**UNIVERSITY OF BUCHAREST**

**FACULTY OF MATHEMATICS AND COMPUTER SCIENCE**

**DOCTORAL SCHOOL OF MATHEMATICS**

# *CONTRIBUTIONS TO SOME CLASSES OF GENERALIZED EQUILIBRIUM PROBLEMS*

## Summary

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*Dedicated to my missing father, Ion Biolan!*

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# INTRODUCTION

In this thesis we study some classes of generalized Nash equilibrium problems. Some characterizations of the solutions corresponding to players which share the same Lagrange Multipliers are given. According to [29], this kind of Nash equilibria concept was introduced by Rosen [93] in 1965 for finite dimensional spaces. In order to obtain the same property for the infinite dimensional approach, we use recent developments of a new duality theory. Regarding its usfulness new theorems are proven and similar kinds of equilibrium for pay-off interval type functions or their extended versions are approached. We also want to apply this special type of Nash Equlibrium conditions obtained above for interval functions to a particular class of interval functions that are applied in economy.We also give some generalizations for some particular problems from multi-period portfolio selection optimization models by using interval analysis. Other research direction consists in extending the concepts of generalized relaxed alpha-monotone application and generalized relaxed beta-monotone application to the generalized relaxed $\alpha - \beta -$monotone application and finally, to generalized relaxed $(r, s) - \alpha - \beta -$monotone application and to generalized $\rho -$mixed relaxed monotone application.

In [29] Faraci extended the Nash Equilibria concept defined by Rosen [93] in 1965 to infinite dimensional spaces. We extend this type of equilibria obtained in [29] to a class of functions, called interval functions.

Generalized Nash equilibrium problems (GNEP's) are noncooperative games in which the strategy of each player can depend on the rival players' strategies. These problems have become popular recently because of their utility for modeling economic problems, as well as routing problems in communication networks.

In the framework of this PhD thesis, by using the new theory, we are able to prove the existence of Lagrange Multipliers for GNEP's in general Banach spaces and to extend the results to the infinite dimension case.

In **Chapter 1, Existence of Equilibrium. The Nash Equilibrium**, the three basic theorems of general equilibrium theory are introduced. Also, some Generalized Nash Equilibrium Problems are presented.In Section 1.1 we establish equilibrium existence results for a "generalized competitive" mechanism. In Section 1.2 we reveal the applications of this result to a "fixed allocation" mechanism. In Section 1.3 the welfare theorems are presentes and Nash equilibrium concept is discussed in Section 1.4. Algorithms for the solution of GNEPs are presented in Section 1.5.

In **Chapter 2, Generalized Nash Equilibrium Problems in infinite dimension and semiinfinite optimization. An interval approach with applications**, we study a special type of Nash equilibria, corresponding to the case when the pay-off functions associated to the players whose objective is to maximize their winning chances are described by interval functions. We prove the existence of optimum interval equilibrium point. The original concepts introduced in this chapter are included in Definitions 2.2, 2.3, 2.4. The original contributions obtained in this chapter are included in Theorems 2.1, 2.2, 2.3, 2.4 and 2.5 and also in the relationships obtained in Section 2.1 for the Gateaux Derivatives, which bring a new approach based on interval modelling combined with the Lagrange Multipliers Rule. The reformulation of some equilibrium problems under more general conditions in Section 2.2 constitute also our original contributions in this field.

In **Chapter 3, Generalized equilibrium problems with relaxed assumptions**, we introduce the new following concepts: generalized relaxed $(r, s) - \alpha - \beta-$monotone application and generalized $\rho-$mixed relaxed monotone application. We extend the concept of relaxed $\alpha-$monotonicity to mixed relaxed $\alpha-\beta-$monotonicity. Finally, this concept is applied with KKM-theory to solve a generalized equilibrium problem. The original contributions obtained in this Section 3.3 are included in Theorems 3.1 and 3.2, regarding existence of the solution for generalized equilibrium problems. The original contributions obtained in this Section 3.3 are included in Theorems 3.3 and 3.4, which brings up new the concept of mixed relaxed $\alpha - \beta-$monotone application and the existence results regarding equilibrium problems associated to this new concept. The original contributions obtained in Section 3.4 are included in Theorems 3.5 and 3.6, which brings up new the concept of $\rho-$mixed relaxed monotone application and $\rho-$convex application and also the existence results regarding equilibrium problems associated to this new concept.

In **Chapter 4, Generalized mixed equilibrium problems**, some existence theorems are obtained and iterative approximation methods for generalized mixed equilibrium problems corresponding to a countable family of nonexpansive mappings are developed. The original contributions obtained in this chapter are included in Theorems 4.1, 4.2, 4.3 and 4.4. These results extend recent results obtained in this field by Kamraksa and Wangkeeree in 2012.

# Chapter 1

# The existence of equilibrium. The Nash equilibrium

This chapter contains some general results regarding equilibrium problems and Nash equilibrium problems.

## 1.1 Generalized Competitive Mechanisms

## 1.2 Application of the Existence Theorem

## 1.3 Welfare Theorems:

**Theorem 1.2.** (First Welfare Theorem). If the preferences are strictly monotone, then any equilibrium of a GCM it is Pareto efficient.

**Theorem 1.4.** (Second Welfare Theorem). Assume that the preferences and the production sets verify the hypotheses in Theorem 1. Then, if $\left\{\left\{\widetilde{x}^h\right\}, \left\{\widetilde{y}^f\right\}\right\}$ is Pareto efficient and $\widetilde{x}^h$ is strictly positive for all $h$, there exist prices $p$ and balanced transfers $\{T^h\}$ (i.e., summing to zero) such that the pair $\left\{\left\{\widetilde{x}^h\right\}, \left\{\widetilde{y}^f\right\}\right\}$ is an equilibrium allocation, with respect to the mechanism that, for each $p$, the consumer h will have the income $p \cdot \omega^h + \sum\limits_f \theta_f^h p \cdot \widetilde{y}^f + T^h$.

## 1.4 Nash Equilibrium

We shall now state Nash's Equilibrium theorem in it's original form, from 1950.

Let $G = (S, u)$ be a finite game of $n$ players in its normal form.

$S = S_1 \times S_2 \times \ldots \times S_n$, $S$ is non-empty, represents the set of feasible strategies, $(S_k)_{k=\overline{1,n}}$ are the sets of individual strategies, $u : S \to \mathbb{R}$ represents the pay-off function.

## 1.5 Generalized Nash Equilibrium Problems

Let us consider a game of $N$ players and $\nu \in \overline{1,N}$. Each player $\nu$ controls his strategy vector:

$x^\nu := \left(x_1^\nu, ..., x_{n_\nu}^\nu\right)^T \in \mathbb{R}^{n_\nu}$,

of $n_v$ decision variables.The vector $x := \left(x^1, ..., x^N\right)^T \in \mathbb{R}^N$

contains the $n = \sum\limits_{\nu=1}^{N} \nu$ decision variables of all players. To emphasize the $\nu-$th player's variables within $x$, we sometimes write $(x^\nu, x^{-\nu})$ instead of $x$, where: $x^{-\nu} := (x^\mu)_{\mu=\overline{1,N},\ \mu\neq\nu}$.

