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Grothendieck Categories, Monoidal Categories and Frobenius Algebras

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ABSTRACT

INTRODUCTION

Grothendieck categories were introduced by Alexander Grothendieck in his paper [32] published in 1957. His aim was to use the homological algebra methods for the study of sheaves. A systematic study of Grothendieck categories was done by Gabriel in his thesis [28]. The study of Grothendieck categories has unified many theories in mathematics. Relevant examples of Grothendieck categories are: the category of left modules over a ring, the category of quasi-coherent sheaves on an algebraic variety, the category of sheaves of abelian groups on a topological space, the category of graded modules over a graded ring, the category of comodules over a coalgebra, the category of generalized Doi-Hopf modules. At the beginning of the theory it was an open question whether any Grothendieck category is equivalent to a category of modules (over a ring). The answer turned out to be negative. For example it is possible to show that certain categories of graded modules are not equivelnt to a category of modules. However, Popescu and Gabriel proved in 1964 a famous theorem that shows that each Grothendieck category is equivalent to a quotient category of a module category, see [29]. More precisely, if $\mathcal{A}$ is a Grothendieck category with a generator G, $R = End_{\mathcal{A}}(G)$ is the endomorphism ring of G and $Mod(R)$ is the category of unitary right R-modules, then the functor $T : \mathcal{A} \to Mod(R)$ defined by $T(X) = Hom_{\mathcal{A}}(G, X)$ on objects X of $\mathcal{A}$ and by $T(f) = Hom_{\mathcal{A}}(G, f) : T(X) \to T(Y)$ on morphisms $f : X \to Y$ in $\mathcal{A}$ is fully faithful and has an exact left adjoint S. The initial proof was rather complicated; especially the part on the exactness of the functor S. In order to simplify the proof, the initial proof was revisited by several authors, among them the most elegant and short proof in chronological order where given by Takeuchi [65], Ulmer [67] and Mitchell [44]. The last one uses an ingenious lemma, that remained in literature as Mitchell Lemma and also the existence of enough injective objects in any Grothendieck category.

The concept of localization was first studied for commutative rings. In this case we consider P a prime ideal of a commutative ring R. The complement of this ideal is denoted by $S_P = R - P$, that is a multiplicative set. Then we can consider the localizing subcategory of $R - Mod$ denoted by

$$\mathcal{C}_P = \{M \in R - Mod | S_P^{-1}M = 0\}$$

It is easy to see that this is the same thing with

$$\mathcal{C}_P = \{M \in R - Mod | Hom_R(M, E(R/P)) = 0\}$$

where $E(R/P)$ denotes the injective hull of R/P. Moreover, the quotient category $R - Mod/\mathcal{C}_P$ is equivalent to the category $R_P - Mod$ of all left modules over the localized ring R_P. When the notion of abelian category was introduced the notion of hereditary torsion theory appeared and it quickly became a powerful instrument for studying categories. A hereditary torsion theory in an arbitrary abelian category $\mathcal{A}$ is the same thing with having a localizing subcategory $\mathcal{C}$ in $\mathcal{A}$. Moreover, for every localizing subcategory $\mathcal{C}$ in $\mathcal{A}$ we have the concept of quotient category $\mathcal{A}/\mathcal{C}$. The natural question to study here is the proximity between any localizing subcategory $\mathcal{C}$ of $\mathcal{A}$ and the localizing subcategories of $\mathcal{C}_P$-type. Cahen was the first who introduced a notion of "stability". He introduced this notion for a localizing subcategory of a module category over a commutative ring. In this case a localizing subcategory is said to be stable if $\mathcal{C} = \bigcap_{P \in X} \mathcal{C}_P$, for a set $X \subseteq Spec(R)$.

