GARCH MODELS AND ENTROPY MEASURES IN FINANCE
PhD Thesis
Abstract

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INTRODUCTION

This thesis addresses the option pricing theory, which is a field of the highest importance in the present research in finance.

The Black-Scholes model (1973) assumes that the volatility of stock prices is a constant function. In practice, analysts always use the market option price to back out the volatility when they use Black-Scholes model. The continuous time stochastic model has an inherent disadvantage to assume that volatility is observable, but it is impossible to exactly filter volatility from discrete observations of spot asset prices in a continuous time stochastic volatility model (Heston and Nandi, 2000). Consequently it is impossible to price an option solely on the basis of the history of asset prices.

The GARCH option pricing model assumes that the conditional volatility of stock prices depends on the past pricing errors. Duan (1995, 1996) argued that most of the existing bivariate diffusion models that had been used to model asset returns and volatility, could be represented as limits of a family of GARCH models. On the other hand the GARCH model has an advantage over the continuous time model that volatility is readily observable in the history of asset prices. Thus the GARCH model is chosen over the continuous time model when comparing the empirical performance of the stochastic option model and the discrete time model. Meaningfully Monte Carlo simulation has performed to examine the empirical performance of the Black-Scholes model and the GARCH option pricing model.

One of the well established facts in the financial modeling states that empirical distributions of log-returns time series are skewed and leptokurtic (fat-tailed). Therefore there are various models which have been introduced that resort to an alternative non-Gaussian assumption. Since the skewness defines the asymmetry of the distribution, it has a significant impact on the shape of the tails. Kurtosis is another parameter of interest under the alternative non-Gaussian fat-tailed assumption. As a result, it is crucial to model skewness and kurtosis as accurately as possible, also considering changes over time.

Entropic reasoning applied to option pricing has been successfully developed mainly after 1999. According to Gulko (1999), the Entropy Pricing Theory suggests that, in informational efficient markets, perfectly uncertain market beliefs must prevail. When the entropy functional is used to index the market uncertainty, then the entropy-maximizing market beliefs must prevail. On optimizing various entropic measures one can derive new stock option pricing models that are similar to Black-Scholes with the log-normal distribution replaced by other probability distributions.

In the first part of the thesis (Chapter1-4) our original research consists in the evaluation of kurtosis for several financial time series models. First we discuss the Black-Scholes model with GARCH volatility, both for "In The Money" (ITM) and "At The Money" (ATM) options. Then we extend the results to the case of doubly stochastic volatility models with asymmetric GARCH innovations such as AGARCH and TGARCH. New results for the mean, variance and kurtosis of the nonlinear volatility models are also obtained. An empirical study is performed in order to compare fitting different models for various real-life stock indexes. We compare models with GARCH, EGARCH, IGARCH, GJR-GARCH, or APARCH innovations.

In the second part of the thesis (Chapter 5-9) we take up the new framework of entropic reasoning applied to option pricing. According to Stutzer (2000), the novelty of this approach lies in the straightforward use to compute the Black-Scholes martingale measure as the product of the relative entropy minimizing conditional risk-neutral probabilities in the return process. Using the entropic reasoning Gulko (1996, 1997) prescribed the risk-free mean and the actual variance to derive the risk-neutral log-normal price distribution, to derive the Black-Scholes formula as a riskless discounted value of the option's payoff at expiration.

We extend Gulko's result by using several entropic measures such as Tsallis, Weighted Tsallis, Kaniadakis, Weighted Kaniadakis, Renyi and Weighted Renyi as objective functions in the new optimization problems defined. We use the obtained risk-neutral price distributions to evaluate the European Call and Put options on a dividend protected stocks. We use some new entropic measures Ubriaco (2009) and Shafee (2007) as optimization objectives subject to traditional constraints in order to construct new risk-neutral price distribution.

The entropic reasoning allows us to approach two different ways of quantifying the statistical heterogeneity of risk-neutral price distribution: entropy-which measures the law's randomness and Gini's index- which measures the law's egalitarianism (or evenness). Two complementary problems are addressed: entropy maximization for specified Gini index value and Gini maximization for specified Shannon entropy. The solutions are presented in terms of Reccatii equations.

Finally, we take up the very recent approach of Hunt and Devolder (2011) for the semi-Markov regime switching interest rate models. The authors discuss a discrete time regime switching binormal-like model of the term structure where the regime switches are governed by a discrete time semi-Markov process. Under market incompleteness, they give an explicit characterization of the minimal entropy martingale measure. We extend the Hunt and Devolder results on using Tsallis and Kaniadakis entropies, as well as Shafee entropy based on Lambert function-to indentify the minimal entropy martingale measures and to find the corresponding risk-neutral price distributions.