Each player $\nu$ has an objective function $\theta_\nu : \mathbb{R}^N \to \mathbb{R}$ that may depend on both the player's decision variables $x^\nu$ and the decision variables $x^{-\nu}$ of the rival players. With respect to the practical setting, the objective function of a player is sometimes called *utility function*, *payoff function* or *loss function*. Moreover, each player's strategy $x^\nu$ has to belong to a set $X_\nu (x^{-\nu}) \subseteq \mathbb{R}^{n_\nu}$ that is allowed to depend on the rival players' strategies. The set $X_\nu (x^{-\nu})$ is called *feasible set* or *strategy space* of player $\nu$. In many applications the feasible set is defined by inequality constraints, i.e., for each $\nu = \overline{1,N}$, there is a continuous function $g^\nu : \mathbb{R}^n \to \mathbb{R}^{n_\nu}$ so that:

$$X_\nu \left(x^{-\nu}\right) = \left\{x^\nu \in \mathbb{R}^{n_\nu} | g\left(x^\nu, x^{-\nu}\right) \leq 0\right\}. \tag{1.1}$$

For any given $x \in \mathbb{R}^n$, let us define:

$X(x) := \prod\limits_{\nu=1}^{N} X_\nu (x^{-\nu}) = \left\{y \in \mathbb{R}^n | y^\nu \in X_\nu (x^{-\nu}),\ \forall \nu = \overline{1,N}\right\}.$

If we fix the rival players' strategies $x^{-\nu}$, the aim of player $\nu$ is to choose a strategy $x^{\nu} \in X_{\nu}\left(x^{-\nu}\right)$ which solves the optimization problem:

$$\min_{\mathrm{x}_{\nu}} \quad \theta_{\nu}(x^{\nu}, x^{-\nu})$$

$$\text{such that} \quad x^{\nu} \in X_{\nu}\left(x^{-\nu}\right)$$

The GNEP is the problem of finding $x^{*} \in X(x^{*})$ such that, for all $\nu = \overline{1, N}$, the following property holds:

$\theta_{\nu}(x^{*,\nu}, x^{*,-\nu}) \leq \theta_{\nu}(x^{\nu}, x^{*,-\nu})$ for all $x^{\nu} \in X_{\nu}(x^{*,-\nu})$.

# Chapter 2

# Generalized Nash Equilibrium Problems in infinite dimension and semiinfinite optimization. An interval approach with applications

## 2.1 A Lagrange multiplier approach using interval functions for Generalized Nash Equilibrium Problems in infinite dimensions

### 2.1.1 Preliminaries

**Definition 2.1.** [3] We say that $\overline{u} = (\overline{u}^1, \overline{u}^2)$ is a generalized Nash equilibrium point or a solution of the GNEP if $\overline{u} \in K$ and the following conditions hold:

$J_1\left(\overline{u}^1, \overline{u}^2\right) = \min\left\{J_1\left(u^1, \overline{u}^2\right); u^1 \in K_1(\overline{u})\right\},$

$J_2\left(\overline{u}^1, \overline{u}^2\right) = \min\left\{J_2\left(\overline{u}^1, u^2\right); u^2 \in K_2(\overline{u})\right\}.$

Let $J_1$ and J$_2$ be two interval functions, $J_1, J_2 : X \to MI(\mathbb{R})$ the utility functions or pay-off functions so that $J_1(\cdot, u_2)$ is convex and Gateaux differentiable for every $u_2 \in X_2$ and $J_2(u_1, \cdot)$ is convex and Gateaux differentiable, for every $u^1 \in X_1$.

**Definition 2.3.** We say that $\overline{u} = (\overline{u}^1, \overline{u}^2)$ is an interval equilibrium point for GNEP if the following conditions hold:

(1) $J_1\left(\overline{u}^1, \overline{u}^2\right) = \min\left\{J_1\left(u^1, \overline{u}^2\right); u^1 \in K_1(\overline{u})\right\}$, where $\overline{u}^2$ is fixed;

(2) $J_2\left(\overline{u}^1, \overline{u}^2\right) = \min\left\{J_2\left(\overline{u}^1, u^2\right); u^2 \in K_2(\overline{u})\right\}$, where $\overline{u}^1$ is fixed;

From well-known results of convex analysis (see e.g. Theorem 3.8 of [4]), $\overline{u} = (\overline{u}^1, \overline{u}^2)$ is considered to be optimum interval for a GNEP interval game if and only if:

$$\begin{aligned}
D_1 J_1^L\left(\overline{u}^1, \overline{u}^2\right)\left(u^1 - \overline{u}^1\right) &\geq 0,\ \forall u^1 \in K_1(\overline{u}) \cap \left\{u^1 : J_1^U\left(u^1, \overline{u}^2\right) \leq J_1^U\left(\overline{u}^1, \overline{u}^2\right)\right\}, && (2.2)\\
D_1 J_1^U\left(\overline{u}^1, \overline{u}^2\right)\left(u^1 - \overline{u}^1\right) &\geq 0,\ \forall u^1 \in K_1(\overline{u}) \cap \left\{u^1 : J_1^L\left(u^1, \overline{u}^2\right) \leq J_1^L\left(\overline{u}^1, \overline{u}^2\right)\right\}, && (2.1)\\
D_2 J_2^L\left(\overline{u}^1, \overline{u}^2\right)\left(u^2 - \overline{u}^2\right) &\geq 0,\ \forall u^2 \in K_2(\overline{u}) \cap \left\{u^2 : J_2^U\left(\overline{u}^1, u^2\right) \leq J_2^U\left(\overline{u}^1, \overline{u}^2\right)\right\},\\
D_2 J_2^U\left(\overline{u}^1, \overline{u}^2\right)\left(u^2 - \overline{u}^2\right) &\geq 0,\ \forall u^2 \in K_2(\overline{u}) \cap \left\{u^2 : J_2^L\left(\overline{u}^1, u^2\right) \leq J_2^L\left(\overline{u}^1, \overline{u}^2\right)\right\},
\end{aligned}$$

where $D_1$ and $D_2$ stand for the Gateaux derivative of $J_1^L\left(\cdot, \overline{u}^2\right), J_1^U\left(\cdot, \overline{u}^2\right)$ and $J_2^U\left(\overline{u}^1, \cdot\right), J_2^L\left(\overline{u}^1, \cdot\right)$, respectively.

Denote by $\Gamma : X \to X_1^* \times X_2^*$,

$$\Gamma\left(u^1, u^2\right) = \begin{pmatrix} D_1 J_1^L\left(u^1, u^2\right) \\ D_1 J_1^U\left(u^1, u^2\right) \\ D_1 J_2^L\left(u^1, u^2\right) \\ D_1 J_2^U\left(u^1, u^2\right) \end{pmatrix}. \tag{2.3}$$

**Definition 2.5.** We say that $L_\psi(\alpha) = \{x : \psi(x) \leq \alpha\}$, where $\alpha \in \mathbb{R}$ is the underlevel subset of the function $\psi : X \to \mathbb{R}$.

Considering this, it is clear that (2.2) is equivalent with the following condition: $\Gamma(\overline{u})^T\left(u - \overline{u}\right) \geq 0, \forall u \in \left(K_1\left(\overline{u}\right) \cap L_{J_1^U}\left(J_1^U\left(\overline{u}^1, \overline{u}^2\right)\right) \cap L_{J_1^L}\left(J_1^L\left(\overline{u}^1, \overline{u}^2\right)\right)\right)$

$\times \left(K_2(\overline{u}) \cap L_{J_2^U}\left(J_2^U(\overline{u}^1,\overline{u}^2)\right) \cap L_{J_2^L}\left(J_2^L(\overline{u}^1,\overline{u}^2)\right)\right).$

Since the convex sets $K_i(\overline{u})$ depend on the solution, one obtains that GNEP for interval games can be formulated equivalently as a quasi-variational inequality. The nature of the optimal sets allows us to reduce the problem to variational inequalities. Solving this associated to $\Gamma$ and the set $K$ (in short: $VI(\Gamma, K)$), means finding a point $\overline{u} = (\overline{u}^1, \overline{u}^2) \in K$ such that we have the following inequality:

$$\Gamma(\overline{u})^T (u - \overline{u}) \geq 0, \forall u \in K. \tag{2.4}$$

**Theorem 2.1.** Every solution of the variational inequality $VI(\Gamma, K)$ is a solution of GNEP interval game.

## 2.2 The Lagrange multipliers rule

A solution of the GNEP interval games can be obtained as a solution of the $VI(\Gamma, K)$. By adopting the reduction method, we can lose solutions of the GNEP interval game.