One of the aims of this thesis is to revisit the Gabriel-Popescu Theorem and its generalizations, and also to introduce a notion of stability in the general case of Grothendieck categories. Another aim is to look at special examples of Grothendieck categories, more precisely categories of corepresentations over Hopf algebras, and to investigate Frobenius algebras and symmetric algebras in such categories. Frobenius algebras originate in the work of F.G. Frobenius on representation theory of finite groups. Their study was initiated by Brauer, Nesbitt and Nakayama and then was continued by Dieudonne,Eilenberg, Azumaya etc. Frobenius algebras have played an important role in Hopf algebra theory, because any finite dimensional Hopf algebra is Frobenius, cohomology rings of compact oriented manifolds, solutions of the quantum Yang-Baxter equation, Jones polynomials and topological quantum field theory. It is a challenging problem to understand the significance of Frobenius algebras in monoidal categories other than the categories of vector spaces. We discuss this problem for the category of right comodules over a Hopf algebra H. Under additional assumptions on H, we also look at symmetric algebras in the category. A special attention is given to the case where H is a group Hopf algebra kG, where G is an arbitrary group; in this situation the category of corepresentations over H is just the category of G-graded vector spaces, and an algebra in the category is a G-graded algebra. One last direction of study in this thesis is also related to group graded algebras. The theory of algebras graded by an arbitrary group has been systematically developed after 1970. The main two sources of inspiration were polynomial algebras with the (usual) positive grading by integers, and group algebras with the standard grading by the considered group. The study of group gradings on several classes of algebras have been of interest in commutative and noncommutative algebras, on Lie algebras and in representation theory. We are interested in gradings on polynomial algebras.

The thesis is divided in four chapters.

Chapter 1: Grothendieck categories. A generalization of Mitchell's Lemma

In 1964 Pierre Gabriel and Nicolae Popescu showed that each Grothendieck category is equivalent to a quotient category of a module category.

More precisely if:

- $\mathcal{A}$ is a Grothendieck category with a generator G
- $R = End_{\mathcal{A}}(G)$ is the endomorphism ring of G
- $Mod(R)$ the category of unitary right R-modules

then the functor $T : \mathcal{A} \to Mod(R)$ defined by:

- $T(X) = Hom_{\mathcal{A}}(G, X)$, where $X \in \mathcal{A}$
- $T(f) = Hom_{\mathcal{A}}(G, f) : T(X) \to T(Y)$ on morphisms $f : X \to$ in $\mathcal{A}$

is fully faithful and has an exact left adjoint S.

Afterward it was revisited by several authors; among the most elegant and short proofs in cronological order are those of Takeuchi, Ulmer and Mitchell. The latter uses an ingenious lemma, referred to as Mitchell Lemma.

Our aim in this chapter is to extend the Mitchell lemma from module categories to functor categories and to show how it can be used in order to obtain in an easier way the Ulmer Theorem on the exactness of S, a generalized Gabriel-Popescu Theorem and a generalized Takeuchi Lemma.

We will use the following notations:

- $\mathcal{A}$ a Grothendieck category
- $\mathcal{U}$ the set of all objects of $\mathcal{A}$
  $Gen(\mathcal{U}) = \{A \in \mathcal{A} | (\exists) f : \bigoplus_{U_i \in \mathcal{U}} U_i \to A \to 0\}$, the full subcategory of $\mathcal{A}$, that is a preabelian category of $\mathcal{A}$ (has kernels and cokernels)
- $\mathcal{A}_{\mathcal{U}} = \{M \in \mathcal{A} | (\forall)\ morfism\ f : \bigoplus_{U \in F} U \to M \in \mathcal{A}$ cu F o submultime finita a lui $\mathcal{U}$, $Ker(f) \in Gen(\mathcal{A})\}$

Let $(\mathcal{U}^{op}, Ab)$ be a category defined in the following way:

- objects: the additive contravariant functors from $\mathcal{U}$ to Ab
- morphisms: the natural transformations between such functors

$(\mathcal{U}^{op}, Ab)$ is a Grothendieck category, whose functors $(h_U)_{U \in \mathcal{U}}$, where h_U form a generating family of finitely generated projective objects for $(\mathcal{U}^{op}, Ab)$.