Chapter 1

GARCH Modelling

1.1 GARCH Models and Volatility

1.1.1 Generalized Autoregressive Conditional Heteroskedastic Model

Definition 1 [7] *The simplest possibility to write ARCH/GARCH models mathematically is as follows repectively.*

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 \ , \omega > 0, \alpha_i \geq 0$$
$$\varepsilon_t = \sigma_t z_t \quad , \ z_t \sim N(0,1)$$

In ARCH framework σ_t is a time varying, positive and measureable function of time $t-1$. A generalized form of ARCH model for a univariate time series X_t is called GARCH and it can be written as follows.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \ , \omega > 0, \alpha_i \geq 0, \beta_j \geq 0$$
$$\varepsilon_t = \sigma_t z_t \ \ , \ z_t \sim N(0,1)$$

1.1.2 IGARCH Model

Definition 2 [7] *Integrated GARCH or IGARCH processes are designed to model persistent changes in the volatility. A GARCH process is stationary with a finite variance if* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ *. A GARCH (p,q) process is called IGARCH process if* $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = 1$.

1.1.3 EGARCH Model

Definition 3 [109] *An EGARCH model is defined by:*

$$\log \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i g(z_{t-i}) + \sum_{j=1}^{q} \beta_j \log \sigma_{t-j}^2$$
$$g(z_{t-i}) = \theta z_{t-i} + \lambda^* \left(|z_{t-i}| - E\,|z_{t-i}| \right)$$

where ω, $\alpha_i, \beta_j, \theta$ and λ^* are real parameters and both z_{t-i} and $|z_{t-i}| - E\,|z_{t-i}|$ are zero mean and identically independently distributed with continuous distributions. Therefore we can write

$E\left(g(z_{t-i})\right)=0$.*We may rewrite* $g(z_{t-i})$ *as follows:*

$$g(z_{t-i})=\begin{cases}(\theta+\lambda^{*})\,z_{t-i} & -\lambda^{*}\left(E\left|z_{t-i}\right|\right) \quad if \quad z_{t-i}\geq 0\\ (\theta-\lambda^{*})\,z_{t-i} & -\lambda^{*}\left(E\left|z_{t-i}\right|\right) \quad if \quad z_{t-i}\leq 0\end{cases}$$

1.1.4 GJR-GARCH Model

Definition 4 [55, 165]*The GJR-GARCH and TGARCH models are defined by.*

$$\sigma_{t}^{2}=\omega+\sum_{i=1}^{p}\left(\alpha_{i}+\gamma I_{t-i}\right)\varepsilon_{t-i}^{2}+\sum_{j=1}^{q}\beta_{j}\sigma_{t-j}^{2}$$

where $I_t = 1$ if $\varepsilon_t < 0$ and $I_t = 0$ if $\varepsilon_t \geq 0$.

1.1.5 APARCH Model

Definition 5 [35] *The asymmetric power ARCH models are defined as follows.*

$$\sigma_{t}^{\delta}=\omega+\sum_{i=1}^{p}\alpha_{i}\left(\left|\varepsilon_{t-i}\right|-\gamma\varepsilon_{t-i}\right)^{\delta}+\sum_{i=1}^{q}\beta_{i}\sigma_{t-i}^{\delta}\ ,i=1,2,..$$
$$\varepsilon_{t}=\sigma_{t}z_{t}\ \ ,z_{t}\ \ \sim N(0,1)$$
$$g(\varepsilon_{t-i})=\left|\varepsilon_{t-i}\right|-\gamma\varepsilon_{t-i}$$

1.2 GARCH Models and Entropy Measures

1.2.1 Problem of Kaniadakis Entropy Measure

Theorem 6 *Consider the optimization problem:*

$$\max H\left(g\right)=E^{g}\left[\frac{g(\mathcal{X})^{k-1}-g(\mathcal{X})^{-k-1}}{2k}\right],k\neq 0$$

subject to

$$E^{g}\left(1\mid\mathcal{F}_{t-1}\right) = 1 \qquad (c\text{-}1)$$
$$E^{g}\left(\mathcal{X}\mid\mathcal{F}_{t-1}\right) = \mu \qquad (c\text{-}2)$$
$$E^{g}\left(\left(\mathcal{X}-\mu\right)^{2}\mid\mathcal{F}_{t-1}\right) = \sigma_{t}^{2} \qquad (c\text{-}3)$$