We want to see now which kind of solutions are preserved for a special set of constraints. We follow the finite dimensional case [5] and prove that a solution of the GNEP interval game is a variational equilibrium iff the shared constraints have the same multipliers. The result is true under any constraints qualification condition.

If $f : X \to \mathbb{R}$ and $\overline{u} \in K$, we say that $\overline{u}$ is a solution of the minimal problem $(P_{f,K})$ [3] if:

$f(\overline{u}) = \min\{f(x) | x \in K\}$.

**Theorem 2.3.** (i) Let $\overline{u}$ be a solution of the $VI(\Gamma, K)$ so that a suitable constraints qualification condition for the $VI(\Gamma, K)$ takes place at $\overline{u}$. Then $\overline{u}$ is a solution of the GNEP–interval game such that both players have the same Lagrange multipliers.

(ii) $\overline{u}$ is a solution of the GNEP-interval game such that a constraints qualification condition takes place at $\overline{u}$ and both players have the same Lagrange multipliers. Then $\overline{u}$ is a solution of the $VI(\Gamma, K)$.

## 2.3 Nonsmooth interval semi-infinite optimization problem using Limiting subdifferentials

### 2.3.1 Preliminaries

**Definition 2.6.** At a point $x^* \in X$, $\xi$ is said to be the subgradient of a convex function $f$ if

$$(x - x^*)^T \xi \leq f(x) - f(x^*),\ \forall x \in X.$$

**Definition 2.7.** At a point $x^* \in X$, $\xi$ is said to be the subgradient of a strictly convex function $f$ if

$$(x - x^*)^T \xi < f(x) - f(x^*),\ \forall x \in X,\ x \neq x^*.$$

**Definition 2.8.** The set of all subgradients of $\phi$ at $x^*$ is called the subdifferential of $\phi$ at $x^*$ and is denoted by $\partial\phi(x^*)$.

We consider the following optimization problem:

$$\begin{aligned} &\min && F(x) \\ &\textit{subject to} && g_i(x) \leq 0, \quad i = \overline{1,m}, \\ &&& x \in C, \end{aligned} \tag{P}$$

where $F(x) = \left[f^L(x), f^U(x)\right]$ is an interval-valued function, $f^L(x), f^U(x)$ and $g_i(x) : X \to \mathbb{R}$ are continuous convex real-valued functions, $X$ is a real, locally convex space and $C$ is a convex subset of $X$.

Let us denote by

$$X^0 = \left\{x \in X \,\middle|\, g_i(x) \leq 0,\ i = \overline{1,m},\ x \in C\right\}$$

the feasible set of primal problem $(P)$.

### 2.3.2 Necessary and sufficient optimality conditions

In this section we give some necessary and/or sufficient optimality conditions for a nonsmooth interval optimization problem.

**Lema 1 (2.1)** *(Sun & Yang, 2013) Let $\overline{x}$ be a feasible solution of problem $(P)$. Then $\overline{x}$ is an optimal solution of problem $(P)$ iff $\overline{x}$ is an optimal solution of the following deterministic optimization problems $(P1)$ and $(P2)$ :*

$$\begin{aligned} \min \quad & f^L(x) \\ \text{subject to} \quad & f^U(x) \leq f^U(\overline{x}) \\ & g(x) \leq 0, \\ & x \in C, \end{aligned} \tag{P1}$$

$$\begin{aligned} \min \quad & f^U(x) \\ \text{subject to} \quad & f^L(x) \leq f^L(\overline{x}) \\ & g(x) \leq 0, \\ & x \in C. \end{aligned} \tag{P2}$$

Now we consider the following deterministic nonsmooth semiinfinite optimization problem considered in [69]:

$$\begin{aligned} \min \quad & \varphi(x) \\ \textit{subject to} \quad & a_i(x) \leq 0, i \in I \quad , \\ & x \in \mathbb{R}^n \end{aligned} \tag{SIP}$$

where $\varphi$ and $a_i, i \in I$, are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$.

**Theorem 2.4.** Let $\overline{x}$ be an optimal solution for the problem M $(SIP)$ and $I_0(\overline{x}) = \{i \in I : g_i(\overline{x}) = 0\}$. We suppose that $\varphi$ and $g_i, i \in I_0(\overline{x})$ are Lipschitz near $\overline{x}$ and $g_i$ for $i \in I \backslash I_0(\overline{x})$ is continuous at $\overline{x}$. T hen there exists a $\lambda = (\lambda_i)_{i \in I}$, where $\lambda_i \geq 0$ and $\geq \lambda_i \neq 0$ for finitely many $i \in I$, such that

$$0 \in \partial_L \varphi(\overline{x}) + \sum_{i \in I} \lambda_i \partial_L g_i(\overline{x})$$

and

$$\lambda_i g_i(\overline{x}) = 0, i \in I.$$

Now we give the following Karush-Kuhn-Yucker necessary optimality conditions for the nonsmooth interval problem:

$$\begin{aligned} \min \quad & F(x) \\ \textit{subject to} \quad & g_i(x) \le 0, i \in I \quad , \\ & x \in C \end{aligned} \tag{ISIP}$$

where $F(x) = \left[f^L(x), f^U(x)\right]$ is an interval-valued function, $f^L(x), f^U(x)$ and $g_i(x) : X \to \mathbb{R}$ are continuous convex real-valued functions, $X$ is a real, locally convex space and $C$ is a convex subset of $X$.

**Theorem 2.5.** Let $\overline{x}$ be an optimal solution of the problem $(ISIP)$. We suppose that $f^L, f^U$ and $g_i, i \in I(\overline{x}) = \{i \in I \,|g_i(\overline{x}) = 0\}$ are Lipschitz near $\overline{x}$ and $g_i$ for $i \notin I(\overline{x})$ is continuous at $\overline{x}$. Then there exists $\xi^* = \left(\xi^{L^*}, \xi^{U^*}\right) > 0$ and $\lambda^* = (\lambda_i^*)_{i\in I} \ge 0, \lambda_i^* \ne 0$ for finitely many $i \in I$, such that

$$0 \in \xi^{L^*}\partial_L f^L(\overline{x}) + \xi^{U^*}\partial_L f^U(\overline{x}) + \sum_{i\in I}\lambda_i^*\partial_L g_i(\overline{x})$$

and

$$\lambda_i^* g_i(\overline{x}) = 0, i \in I.$$

**Theorem 2.6.** Let $\overline{x}$ be a feasible solution of the problem $(ISIP)$, such that there exists $\xi^{L^*} > 0, \xi^{U^*} > 0, \lambda^* = (\lambda_i^*)_{i\in I} \ge 0, \lambda^* \ne 0$ with $\lambda_i^* \ne 0$ for finitely many $i \in I$, such that

$$0 \in \xi^{L^*}\partial_L f^{L^*}(\overline{x}) + \xi^{U^*}\partial_L f^U(\overline{x}) + \sum_{i\in I}\lambda_i^*\partial_L g_i(\overline{x}), \tag{2.2}$$

$$\lambda_i^* g_i(\overline{x}) = 0, i \in I.$$

If $f^L$ and $f^U$ are $(\phi, \rho_{f^L}), (\phi, \rho_{f^U})$ strict pseudo invex, $g_i, i \in I(\overline{x})$ are $(\phi, \rho_{g_i})$ quasiinvex at $\overline{x}$ and

$$\rho_{f^L} + \rho_{f^U} \geqq 0, \tag{2.3}$$

then $\overline{x}$ is an optimal solution for the problem $(ISIP)$.