Consider the functor $T : \mathcal{A} \to (\mathcal{U}^{op}, Ab)$ defined by:

- $T(X) = Hom_{\mathcal{A}}(-, X)|\mathcal{U}$ on objects $X \in \mathcal{A}$
- $T(f) = Hom_{\mathcal{A}}(-, f)|\mathcal{U} : T(X) \to T(Y)$ on morphism $f : X \to Y$ in $\mathcal{A}$

It is known that T has a left adjoint $S : (\mathcal{U}^{op}, Ab) \to \mathcal{A}$. In particular we have that $S(h_U) = U$, $(\forall) U \in \mathcal{U}$.

The most important result of this chapter is the following:

Lemma 1. *(Generalized Mitchell Lemma) Consider:*

- *$\mathcal{U}$ be a set of objects of $\mathcal{A}$*
- *A, B objects of $\mathcal{A}$ with $A \in \mathcal{A}_{\mathcal{U}}$*
- *M_A a subobject of $T(A)$*
- *$G : M_A \to T(B)$ a morphism in $(\mathcal{U}^{op}, Ab)$*

We denote:

- $M = \bigcup_{U \in \mathcal{U}} M_A(U)$

- $(\forall)m \in M$ *take* $U_m \in \mathcal{U}$ *for which* $m \in M_A(U)$
- $(\forall)m \in M$, $p_m : \bigoplus_{m \in M} U_m \to U_m$ *the canonical projection*
- $\psi : \bigoplus_{m \in M} U_m \to A$ *the unique morphism with* $\psi u_m = m$, $(\forall)m \in M$
- $\phi : \bigoplus_{m \in M} U_m \to B$ *the unique morphism with* $\phi u_m = G_U(m)$, $(\forall)m \in M_A(U)$

The ϕ *factors through* $Im(\psi)$.

We will have the following three important applications, that follow directly from Mitchell Lemma.

Theorem 2. *(Ulmer) The functor* $S : (\mathcal{U}^{op}, Ab) \to \mathcal{A}$ *is exact if and only if* $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$.

Theorem 3. *(Generalized Gabriel Popescu) Assume that* $\mathcal{U}$ *is a family of* $\mathcal{A}$. *Then* $T : \mathcal{A} \to (\mathcal{U}^{op}, Ab)$ *is a full and faithful functor, and its left adjoint* $S : (\mathcal{U}^{op}, Ab) \to \mathcal{A}$ *is exact.*

Corollary 4. *(Takeuchi Lemma): Let* $\mathcal{A}$ *be an object of* $\mathcal{A}$, *let* Y_A *be a subobject of* $T(A)$ *and denote by* $i : Y_A \to T(A)$ *the inclusion morphism.*

(i) If $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$ *then* $S(i)$ *is a monomorphism*

(ii) If $A \in \mathcal{A}_{\mathcal{U}} \bigcap Gen(\mathcal{U})$, *then the canonical morphism* $\nu_A : ST(A) \to A$ *is an isomorphism.*

Chapter 2: Locally stable Grothendieck categories

Consider:

- $\mathcal{A}$ a Grothendieck category
- $\mathcal{C}$ a localizing subcategory of $\mathcal{A}$
- $t_{\mathcal{C}} : \mathcal{A} \to \mathcal{C}$ the torsion functor
- $T_{\mathcal{C}} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ the canonical functor
- $S_{\mathcal{C}} : \mathcal{A}/\mathcal{C} \to \mathcal{A}$ the right adjoint of the functor $T_{\mathcal{C}}$

