\cdot\|, \mathcal{C}, L^p]$ the extension of the system $\mathcal{A}^\omega[X, \|\cdot\|, \mathcal{C}]$ by the constant c_p of type 1, together with the axiom $1_{\mathbb{R}} \leq_{\mathbb{R}} c_p$ and the axiom B from above. From the above discussion, the following soundness theorem holds.

Theorem 25 (cf. [26, Propositions 3.5 and 7.12])**.** *Let X be a Banach space and $p \geq 1$. Denote by $\mathcal{S}^{\omega,X}$ its associated set-theoretic model and let the constant c_p in our extended signature take as a value the canonical representation of the real number p. Then $\mathcal{S}^{\omega,X}$ is a model of $\mathcal{A}^\omega[X, \|\cdot\|, \mathcal{C}, L^p]$ iff X is isomorphic to some $L^p(\Omega, \mathcal{F}, \mu)$ space.*

In a parallel way to the one suggested in [29], by some similar arguments to the ones used above to construct the required higher-order system, one could perform reasonable transformations to the formulas in Table 1, obtaining a new, concrete proof of the following classical result of Henson (shown in [27] using ultraproducts).

Theorem 26. *The subclass of Banach spaces which are isomorphic to spaces of the form $L^p(\mu)$ is axiomatizable in positive-bounded logic.*

Analogously to the treatment done in [26] for the classes of Banach lattices, we may now state the corresponding metatheorem for the system devised above.

Theorem 27 (Logical metatheorem for $L^p(\mu)$ Banach spaces, cf. [26, Theorems 5.13 and 7.13])**.** *Let $\rho \in \mathbf{T}^X$ be an admissible type. Let $B_\forall(x,u)$ be a $\forall$-formula with at most x, u free and $C_\exists(x,v)$ an $\exists$-formula with at most x, v free. Let Δ be a set of Δ-sentences. Suppose that:*

$$\mathcal{A}^\omega[X, \|\cdot\|, \mathcal{C}, L^p] + \Delta \vdash \forall x^\rho (\forall u^0 B_\forall(x,u) \to \exists v^0 C_\exists(x,v)).$$

Then one can extract a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$, whose restriction to the strongly majorizable functionals of $S_{\widehat{\rho}}$ is a bar-recursively computable functional of $\mathcal{M}^\omega$, such that for all $L^p(\mu)$ Banach spaces $(X, \|\|)$ having the property that any associated set-theoretic model of it satisfies Δ, we have that for all $x \in S_\rho$ and $x^* \in S_{\widehat{\rho}}$ such that $x^* \gtrsim_\rho x$, the following holds:

$$\forall u \leq \Phi(x^*) B_\forall(x,u) \to \exists v \leq \Phi(x^*) C_\exists(x,v).$$

In addition, this system admits an internal proof that the standard modulus of uniform convexity is valid for this class of spaces. We have the following.

Theorem 28. *Provably in the system $\mathcal{A}^\omega[X, \|\cdot\|, \mathcal{C}, L^p] + \{2 \leq_{\mathbb{R}} c_p; C_1; C_2\}$, the function $\eta : (0,2] \to (0,\infty)$, defined, for any $\varepsilon > 0$, by $\eta(\varepsilon) := 1 - (1 - (\frac{\varepsilon}{2})^p)^{1/p}$, is a modulus of uniform convexity.*

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