**Definition 2.9.** $\overline{x}$ is a local optimal solution of the problem $(ISIP)$ if there exists $\delta > 0$ such that $\overline{x}$ is an optimal solution in $B_\delta(\overline{x})$ the admissible set for $(ISIP)$.

**Theorem 2.7.** Let $\overline{x}$ be a feasible solution of the problem $(ISIP)$. Suppose that $f^L, f^U$ and $g_i, i \in I(\overline{x})$ are invex near $\overline{x}$. Also we assume $D^L = \varnothing,\ D^U = \varnothing$. Then $\overline{x}$ is a local optimal solution of the problem $(ISIP)$.

### 2.3.3 Duality. A Wolfe-type interval dual problem

**Theorem 2.8.** (Weak Duality) Let $x$ be feasible solution for $(SIP)$ and $(u, \xi^L, \xi^U, \lambda)$ be a feasible solution for (**??**). We suppose that $f^L, f^U$ and $g_i, i \in I$ are $(\phi, \rho^L), (\phi, \rho^U)$ and $(\phi, \rho_i), i \in I$ invex respectively, with $\xi^L \rho^L + \xi^U \rho^U + \sum\limits_{i \in I} \lambda_i g_i \geq 0$. If $g_i$ for $i \notin I(x) = \{i : g_i(x) = 0\}$ is continuous at $x$, then

$$f(x) \not< f(u) + \sum_{i \in I} \lambda_i g_i(u).$$

**Theorem 2.9.** (Strong duality) Let $\overline{x}$ be an optimal solution for the $(ISIP)$, $f^L, f^U$ and $g_i, i \in I$ satisfy the hypothesis of the weak duality theorem. If the problems $P^L(\overline{x})$ and $P^U(\overline{x})$ [107] satisfy a suitable constraint qualification, then there exists $\overline{\lambda} = (\overline{\lambda}_i)_{i \in I}, \overline{\xi}^L, \overline{\xi}^U > 0$ so that $\left(\overline{x}, \overline{\xi}^L, \overline{\xi}^U, \overline{\lambda}\right)$ is an optimal solution for (**??**) and the respective objective values are equal.

**Theorem 2.10.** (strict convex duality) Let $\widetilde{x}$ and $\left(\overline{x}, \overline{\xi}^L, \overline{\xi}^U, \overline{\lambda}\right)$ be an optimal solution for $(ISIP)$ and (**??**) respectively. We suppose that $f^L, f^U$ and $g_i, i \in I$ are $(\phi, \rho^L), (\phi, \rho^U)$ and $(\phi, \rho_i), i \in I$ respectively convex functions and for any feasible solution $x$ for $(ISIP)$, $g_i$ is continuous at $x$ for any $i \in I(x) = \{i : g_i(x) = 0\}$. If some constraint qualifications are satisfied by the problems $\left[P^L(\overline{x}), P^U(\overline{x})\right]$ and $f^L$

is $(\phi, \rho^L)$ strict convex or $f^U$ is $(\phi, \rho^U)$ strict convex or there exists $\alpha \in \{L, U\}$ such that $f^\alpha$ is $(\phi, \rho^\alpha)$ strict convex at $\overline{x}$ w.r.t. $\eta$, then $\widetilde{x} = \overline{x}$.

**Theorem 2.11.** Let $\widetilde{x}$ and $\left(\overline{x}, \overline{\xi}^L, \overline{\xi}^U, \overline{\lambda}\right)$ be feasible solutions for $(ISIP)$ and (**??**) respectively, such that $\overline{\xi}^L f^L(\widetilde{x}) + \overline{\xi}^U f^U(\widetilde{x}) \leq \overline{\xi}^L f^L(\overline{x}) + \overline{\xi}^U f^U(\overline{x}) + \sum_{i \in I} \overline{\lambda}_i g_i(\overline{x})$ and the application $x \rightsquigarrow \overline{\xi}^L f^L + \overline{\xi}^U f^U + \sum_{i \in I} \overline{\lambda}_i g_i$ is $(\phi, \rho)$ strict convex at $\overline{x}$, with $\rho > 0$. Then $\widetilde{x} = \overline{x}$ and $\overline{x}$ is an optimal solution for $(ISIP)$.

## 2.4 Interval Functions And Applications To Economy of Interval GNEP

### 2.4.1 The Mathematical Model

Let $X$ and $Y$ be two Banach spaces and let $Z = X \times Y$ the product space and let $z = (x, y)$ an element of $Z$. The variable $x$ corresponds to the first player and the variable $y$ corresponds to the second one. Let $C \subset Z$ a non-empty convex set and let $f, g : X \to \mathbb{R}$ be two functionals, also known as the utility functions or the pay-off functions so that $f(\cdot, y)$ it is convex and Gateaux differentiable for every $y \in Y$ and $g(x, \cdot)$ is convex and Gateaux differentiable, for every $x \in X$.

For every $z = (x, y)$ the sets of the feasible strategies of the two players are of the following type:

$C_1(z) = \{x' \in X : (x', y) \in C\} \subset X,$

$C_2(z) = \{y' \in Y : (x, y') \in C\} \subset Y.$

The purpose of each player, given the strategy of the rival, it is to choose a strategy which minimizes the function $f$ or $g$ on its feasible set.

### 2.4.2 The Economic Model

The aim of this section is to prove that, if a trader has a moment in time when he usually enters in a period when he has positive profit $(W_T(x) > 0)$or negative profit $(W_T(x) < 0)$ , this moment in time can be modeled as an equilibrium point and can be determined given the interval variables from below [62]. So as there, we consider the following economic model for which if $W_T$ has a continuous form and it is Gateaux-Differentiable we can apply conditions (**??**) and Theorem 2.1 in order to obtain an optimum interval point. The model introduced in [62] considers a financial market with $n$ risky assets for trading. An investor intends to invest his wealth $W_0$ among the $n$ risky assets at the beginning of period 1 for constructing a $T-$period investment. The investor can reallocate his wealth at the beginning of each of the following $T-1$ consecutive time periods. It is assumed that the returns, risk and turnover rates of assets are interval numbers and the returns of portfolios in $T$ different periods are independent of each other. We denote by:

$r_{t,i}$ the return of risky asset $i$ at period $t$, where $r_{t,i} = \left[r^L_{t,i}, r^D_{t,i}\right]$, obviously: $r^L_{t,i} \leq r^D_{t,i}$;

$\delta_{i,j,t} = \left[\delta^L_{i,j,t}, \delta^D_{i,j,t}\right]$ the covariance between $r_{t,i}$ and $r_{t,j}$, where $\delta^L_{i,j,t} \leq \delta^D_{i,j,t}$;

$c_{t,i} = \left[c^L_{t,i}, c^D_{t,i}\right]$ the transaction cost rate of risky asset $i$ at period $t$;

$x_{t,i}$ the investment proportion of risky asset $i$ at period $t$;

$d_{t,i} = \left[d^L_{t,i}, d^D_{t,i}\right]$ is given;

$C_t = \sum\limits_{i=1}^{n} c_{t,i}x_{t,i} + d_{t,i}$ the transaction cost rate, $C_t = \left[C^L_t, C^D_t\right] = \left[\sum\limits_{i=1}^{n} c^L_{t,i}x^L_{t,i} + d^L_{t,i}, \sum\limits_{i=1}^{n} c^U_{t,i}x^U_{t,i} + d^U_{t,i}\right]$

$\delta_t = \left[\delta^L_t, \delta^D_t\right]$ the preset maximum risk tolerance interval of the portfolio at the period $t$, where $\delta^L_t \leq \delta^D_t$;

$l_{t,i} = \left[l^L_{t,i}, l^D_{t,i}\right]$ the interval turnover rate of risky asset i, where $l^L_{t,i} \leq l^D_{t,i}$;

$l_t = \left[l^L_t, l^D_t\right]$ the preset minimum expected interval valued turnover rate of the porfolio at period $t$, with $l^L_t \leq l^D_t$;

$e_t$ the preset minimum diversification degree of the $t-$th period portfolio;

$$W_T(x) = \left[W_0 \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}r^L_{t,i} - C^L_t\right), W_0 \prod_{t=1}^{T}\left(\sum_{i=1}^{n} x_{t,i}r^D_{t,i} - C^D_t\right)\right]$$

the available wealth at the end of the period $t, t = \overline{1,T}$.