We define the spectrum of a Grothendieck category as follows:

$$Sp(\mathcal{A}) = \{I \in \mathcal{A} | I \ indecompozabil \ injectiv\}$$

To any indecomposable injective object I of $Sp(\mathcal{A})$ we can associate the localizing subcategory

$$\mathcal{A}_I = \{M \in \mathcal{A} | Hom_{\mathcal{A}}(M, I) = 0\}$$

Definition 5. *A subobject A of an object $B \in \mathcal{A}$ is irreducible in B if it cannot be written as the intersection of thwo strictly larger subobjects of B, or equivalently, if $A = M \bigcap N$, where M and N are two subobjects of B, then $A = M$ or $A = N$.*

Proposition 6. *$\mathcal{A}_I$ is an irreducible element in $Tors(\mathcal{A})$, where $Tors(\mathcal{A})$ is the set of localizing subactegories.*

Definition 7. *Let $\mathcal{A}$ be a Grothendieck category. A localizing subcategory $\mathcal{C}$ of $\mathcal{A}$ is said to be stable if there exists a subset Λ of $Sp(\mathcal{A}$ sucht that*

$$\mathcal{C} = \bigcap_{I \in \Lambda} \mathcal{A}_I$$

In this case, we write $\mathcal{C}_\Lambda = \mathcal{C}$. If every localizing subcategory $\mathcal{C}$ of $\mathcal{A}$ is stable, then $\mathcal{A}$ is called a locally stable category. If $\mathcal{A} = R - Mod$ is a locally stable category then R is a left locally stable ring.

A Grothendieck category $\mathcal{A}$ is called *locally coirreducible* if every nonzero object contains a nonzero coirreducible subobject. This is equivalent to saying that every injective object of $\mathcal{A}$ is the injective hull of a direct sum of indecomposable injective objects. $\mathcal{A}$ is called atomical if it has two localizing subcategories, namely $\{0\}$ and $\mathcal{A}$. We say that $\mathcal{A}$ has Gabriel dimension if any object of $\mathcal{A}$ has Gabriel dimension.

Using this definitions we give some examples of locally stable Grothendieck categories.

Proposition 8. *Let $\mathcal{A}$ be a Grothendieck category verifying one of the following conditions:*

1. *$\mathcal{A}$ is locally coirreducible*
2. *$\mathcal{A}$ has Gabriel dimension*
3. *$\mathcal{A}$ is atomical with spectrum $Sp(\mathcal{A} \neq$*

Then $\mathcal{A}$ is locally stable category.

The main result of this chapter gives a connection between the stability of the Grothendieck category $\mathcal{A}$ and the stability of the subcategory $\mathcal{C}$ and the quotient category $\mathcal{A}/\mathcal{C}$.

Theorem 9. *Let $\mathcal{A}$ be a Grothendieck category and let $\mathcal{C}$ be a localizing subcategory of $\mathcal{A}$. Then $\mathcal{A}$ is locally stable if and only if $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are locally stable.*

In the following sections of this chapter we will consider some particular cases of Grothendieck categories.

Let $\mathcal{A}$ a locally finitely generated Grothendieck category. $\mathcal{A}$ is called a V-category if every simple object of $\mathcal{A}$ is an injective object. The main theorem of this section states that some localizing subcategories of a V-category are stable. A localizing subcategory $\mathcal{C}$ of $\mathcal{A}$ is called a TTF-class if $\mathcal{C}$ is closed under arbitrary direct products.

We will also consider the category of comodules over a coalgebra. Let C be a coalgebra over a field k. We denote by $\mathcal{M}^C$ the k-linear category of all right C-comodules. The Hom bifunctor in this category is denoted by $Com_C(-,-)$. To every localizing subcategory $\mathcal{T}$ of $\mathcal{M}^C$ we can associate the quotient category $\mathcal{M}^C/\mathcal{T}$, a k-abelian category determined up to equivalence by an exact functor $T : \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$. For any simple right C-comodule S we can consider its injective hull $E(S)$ which is an indecomposable injective object in $\mathcal{M}^C$. We denote by

$$\mathcal{T}_{E(S)} = \{M \in \mathcal{M}^C | Com_{\mathcal{C}}(M, E(S)) = 0\}$$

the localzing subcategory of $\mathcal{M}^C$ associated to $E(S)$. The main theorem of this section states that:

Theorem 10. *Let $\{S_i | i \in I\}$ be a complete set of representatives of the isomorphism types of simple right C-comodules. If $\mathcal{T}$ is a localizing subcategory of $\mathcal{M}^C$, then*

$$\mathcal{T} = \bigcap_{i \in J \subseteq I} \mathcal{T}_{E(S_i)},$$

where $J = \{i \in I | S_i \notin \mathcal{T}\}$. Moreover, this intersection is irreducible.

Chapter 3: Frobenius algebras and symmetric algebras in monoidal categories

We will work over a field k. We say that a Frobenius algebra is a finite dimensional algebra A such that $A \simeq A^*$ as left(or, equivalently right) A-modules. This is an important concept for:representation theory, Hopf algebra theory, quantum group theory, the topological quantum field theory and others.

An equivalent characterization for the notion of Frobenius algebra was given by Abrams. So, Abrams proved that an algebra A is Frobenius $\Leftrightarrow$ there exists a coalgebra structure $(A, \delta_A, \varepsilon_A)$ on the k-vector space A such that the comultiplication δ_A is a morphism of A,A-bimodules. This equivalent definition of the Frobenius property makes sense in any monoidal category, which allowed the introduction of the concept of Frobenius algebra in a monoidal category. The study of Frobenius algebras in monoidal categories was initiated by M. Murger(2003), R. Street(2004) and S. Yamagami(2004). This is an important concept that appears in the theory of Morita equivalences of tensor categories, conformal qunatum field theory and others.

Among the Frobenius algebras there are some algebras that distinguish themselves by having more symmetry. This are the symmetric algebras. We say that a symmetric algebra is a finite dimensional algebra A such that $A \simeq A^*$ as A,A-bimodules. It is obvious that any symmetric algebra is a Frobenius algebra. However, the converse does not hold. This is an important concept used in topological quantum field theory and block theory of group algebras in positive characteristic.

Let H an Hopf algebra, and A a finite dimensional algebra in $\mathcal{M}^H$, that is a right H-comodul algebra. If $A \in_A \mathcal{M}^H$ it follows that its dual $A^* \in \mathcal{M}^H_A$. Also if $A \in \mathcal{M}^H_A$ it does not follow that $A^* \in_A \mathcal{M}^H$, but it results that $A^* \in_{A^{(S^2)}} \mathcal{M}^H$, where $A^{(S^2)}$ is just A as an algebra and has the right H-coaction given by $a \mapsto \sum a_0 \otimes S^2(a_1)$. The right H-coaction of A is given by $\sum a_0 \otimes a_1$. Now we will give the definition of left H-Frobenius and right H-Frobenius.

Definition 11. *The finite dimensional right H-comodul algebra A is called*

- *left H-Frobenius if $A^{(S^2)} \simeq A^*$ in ${}_{A^{(S^2)}}\mathcal{M}^H$*
- *right H-Frobenius if $A \simeq A^*$ in $\mathcal{M}^H_A$*

The following characterizes the left H-Frobenius property. The first four equivalent conditions are in the spirit of the classical ones for Frobenius algebras, taking also care of the H-coaction. The last condition shows the connection to the concept of Frobenius algebra in the category of corepresentations of H.