Then the solution of the optimization problem is given by:

$$g=g\left(s_{t}\right)=\left(\frac{k\left(\lambda_{0}+\lambda_{1}\mu+\lambda_{2}\sigma_{t}^{2}\right)+\sqrt{k^{2}\left(\left(\lambda_{0}+\lambda_{1}\mu+\lambda_{2}\sigma_{t}^{2}\right)^{2}-1\right)+1}}{k+1}\right)^{\frac{1}{k}}$$

1.2.2 Problem of Renyi Entropy Measure

Theorem 7 *Consider the optimization problem:*

$$\max H(g) = \frac{1}{1-r} \log E^g \left[g^{r-1}(\mathcal{X}) \right], r \neq 1$$

subject to c-1, c-2 and c-3 of previous problem. Then the solution of maximum entropy density in functional form is given by:

$$g(s_t) = \frac{\left[1 - \frac{1-r}{r} \left(\beta_1 (\mu - s_t) - \beta_2 (\sigma_t^2 - s_t^2)\right)\right]^{\frac{1}{r-1}}}{\int_0^\infty \left[1 - \frac{1-r}{r} \left(\beta_1 (\mu - s_t) - \beta_2 (\sigma_t^2 - s_t^2)\right)\right]^{\frac{1}{r-1}} ds_t}$$

1.2.3 Problem of Shafee Entropy Measure

Theorem 8 *Consider the optimization problem:*

$$\max H(g) = -E^g \left[g^{a-1}(\mathcal{X}) \ln \ g(\mathcal{X}) \right], \ a > 0$$

subject to c-1, c-2 and c-3 of previous problem.

Then of solutions of $g(s_t)$ is given by:

$$g(s_t) = \left[\frac{aW(\gamma)(1-a) \exp\left\{ -\left(\frac{1-a}{a}\right)\right\}}{\gamma(1-a)} \right]^{\frac{1}{1-a}}$$

where W is a Lambert function and $\gamma = \lambda_0 + \lambda_1 \mu + \lambda_2 \sigma_t^2$, $\lambda_0, \lambda_1, \lambda_2$ are the Lagrange multipliers which can be determined using some specified constraints. An alternate solution can be written as:

$$g(s_t) = (\psi')^{-1} \left(\lambda_0 + \lambda_1 \mu + \lambda_2 \sigma_t^2\right)$$

1.2.4 Problem of Ubriaco Entropy Measure

Theorem 9 *Consider the optimization problem:*

$$\max H(g) = -E^g \left(\ln \frac{1}{g(\mathcal{X})} \right)^d, d > 0$$

subject to c-1, c-2 and c-3 of previous problem.

Then the solution for $g(x)$ can be written as:

$$g(s_t) = (\psi')^{-1} \left(\lambda_0 + \lambda_1 \mu + \lambda_2 \sigma_t^2\right)$$

Chapter 2

Option Pricing and GARCH Volatility

2.1 Black-Scholes Model with GARCH Volatility

The call option for this model can be written as:

$$C = SE_{\theta_t}\left[\phi(\frac{\log(\frac{S}{K})+\mathbf{r}T+\frac{1}{2}\theta_t^2}{\theta_t})\right] - Ke^{-\mathbf{r}T}E_{\theta_t}\left[\phi(\frac{\log(\frac{S}{K})+\mathbf{r}T-\frac{1}{2}\theta_t^2}{\theta_t})\right] \quad (2.9)$$

$$C = SE_{\theta_t}\left[\phi(d_1)\right] - Ke^{-\mathbf{r}T}E_{\theta_t}\left[\phi(d_2)\right] \quad (2.10)$$

The put option for this model can be written as:

$$P = Ke^{-\mathbf{r}T}E_{\theta_t}\left[\phi(-d_2)\right] \ - SE_{\theta_t}\left[\phi(-d_1)\right] \quad (2.11)$$

Theorem 10 *[59]For any twice differentiable functions $f(x)$ and $g(x)$, the call price (3.10) can be written as:*

$$C = S\left(f[E(\theta_t^2)] + \frac{1}{2}f''[E(\theta_t^2)](\frac{1}{3}k^{(y)}-1)E^2(\theta_t^2)\right) - Ke^{-\mathbf{r}T}\left(g[E(\theta_t^2)] + \frac{1}{2}g''[E(\theta_t^2)](\frac{1}{3}k^{(y)}-1)E^2(\theta_t^2)\right) \quad (2.12)$$

where $k^{(y)} = \frac{E(y_t^4)}{E^2(y_t^2)}$,the kurtosis of the observed logreturn y_t. We also have:

$$f[E(\theta_t^2)] = \phi\left(\frac{\log(\frac{S}{K})+\mathbf{r}T+\frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2}}\right) \quad (2.13)$$

$$g[E(\theta_t^2)] = \phi\left(\frac{\log(\frac{S}{K})+\mathbf{r}T-\frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2}}\right) \quad (2.14)$$