Let $W_T(x)^L = W_0 \prod\limits_{t=1}^{T}\left(\sum\limits_{i=1}^{n} x_{t,i}r^L_{t,i} - C^L_t\right), W_T(x)^L = W_0 \prod\limits_{t=1}^{T}\left(\sum\limits_{i=1}^{n} x_{t,i}r^D_{t,i} - C^D_t\right).$

Then, the optimality conditions from (2.2) are:

$$\left\{\begin{array}{c} 0 \leq \sum_{t=1}^{T}\sum_{i=1}^{n} W_T(x)^L \dfrac{r_{t,i}^L - c_{t,i}^L}{\sum_{i=1}^{n} x_{t,i} r_{t,i}^L - C_t^L} \left(r_{t,i}^L - \overline{r_{t,i}^L}\right) \\ 0 \leq \sum_{t=1}^{T}\sum_{i=1}^{n} W_T(x)^D \dfrac{r_{t,i}^D - c_{t,i}^D}{\sum_{i=1}^{n} x_{t,i} r_{t,i}^D - C_t^D} \left(r_{t,i}^D - \overline{r_{t,i}^D}\right) \\ \left[\overline{r_{t,i}^L}, \overline{r_{t,i}^D}\right] \geq 0 \end{array}\right\} \quad (6)$$

An interval multi-period selection model is now ready to be formulated as follows:

$$\begin{array}{ll} \max & W_T(x) \\ \text{s.t.} & \left[\sum_{i=1}^{n} x_{t,i} r_{t,i}^L - C_t^L, \sum_{i=1}^{n} x_{t,i} r_{t,i}^D - C_t^D\right] \geq \left[R_t^L, R_t^D\right] \\ & \left[\sum_{i=1}^{n}\sum_{k=1}^{n} x_{t,i} x_{t,k} \delta_{i,k,t}^L, \sum_{i=1}^{n}\sum_{k=1}^{n} x_{t,i} x_{t,k} \delta_{i,k,t}^D\right] \leq \delta_t \\ & \left[\sum_{i=1}^{n} x_{t,i} l_{t,i}^L, \sum_{i=1}^{n} x_{t,i} l_{t,i}^D\right] \geq l_t \\ & -\sum_{i=1}^{n} x_{t,i} \ln(x_{t,i}) \geq e_t \\ & \sum_{i=1}^{n} x_{t,i} = 1 \\ & x_{t,i} \geq 0, i = \overline{1,n}, t = \overline{1,T} \end{array} \quad (7)$$

Problem (P1) can be reformulated like:

$$\begin{array}{ll} \max & W_T(x)^L \\ \max & W_T(x)^D \quad (8) \\ \text{s.t.} & x \in \Omega \end{array}$$

Conditions from (8) represent problem (P2). Again, the above problem can be rewriten like:

$$\begin{array}{ll} \max & f(x) = \lambda W_T(x)^L + (1-\lambda) W_T(x)^D \\ \text{s.t.} & x \in \Omega \end{array} \quad (9)$$

Conditions (9) above, are called problem (P3) can also be transformed, adding the restrictions from [62], into a crisp form nonlinear programming problem:

$$
\begin{aligned}
\max \quad & f(x) = \lambda W_T(x)^L + (1-\lambda) W_T(x)^D \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{t,i} r_{t,i}^D - \left( \sum_{i=1}^{n} c_{t,i}^D x_{t,i} + d_{t,i}^D \right) \geq R_t^L \\
& \sum_{i=1}^{n} \sum_{k=1}^{n} x_{t,i} x_{t,k} \delta_{i,k,t}^L \leq \delta_t^D \\
& \sum_{i=1}^{n} x_{t,i} l_{t,i}^D \geq l_t^L \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10) \\
& -\sum_{i=1}^{n} x_{t,i} \ln(x_{t,i}) \geq e_t, t = \overline{1,T} \\
& \sum_{i=1}^{n} x_{t,i} = 1, t = \overline{1,T} \\
& x_{t,i} \geq 0, i = \overline{1,n}, t = \overline{1,T}
\end{aligned}
$$

Conditions (10) form problem (P4), and it can be rewritten into the following form nonlinear programming problem (P5):

$$
\begin{aligned}
\max \quad & f(x) = \lambda W_T(x)^L + (1-\lambda) W_T(x)^D \\
\text{s.t.} \quad & R_t^L - \left( \sum_{i=1}^{n} x_{t,i} r_{t,i}^D - \left( \sum_{i=1}^{n} c_{t,i}^D x_{t,i} + d_{t,i}^D \right) \right) \leq 0, t = \overline{1,T} \\
& \sum_{i=1}^{n} \sum_{k=1}^{n} x_{t,i} x_{t,k} \delta_{i,k,t}^L - \delta_t^D \leq 0, t = \overline{1,T} \\
& l_t^L - \sum_{i=1}^{n} x_{t,i} l_{t,i}^D \leq 0, t = \overline{1,T} \\
& e_t + \sum_{i=1}^{n} x_{t,i} \ln(x_{t,i}) \leq 0, t = \overline{1,T} \\
& \sum_{i=1}^{n} x_{t,i} - 1 = 0, t = \overline{1,T} \\
& -x_{t,i} \leq 0, i = \overline{1,n}, t = \overline{1,T}
\end{aligned}
$$

# Chapter 3

# Generalized equilibrium problems with relaxed assumptions

## 3.1 Mathematical Background

Let $K$ be a non-empty subset of a real Banach space $X$. Let $\phi : K \times K \to \mathbb{R}$ be a real valued function and let $f : K \times K \to \mathbb{R}$ be an equilibrium function, i.e. $f(x, x) = 0$, for all $x \in K$.

We will now consider the following generalized equilibrium problem: find $\overline{x} \in K$ in order to have the folllowing relation:

$$f(\overline{x}, y) + \phi(\overline{x}, y) - \phi(\overline{x}, \overline{x}) \geq 0, \forall y \in K \tag{3.1}$$

**Definition 3.1.** A real valued function $f$ defined on a convex subset $K$ of $X$ is said to be hemicontinuous if

$$\lim_{t \to 0^+} f\left(tx + (1-t)y\right) = f(y), \text{ for each } x, y \in K. \tag{3.3}$$

**Definition 3.2.** Let $f : X \to 2^X$ be a multivalued mapping.Then $f$ is said to be a KKM-mapping if, for any finite subset $\{y_1, y_2, ..., y_n\}$ of $K$, $\text{co}\{y_1, y_2, ..., y_n\} \subset \bigcup_{i=1}^{n} f(y_i)$, where co denotes the convex hull.

**Lemma 3.1.** Let $K$ be a nonempty subset of a topological vector space $X$ and let $f : X \to 2^X$ be a KKM mapping. If $f(y)$ is closed in $X$ for all $y \in K$ and compact for at least one $y \in K$, then $\bigcap_{i=1}^{k} f(y) = \varnothing$.