Theorem 12. *Let A be a finite dimensional right H-comodule algebra. The following assertions are equivalent.*

1. *A is left H-Frobenius*
2. *There exists a non-degenerate associative bilinear form $B : A \times A \to k$ with the property that $B(b, (h^*S^2) \cdot a) = B((h^*S) \cdot b, a)$ for any $a, b \in A$ and any $h^* \in H^*$.*
3. *A has a hyperplane $\mathcal{H}$ which does not contain any non-zero left ideal of A, and $(h^*S^2) \cdot A \subseteq \mathcal{H}$ for any $h^* \in H^*$ with $h^*(1) = 0$*
4. *A has a hyperplane $\mathcal{H}$ which does not contain any non-zero subobject of $A^{(S^2)}$ in ${}_{A^{(S^2)}}\mathcal{M}^H$, and $(h^*S^2) \cdot A \subseteq \mathcal{H}$ for any $h^* \in H^*$ with $h^*(1) = 0$*
5. *$A^{(S^2)}$ is a Frobenius algebra in the monoidal category $\mathcal{M}^H$.*

We have a similar theorem for the notion of right H-Frobenius. Also there exists a connection between left H-Frobenius and right H-Frobenius. The connection is given by the following:

Theorem 13. *Let H be a Hopf algebra, and let A be a finite dimensional right H-comodule algebra. The following two assertions hold.*

1. *If A is right H-Frobenius, then A is left H-Frobenius*
2. *If the antipode S is injective and A is left H-Frobenius, then A is also right H-Frobenius.*

The next thing is to particularize the Hopf algebra H to kG, where G is an arbitrary group. In this case $\mathcal{M}^{kG}$ will be the monoidal category of G-graded vector spaces, and an algebra A in this category will be just a G-graded algebra.

Proposition 14. *Let A be a finite dimensional G-graded algebra, and let $\sigma \in G$. The following conditions are equivalent.*

1. *$A(\sigma) \simeq A^*$ in A-gr*
2. *$(\sigma)A \simeq A^*$ in gr-A*

Using the above equivalence we can give the definition for σ-graded Frobenius algebra. So we say that a finite dimensional G-graded algebra A is called σ-graded Frobenius if it satisfies the equivalent conditions in the above proposition. Clearly, e-graded Frobenius means just graded Frobenius.

We study basic properties of σ-graded Frobenius and give several characterizations of them. One of our main results of this section says that:

Theorem 15. *Let $A = \bigoplus_{g \in G} A_g$ be a finite dimensional G-graded algebra, and let $\sigma \in G$. The following assertions are equivalent.*

1. *A is σ-graded Frobenius*
2. *$A_\sigma \simeq A_e^*$ as left A_e-modules, and A is left σ-faithful*
3. *$A_\sigma \simeq A_e^*$ as right A_e-modules, and A is right σ-faithful*

In particular this theorem gives the structure of Frobenius algebras in the category of graded vector spaces.

Among graded Frobenius algebras there are some objects with more symmetry: the graded symmetric algebras, which are just the symmetric algebras in the sovereign category of graded vector spaces. We say that a finite dimensional graded algebra A is called graded symmetric if A and A^* are isomorphic as graded left-A, right-A bimodules. Afterwards we give a theorem that characterizes the property of being graded symmetric.

Theorem 16. *Let $A = \bigoplus_{g \in G} A_g$ be a finite dimensional G-graded algebra. The following assertions are equivalent.*

1. *A is graded symmetric*
2. *There exists a non-degenerate associative symmetric bilinear form $B : A \times A \to k$ such that $B(r_\tau, r_\nu) = 0$ for any $r_\tau \in A_\tau$, $r_\nu \in A_\nu$, with $\tau\nu \neq e$*
3. *There exists a linear map $\lambda : A \to k$ such that $\lambda(xy) = \lambda(yx)$ for any $x, y \in A$, $Ker(\lambda)$ does not contain non-zero graded left ideals, and $\lambda(x_\sigma y_\tau) = 0$ for any $x_\sigma \in A_\sigma$, $x_\tau \in A_\tau$ with $\sigma\tau \neq e$.*

From an example of Eilenberg and Nakayamma it is known that every semisimple algebra is symmetric. Taking into account what happens in the monoidal category of vector spaces, it is a natural question to ask whether finite dimensional graded semisimple algebras are graded symmetric. The first step is to look at graded division algebras. We show that this is indeed the case, provided that the characteristic of k does not divide the dimension of A, in particular is always true in characteristic 0. Using the structure of graded semisimple algebras, we uncover a class of such objects that are graded symmetric; in particular any finite dimensional graded semisimple algebras is graded symmetric if char k= 0.