**Definition 3.3.** Let $X$ be a Banach space. A mapping $f : X \to \mathbb{R}$ is said to be weakly lower semicontinuous at $x_0 \in X$, if the following stands as true:

$$f(x_0) \leq \lim_{n\to\infty} \inf f(x_n), \tag{3.4}$$

for any sequence $\{x_n\}$ of $X$ such that $x_n \to x_0$.

**Definition 3.4.** Let $X$ be a Banach space. A mapping $f : X \to \mathbb{R}$ is said to be weakly upper semicontinuous at $x_0 \in X$, if the following stands as true:

$$f(x_0) \geq \lim_{n\to\infty} \inf f(x_n), \tag{3.5}$$

for any sequence $\{x_n\}$ of $X$ such that $x_n \to x_0$.

**Definition 3.5.** A mapping $f : K \times K \to \mathbb{R}$ is said to be mixed relaxed $\alpha - \beta-$monotone, if there exist the mappings $\alpha : K \to \mathbb{R}$ with $\alpha(tx) = t^p\alpha(x)$, for all $t > 0$ and $\beta : K \times K \to \mathbb{R}$, such that

$$f(x,y) + f(y,x) \leq \alpha(y-x) + \beta(x,y), \forall x, y \in K, \tag{3.6}$$

where

$$\lim_{t\to 0} \left[ \frac{t^p\alpha(y-x)}{t} + \frac{\beta(x, ty + (1-t)x)}{t} \right] = 0 \tag{3.7}$$

and $p > 1$ is a constant.

**Definition 3.6.** A function $f$ is said to be $(r,s)-(\alpha,\beta)$ monotone if the following holds:

$\frac{1}{r}\left[e^{rf(x,y)} - 1\right] + \frac{1}{r}\left[e^{rf(x,y)} - 1\right] \leq \alpha(y-x) + \beta(x,y)$, $r \leq s$, meaning:

$f_r(x,y) + f_s(y,x) \leq \alpha(y-x) + \beta(x,y)$, $r \leq s$.

**Definition 3.7.** A mapping $\phi : K \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be 0-diagonally convex if, for any finite subset $\{x_1, x_2, ..., x_n\}$ of $K$ and $\lambda_i \geq 0, i = \overline{1, n}$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\overline{x} = \sum_{i=1}^{n} \lambda_i x_i$, one has :

$$\sum_{i=1}^{n} \lambda_i \phi(\overline{x}, x_i) \geq 0. \tag{3.13}$$

## 3.2 Existence of Solution for Generalized Equilibrium Problem

**Theorem 3.1.** Suppose $f : K \times K \to \mathbb{R}$ is mixed relaxed $\alpha - \beta-$monotone, hemicontinuous in the first argument with $f(x, x) = 0$, for all $x \in K$. Let $\phi : K \times K \to \mathbb{R}$ be convex in the second argument. Then, the generalized equilibrium problem (3.1) is equivalent with the following problem. Find $\overline{x} \in K$ such that:

$$f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y), \forall y \in K, \tag{3.14}$$

where $\alpha(tx) = t^p \alpha(x)$ and $p > 1$ is a constant.

**Theorem 3.2.** Let $K$ be a nonempty bounded closed subset of a real Banach space $X$. Let $f : K \times K \to \mathbb{R}$ be a mixed relaxed $\alpha - \beta-$monotone, hemicontinuous in the first argument, convex in the second argument with $f(x, x) = 0$, 0-diagonally convex and lower semicontinuous. Let $\phi, \psi : K \times K \to \mathbb{R}$, $\phi$ be convex in the second argument, $\psi(x, y) = \phi(x, y) - \phi(x, x)$ and $\psi$ be 0-diagonally convex, and lower semicontinuous; $\alpha : K \to \mathbb{R}$ is weakly upper semicontinuous and $\beta : K \times K \to \mathbb{R}$ is weakly upper semicontinuous in the second argument.Then the mixed equilibrium problem (4.2) admits a solution.

## 3.3 Existence of Solution for Generalized Equilibrium Problem for $(r, s) - \alpha - \beta-$monotone functions

**Definition 3.8.** A mapping $\phi : K \times K \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be 0-diagonally convex if, for any finite subset $\{x_1, x_2, ..., x_n\}$ of $K$ and $\lambda_i \geq 0, i = \overline{1, n}$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\overline{x} = \sum_{i=1}^{n} \lambda_i x_i$, one has :

$$\sum_{i=1}^{n} \lambda_i \phi(\overline{x}, x_i) \geq 0. \tag{3.13}$$

**Theorem 3.3.** Suppose $f_r : K \times K \to \mathbb{R}$ is mixed relaxed $\alpha - \beta-$monotone, hemicontinuous in the first argument with $f_r(x, x) = 0$, for all $x \in K$. Let $\phi : K \times K \to \mathbb{R}$ be convex in the second argument. Then, generalized equilibrium problem (3.1) is equivalent with the following problem. Find $\overline{x} \in K$ such that:

$$\frac{1}{r}\left[e^{rf(y,\overline{x})} - 1\right] + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq \alpha(y - \overline{x}) + \beta(\overline{x}, y), \forall y \in K, \tag{3.14}$$

where $\alpha(tx) = t^p \alpha(x)$ and $p > 1$ is a constant.

**Theorem 3.4.** Let $K$ be a nonempty bounded closed subset of a real Banach space $X$. Let $f_r : K \times K \to \mathbb{R}$ be a mixed relaxed $\alpha - \beta-$monotone, hemicontinuous in the first argument, convex in the second argument with $f_r(x, x) = 0$, 0-diagonally convex and lower semicontinuous. Let $\phi, \psi : K \times K \to \mathbb{R}$, $\phi$ be convex in the second argument, $\psi(x, y) = \phi(x, y) - \phi(x, x)$ and $\psi$ be 0-diagonally convex, and lower semicontinuous; $\alpha : K \to \mathbb{R}$ is weakly upper semicontinuous and $\beta : K \times K \to \mathbb{R}$ is weakly upper semicontinuous in the second argument.Then the mixed equilibrium problem (4.2) admits a solution.

## 3.4 On relaxed monotonicity using $\rho-$mixed relaxed monotone functions

**Definition 3.9.** $\varphi : K \times K \to \mathbb{R}$ is called $\rho-$diagonally convex, if for any finite subset $\{x_1, x_2, ..., x_n\}$ on $K$ and $\lambda_i \geq 0$, $i = \overline{1,n}$, with $\sum_{i=1}^{n} \lambda_i = 1$ and $\overline{x} = \sum_{i=1}^{n} \lambda_i x_i$, we have

$$\sum_{i=1}^{n} \lambda_i \varphi(\overline{x}, x_i) \geq -\rho \min_{i=\overline{1,n}} d(\overline{x}, x_i).$$

**Theorem 3.5.** Let $f : K \times K \to \mathbb{R}$ be $\rho_1-$mixed relaxed monotone, in first argument, $\rho_2-$convex in the second argument, with $f(x,x) = 0, \forall x \in X$.

Let $\varphi : K \times K \to \mathbb{R}$ be $\rho_3-$convex in the second argument.

Then, the generalized equilibrium problem (3.1) from Section 3.1 is equivalent with the following problem: find $\overline{x} \in K$ such that:

$f(y, \overline{x}) + \phi(\overline{x}, \overline{x}) - \phi(\overline{x}, y) \leq \rho_0 d(\overline{x}, y), \forall y \in K$, where $\rho_0 \in \mathbb{R}$.

**Theorem 3.6.** Let $K$ be a nonempty bounded closed subset of a real Banach space $X$. Let $f : K \times K \to \mathbb{R}$, be $\rho_1-$mixed relaxed monotone, hemicontinuous in the first argument, $\rho_2-$convex in the second argument with $f(x,x) = 0$, $\rho_3-$diagonally convex and weakly lower semicontinuous. Let $\phi, \psi : K \times K \to \mathbb{R}$, $\phi$ be $\rho_4-$convex in the second argument, $\psi(x,y) = \phi(x,y) - \phi(x,x)$ and $\psi$ be $\rho_5-$diagonally convex, and weakly lower semicontinuous. Let $d : K \times K \to \mathbb{R}$, $d \geq 0$ be weakly upper semicontinuous in the second argument, and $\rho_1 \leq \rho_0$, $\rho_3 + \rho_5 \leq 0$.

Then the mixed equilibrium problem (3.1) admits a solution.

# Chapter 4

# Generalized mixed equilibrium problems

## 4.1 Problem statement and state of the art

Let $\rho \in \mathbb{R}$.

**Definition 4.1.** A mapping $T : C \to E^*$ is said to be relaxed $(\rho, \eta - \delta)$ monotone if there exist a mapping $\eta : C \times C \to E^*$ and a function $\delta : E \times E \to \mathbb{R}$ such that

$$(Tx - Ty, \eta(x, y)) \geq \rho\delta(x, y), \quad x, y \in C$$

## 4.2 Preliminaries

Let $E$ be a real Banach space and let $U = \{x \in E : ||x|| = 1\}$ be the unit sphere of $E$.

**Definition 4.2.** A Banach space $E$ is said to be strictly convex if for any $x, y \in U$,

$$x \neq y \text{ implies } ||x + y|| < 2$$

**Definition 4.3.** A Banach space $E$ is said to be uniformly convex if and only if $\lambda(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where $\lambda : [0, 2] \to [0, 1]$ called the modulus of convexity of E is defined as follows

$$\lambda(\epsilon) = inf\left\{1 - \|\frac{x+y}{2}\| : x, y \in E, ||x|| = ||y|| = 1, ||x - y|| \geq 1\right\}$$

**Definition 4.4.** A Banach space $E$ is said to be smooth if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{4.1}$$

exists for all $x, y \in U$.

**Theorem 4.1.** [1] Let $C$ be a nonempty convex subset of a smooth Banach space E and let $x \in E$ and $y \in C$. Then the following are equivalent:

1. $y$ is a best approximation to $x : y = P_C x$
2. $y$ is a solution of the variational inequality:

$$< y - z, J(x - y) > \geq 0 \quad \forall z \in C$$

where $J$ is a duality mapping and $P_C$ is the metricprojection from $E$ onto $C$.

For solving the mixed equilibrium problem, let us assume the following conditions for a bifunction $f$:

$(i_1)$ $f(x, x) = 0$, $\forall x \in C$

$(i_2)$ $f$ is $\rho_1$-monotone, i.e. $f(x, y) + f(y, x) \leq \rho_1 d(x, y)$, for all $x, y \in C$, $d : C \times C \to \mathbb{R}_+$ and $\rho_1 \in \mathbb{R}$.

$(i_3)$ For all $y \in C$, $f(\cdot, y)$ is weakly upper semicontinuous

$(i_4)$ For all $x \in C$, $f(x, \cdot)$ is $\rho_2-$convex, $\rho_2 \in \mathbb{R}$.

**Lemma 4.1.** [125] Let $E$ be a uniformly convex Banach space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < b \leq \alpha_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$

and $\{y_n\}$ be sequences in E such that $\limsup_{n\to\infty}||x_n|| \leq d\sup$, $\limsup_{n\to\infty}||y_n|| \leq d$ and $\lim_{n\to\infty}||\alpha_n x_n + (1-\alpha_n)y_n|| = d$. Then $\lim_{n\to\infty}||x_n - y_n|| = 0$.

**Lemma 4.2.** [16] Let $C$ be a bounded, closed and convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing,convex and continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and

$$\gamma\left(\|S\left(\sum_{i=1}^{n}\theta_i x_i\right) - \sum_{i=1}^{n}\theta_i S x_i\|\right) \leq \max_{1\leq j\leq k\leq n}(||x_j - x_k|| - ||Sx_j - Sx_k||)$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, ..., x_n\} \subset C$, $\{\theta_1, \theta_2, ..., \theta_n\} \subset [0,1]$ with $\sum_{i=1}^{n}\theta_i = 1$ and nonexpansive mapping S of C into E.

**Lemma 4.3.** [50] Let $C$ be a nonempty, closed and convex subset of a uniformly convex Banach space E and let $[S_n]$ be a family of nonexpansive mappings of C into itself such that $F = \bigcap\limits_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $[\beta_n^k]$ be a family of nonnegative numbers with indices $n.k \in \mathbb{N}$ with $k \leq n$ such that

1. $\sum\limits_{k=1}^{n}\beta_n^k = 1$ for every $n \in \mathbb{N}$
2. $\lim\limits_{n\to\infty}\beta_n^k > 0$ for every $k \in \mathbb{N}$

and let $T_n = \alpha_n I + (1-\alpha_n)\sum\limits_{k=1}^{n}\beta_n^k S_k$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ with $a \leq b$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap\limits_{n=1}^{\infty} F(S_n) = F$ and satisfies the NST-condition.

**Definition.** [44] Let $B$ be a subset of topological vector space $X$. A mapping $G : B \to 2^X$ is called a KKM mapping if $co\{x_1, x_2, ..., x_m\} \subset \bigcup\limits_{i=1}^{m} G(x_i)$ for $x_i \in B$ and $i = \overline{1, m}$, where $coA$ denotes the convex hull of the set $A$.

**Lemma 4.4.** [27] Let $B$ be a nonempty subset of a Hausdorff topological vector space $X$ and let $G : B \to 2^X$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap\limits_{x\in B} G(x) \neq \emptyset$.

## 4.3 Existence results of generalized mixed equilibrium problem

**Theorem 4.1.** Let $B$ be a smooth, strictly convex and reflexive Banach space, with a nonempty, bounded, closed and convex subset $C$ of $E$.

Also, we consider:

$(j_1)$ a mapping $T : C \to E^* \ \eta-$ hemicontinuous and $\delta - \eta$ relaxed monotone

$(j_2)$ a bifunction $f : C \times C \to \mathbb{R}$ satisfying $(i_1)-(i_4)$.

$(j_3)$ a lower semicontinuous $\rho_3$-convex function, $\varphi : C \to \mathbb{R}$

Let $r > 0$ and $z \in C$ and we supppose that

$(j_4)$ $\eta(x, x) = 0, \forall x \in C$

$(j_5)$ For any fixed $u, v \in C$, the mapping $x \to < Tv, \eta(x, u) >$ is $\rho_4-$convex

$(j_6)$ $\lim\limits_{t\to 0} \dfrac{\delta(x,(1-t)x+ty)}{t} = 0$ for all $x, y \in C$

If $\rho_2 + \rho_3 + \rho_4 \geq 0$, then the following problems 4.2 and 4.3 are equivalent:

$$\text{Find } x \in C \text{ such that } f(x, y)+\varphi(y)+ < Tx, \eta(y, x) > +\frac{1}{r} < y-x, J(x-z) \geq \varphi(x), \forall y \in C \tag{4.2}$$

$$\text{Find } x \in C \text{ such that} f(x, y)+ < Ty, \eta(y, x) > +\varphi(y)+\frac{1}{r} < y-x, J(x-z) >\geq \varphi(x)+\delta(x, y), \forall y \in C \tag{4.3}$$

**Theorem 4.2.** Let $C$ be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an $\eta$-hemicontinuous and relaxed $\eta - \delta$ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (a),(c) and (d) and let $\varphi$ be a lower semicontinuous and $\rho_3$-convex function from $C$ to $\mathbb{R}$. Let $r > 0$ and $z \in C$. Assume that