In one section of this chapter we also discuss the concept of graded Frobenius in relation to Frobenius functors. It is presented at the beginning an already known theorem, that gives the connection between Frobenius algebras and Frobenius functors and afterwards we prove the same theorem, but in the graded case.

Theorem 17. *A is graded Frobenius if and only if the forgetful functor $U : A - gr \to k - gr$ is a Frobenius functor.*

Finally, we give a new proof of a result of Bergen using Frobenius functors, stating that if H is a finite dimensional Hopf algebra acting on a finite dimensional algebra A, then the smash product $A\#H$ is Frobenius if and only if so is A.

The symmetric algebras can be defined in several monoidal categories, like the sovereign monoidal categories. However, it is not clear how one could define symmetric algebras in an arbitrary monoidal category. So, the question is for which Hopf algebra H, we can give "a good definition" for the concept of symmetric algebra in the category of right H-comodule algebra. The answer is for cosovereign Hopf algebras.

Definition 18. ***Cosovereign Hopf algebra*** *is a Hopf algebra H with a character u on H(i.e. u= grouplike on H^*) such that*

$$S^2(h) = \sum u^{-1}(h_1)u(h_3)h_2, \; \forall h \in H$$

We say that u is a sovereign character on H

The map $f : A \to A^{(S^2)}$, $f(a) = u^{-1} \cdot a$, is an isomorphism of right H-comodule algebras, and its induces an isomorphism of categories $F :_{A^{(S^2)}} \mathcal{M}^H \to_A \mathcal{M}^H$. By restriction this induces an isomorphism of categories(also denoted by F), $F :_{A^{(S^2)}} \mathcal{M}_A^H \to_A \mathcal{M}_A^H$. Now, if $A^* \in_{A^{(S^2)}} \mathcal{M}_A^H$ then $F(A^*) \in_A \mathcal{M}^H$. Using the observations above we can define symmetric algebras in the category $\mathcal{M}^H$ with respect to u.

Definition 19. *Let H be a cosovereign Hopf algebra with u as a sovereign character. A finite dimensional right H-comodule algebra A is a symmetric algebra with respect to u if*

$$F(A^*) \simeq A \text{ in the category } {}_A\mathcal{M}^A$$

We say simply that A is (H, u)-symmetric.

Afterwards we give explicit characterizations of this property in $\mathcal{M}^H$ and show that the definition of symmetry depends on the character. It is possible that a cosovereign Hopf algebra H has two sovereign characters u and v, such that a right H-comodule algebra A is (H, u)-symmetric, but not (H, v)-symmetric. Also, we use a modified version of the trivial extension construction to give examples of (H, u)-symmetric algebras of corepresentations.

Theorem 20. *Let H be a cosovereign Hopf algebra with sovereign character u, A a right H-comodule algebra, and $\varepsilon(A) = A \bigoplus F(A^*)$, with the direct sum structure of a right H-comodule and the algebra structure obtained by the trivial extension of A and the A,A-bimodule $F(A^*)$. Then*

$$\varepsilon(A) = A \bigoplus F(A^*)$$

is a right H-comodule algebra which is (H, u)-symmetric.

In the case where H is involuntary, i.e. $S^2 = Id$, H is cosovereign if we take $u = \varepsilon$, the counit of H, and in this case it is clear that an (H, ε)-symmetric algebra is also symmetric as a k-algebra. However, we show that in general H may be (H, u)-symmetric, without being symmetric as a k-algebra.