1. $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in C$

2. for any fixed $u, v \in C$, the mapping $x \to < Tv, \eta(x,u) >$ is $\rho_4$-convex and lower semicontinuous

3. $\xi : E \to \mathbb{R}$ is weakly lower semicontinuous; that is ,for any net $\{x_\beta\}$, $x_\beta$ converges to x in $\sigma(E, E^*)$ which implies that $\xi(x) \leq \liminf \xi(x_\beta)$

Then, the solution set of the problem (4.2) is nonempty: that is, there exists $x_0 \in C$ such that

$$\text{Find } x \in C \text{ such that } f(x,y)+\varphi(y)+ < Tx, \eta(y,x) > +\frac{1}{r} < y-x, J(x-z) \geq \varphi(x), \forall y \in C \tag{4.4}$$

**Theorem 4.3.** Let $C$ be a nonempty bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be a $\eta$-hemicontinuous and relaxed $\eta - \delta$ monotone mapping. Let $f$ be a bitfunction from $C \times C$ to $\mathbb{R}$ satisfying (a)-(d) and let $\varphi$ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and define a mapping $\phi_r : E \to C$ as follows:

$$\phi_r(x) = \left\{ z \in C : f(z,y)+ < Tz, \eta(y,z) > +\varphi(y)+\frac{1}{r} < y-z, J(z-x) >\geq \varphi(z), \forall y \in C \right\} \tag{4.5}$$

for all $x \in E$. Assume that

1. $\eta(x,y) + \eta(y,x) = 0$, for all $x, y \in C$

2. for any fixed $u, v \in C$, the mapping $x \to < Tv, \eta(x,u) >$ is $\rho_4$-convex and lower semicontinuous and the mapping $x \to < Tu, \eta(v,x) >$ is lower semicontinuous

3. $\xi : E \to \mathcal{R}$ is weakly lower semicontinuous

4. for any $x, y \in C$, $\delta(x,y) + \delta(y,x) \geq 0$

Then, the following holds:

1. $\phi_r$ is single valued

2. $< \phi_r x - \phi_r y, J(\phi_r x - x) > \leq < \phi_r x - \phi_r y, J(\phi_r y - y) >$ for all $x, y \in E$

3. $F(\phi_r) = EP(f, T)$

4. $EP(f, T)$ is closed and convex

## 4.4 A hybrid projection algorithm

Let $E$ be a uniformly convex and smooth Banach space and let $C$ be a nonempty, bounded, closed and convex subset of $E$. Also let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ and the mapping $T : C \to E^*$.

**Theorem 4.4.** We suppose that the bifunction f satisfy the (a)-(d) assumption and $T$ is a $\eta - \delta$ relaxed monotone mapping.

If $\{S_n\}_{n\geq 0}$ is a sequence of nonexpansive mappings with the NST-condition, $S_n : E \to C$, such that $\Omega \neq \emptyset$, where $\Omega = \cap_{n\geq 0} F(S_n) \cap EP(f, T)$, and $\{x_n\}_{n\geq 0}$ is the sequence from $C$, given by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \bar{co}\{z \in C : ||z - S_n z|| \leq t_n ||x_n - S_n x_n||\}, \quad n \geq 0 \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + < Tu_n, \eta(y, u_n) > + \frac{1}{r_n} < y - u_n, J(u_n - x_n) > \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : < u_n - z, J(x_n - u_n) > \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{4.6}$$

$0 < t_n < 1, 0 < r_n < 1$ for any n and $\lim_{n\to\infty} t_n = 0$, $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}_n$ converges strongly to $P_{S_0} x_0$ sau $P_\Omega x_0$.

We suppose $f, T$ and $\Omega$ are as in Theorem 4.1. Let $\{\beta_n^k\}_{n,k}, n, k \geq 1, k \leq n, \beta_{n^k} \geq 0$ such that $\lim\limits_{n\to\infty} \beta_n^k > 0$ for any $k \geq 1$ and $a_n = 1$, for any $n \geq 1$, where $a_n = \sum\limits_{k=1}^{n} \beta_n^k$.

Let us construct a sequence $\{G_n\}_{n\geq 1}$, $G_n = \alpha_n I + (1-\alpha_n)b_n$, with $b_n = \sum_{k=1}^{n} \beta_n^k S_k$, $a_0 < \alpha_n < b_0$ for $n \geq 1$, for some $0 < a_0 \leq b_0 < 1$, where $\{S_n\}_{n\geq 0}$ is a sequence of nonexpansive mappings of $C$ into itself.

If $\{t_n\}_n, \{r_n\}_n$ are two sequences with $0 < t_n < 1, 0 < r_n < 1$ for any $n \geq 1$ and $\lim_{n\to\infty} t_n = 0$, $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}_{n\geq 0}$ given by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \bar{co}\{z \in C : ||z - G_n z|| \leq t_n ||x_n - G_n x_n||\}, \quad n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + < Tu_n, \eta(y, u_n) > + \frac{1}{r_n} < y - u_n, J(u_n - x_n) > \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} :< u_n - z, J(x_n - u_n) > \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0 \end{cases}$$

converges strongly to $P_\Omega x_0$.

We suppose f and T as in Theorem 4.1. Let $\Omega \neq \emptyset$, $\Omega = F(S) \cap EP(f, T)$ and $\{x_n\}_n \subset C$ a sequence given by

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \bar{co}\{z \in C : ||z - Sz|| \leq t_n ||x_n - Sx_n||\}, \quad n \geq 0, \\ u_n \in C \text{such that} \\ f(u_n, y) + \varphi(y) + < Tu_n, \eta(y, u_n) > + \frac{1}{r_n} < y - u_n, J(u_n - x_n) > \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} :< u_n - z, J(x_n - u_n) > \geq 0\} \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}_n$ and $\{r_n\}_n$ are two sequences as in Theorem 4.1. Then, the sequence $\{x_n\}_n$ converges strongly to $P_\Omega x_0$.

We suppose that $f$ is as in Theorem 4.1. For $\{t_n\}_n$ and $\{r_n\}_n$ as in Theorem 4.1 and the sequence $\{x_n\}_{n\geq 0}$ given by

$$
\begin{cases}
x_0 \in C, D_0 = C, \\
C_n = \bar{co}\{z \in C : ||z - S_n z|| \leq t_n ||x_n - S_n x_n||\}, \quad n \geq 0, \\
u_n \in C \text{such that} \\
f(u_n, y) + \frac{1}{r_n} < y - u_n, J(u_n - x_n) > \geq 0, \forall y \in C, n \geq 0, \\
D_n = \{z \in D_{n-1} :< u_n - z, J(x_n - u_n) > \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0
\end{cases}
$$

we have that $\{x_n\}_n$ converges strongly to $P_\Omega x_0$.

Let E be a uniformly convex and smooth Banach space, $C \neq \emptyset$ a bounded, closed and convex subset of E. Also let $\{S_n\}_{n \geq 0}$ be a sequence of nonexpansive mappings, $S_n : C \to C$, satisfies the NST-condition and $\Omega \neq \emptyset$, $\Omega = \cap_{n \geq 0} F(S_n)$.
If $\{x_n\}_n$ is a sequence given by

$$
\begin{cases}
x_0 \in C, D_0 = C, \\
C_n = \bar{co}\{z \in C : ||z - S_n z|| \leq t_n ||x_n - S_n x_n||\}, \quad n \geq 0, \\
x_{n+1} = P_{C_n} x_0, \quad n \geq 0,
\end{cases}
$$

where $0 < t_n < 1$ for any $n$ and $\lim_{n \to \infty} t_n = 0$, then $\{x_n\}_n$ converges strongly to $P_\Omega x_0$.

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