Given a finite dimensional algebra A in the category $\mathcal{M}^H$, where H is a finite dimensional Hopf algebra, one can construct the smash product $A\#H^*$. Smash product constructions are of great relevance since they describe the algebra structure in a process of bosonization, which associates for instance a Hopf algebra to a Hopf superalgebra. It was proved by Bergen as I said earlie that A is Frobenius if and only if so is $A\#H^*$. On the other hand, we show in an example that such a good connection does not hold for the symmetry property. Our main aim in this section is to study the connection between A being a Frobenius algebra in $\mathcal{M}^H$ and $A\#H^*$ being a Frobenius algebra in $\mathcal{M}^{H^*}$. We show that just an implication holds.

Theorem 21. *Let H be a finite dimensional Hopf algebra and A be a finite dimensional right H-comodule algebra. Then if A is a Frobenius algebra in $\mathcal{M}^H$, $A\#H^*$ is a Frobenius algebra in $\mathcal{M}^{H^*}$.*

The next aim we have is to establish a good transfer of the symmetry property between A and $A\#H^*$.

Theorem 22. *Let H be a finite dimensional Hopf algebra with antipode S, and let g and α be the distinguish grouplike elements of H and H^*. We assume that $S^2(h) = g^{-1}hg = \sum \alpha^{-1}(h_1)\alpha(h_3)h_2$, $\forall h \in H$. If A is a right H-comodule algebra, then*

$$A \textit{ is } (H,\alpha)\textit{-symmetric} \Leftrightarrow A\#H^* \textit{ is } (H^*, g)\textit{-symmetric.}$$

If A is a right H-comodule algebra which is Frobenius (respectively symmetric) as an algebra it is a natural question to ask whether this property transfers to the subalgebra of coinvariants. It is easy to see that the answer is no. However, we show that under some extra condition the transfer takes place.

Group gradings on polynomial algebras

Let k be a field and let $A = k[X_1, X_2, ..., X_n]$ be the algebra of polynomials in n indeterminates. Let G be a group, with the opperation denoted additively. Here we can always assumte that G is abelian.

We study a special class of linear gradings, called good gradings, characterized by the fact that any indeterminate is a homogenous element of nontrivial degree.Our aim main result is the following:

Theorem 23. *Let R be an IBN ring, and let $u, v : \{1, 2, .., n\} \to G - \{0\}$ be functions, where G is an abelian group. Let $\mathcal{A} = R[X_1, X_2, ..., X_n]$ be the G-grading ring such that X_i is homogenous of degree $u(i)$, for any $1 \leq i \leq n$, and the elements of R has degree 0. Let $\mathcal{B} = R[X_1, ..., X_n]$ with the G-grading defined similarly, by using v. If there exists an isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ of G-graded rings such that $\phi(r) = r$ for any $r \in R$, then there exists a permutation $\sigma \in S_n$ such that $v = u\sigma$.*

The original contributions of the thesis are contained in the papers:

1. S. Crivei, C. Năstăsescu, L. Năstăsescu, A generalization of the Mitchell Lemma: The Ulmer Theorem and the Gabriel-Popescu Theorem revisited, J.Pure Appl Algebra **216**, (2012), 2126-2129

2. F. Castano-Iglesias, C. Năstăsescu and L. Năstăsescu, Locally stable Grothendieck categories: Applications, Appl. Categor. Struct. **21** (2013), 105-118

3. S. Dăscălescu, C. Năstăsescu and L. Năstăsescu, Frobenius algebras of corepresentations and group graded vector spaces, J. Algebra **406** (2014), 226-250

4. S. Dăscălescu, C. Năstăsescu and L. Năstăsescu, Are graded semisimple algebras symmetric?, preprint, arXiv:1504.04868

5. S. Dăscălescu, C. Năstăsescu and L. Năstăsescu, Symmetric algebras in categories of corepresentations and smash products, Journal of Algebra **465** (2016), 62-80

6. S. Dăscălescu, C. Năstăsescu and L. Năstăsescu, Group gradings on polynomial algebras, Communications in Algebra, **44** (2016), no.8, 3340-3348

We also use standard notations from the following books: [18], [37], [38], [43], [49] and [53].

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