2.2 Kurtosis in BS-Model with GARCH Volatility.

2.2.1 In The Money Option (ITM)

Theorem 11 *For any twice differentiable functions $f(x)$ and $g(x)$, the call price can be written as:*

$$\begin{aligned} C &= S\left(f[E(\theta_t^2)] + \frac{1}{2}f''[E(\theta_t^2)](\frac{1}{3}k^{(y)} - 1)E^2(\theta_t^2)\right) \\ &\quad -Ke^{-rT}\left(g[E(\theta_t^2)] + \frac{1}{2}g''[E(\theta_t^2)](\frac{1}{3}k^{(y)} - 1)E^2(\theta_t^2)\right) \end{aligned}$$

where $k^{(y)} = \frac{E(y_t^4)}{E^2(y_t^2)}$,the kurtosis of the observed logreturn y_t.We can find the kurtosis $k^{(y)}$ as:

$$k^{(y)} = \left(\frac{1536\sqrt{2\pi}Cx^3\sqrt{x} - 768(\sqrt{2\pi}+2)(S-X)x^4}{F(x)}\right) + 3$$

where C is value of call option, x is volatiltiy, S is stock price, K is strike price, r is interest rate, T is time to expiry, $u = \log\left(\frac{S}{k}\right) + rT$ and

$$\begin{aligned} F(x) &= [16u^4(S-X) - 64u^3(S+X)]x + [\left(32u^2 - 96u^3\right)(S-X)]x^2 + \\ &\quad [(196u - 8u^2-)(S-X) + (16u + 100u^2)(S+X)]x^3 - \\ &\quad [(32-2u)(S+X) + (8+8u)(S-X)]x^4 + [3(S-X)]x^5 + \\ &\quad 32u^5(S+X) \end{aligned}$$

2.2.2 At The Money Option (ATM)

Theorem 12 *Consider case-2 when stock price S and strike price K are identical i.e. $S = K$ and $r = 0$ also if $E(\theta_t^2) = x$, then we can write as follows:*

$$f(x) = \phi(d_1) = \phi\left(\frac{\sqrt{x}}{2}\right),\ g(x) = \phi(d_2) = \phi\left(\frac{-\sqrt{x}}{2}\right)$$

Then we have following results:

1. $f''(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x}{8}}\left[\frac{-1}{8x\sqrt{x}} - \frac{1}{32\sqrt{x}}\right]$
2. $g''(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x}{8}}\left[\frac{1}{8x\sqrt{x}} + \frac{1}{32\sqrt{x}}\right]$

   and we have the kurtosis $k^{(y)}$ given by:

$$k^{(y)} = \frac{96\left(\sqrt{\frac{2\pi}{x}}\frac{C}{S} - 1\right)}{(x-4)(x+8)} + 3$$

2.2.3 A Simple Case for ATM and ITM Options

Theorem 13 *Consider the call option of BS-model with GARCH volatiltiy as follows:*

$$C = SE_{\theta_t}[\phi(d_1)] - Ke^{-rT}E_{\theta_t}[\phi(d_2)]$$

where

$$E_{\theta_t}[\ \phi(d_1)] = \quad \phi\left[\frac{\log(\frac{S}{K}) + rT + \frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}\right]$$

$$E_{\theta_t}[\ \phi(d_2)] = \quad \phi\left[\frac{\log(\frac{S}{K}) + rT - \frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}\right]$$

Then we have following results:

1. *In the case of BS-model with GARCH volatility for ITM i.e $S \neq K$,the implied volatiltiy $v = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a = (S + X)$, $b = \sqrt{2\pi}[(S - X) - 2C]$ and $c = 2H(S - X)$ and $X = Ke^{-rT}$, $H = \log(\frac{S}{K}) + rT$*

2. *In the case of ATM i.e $S = K$,we have $v = \frac{-b}{a}, b = \sqrt{2\pi}[(S - X) - 2C], a = (S + X)$,and $X = Ke^{-rT}$*

3. *We have same results for original Black-Scholes model for above two cases but the value of call option in under GARCH volatiltiy is more significant.*

Chapter 3

Doubly Stochastic Models with GARCH process

3.1 Doubly Stochastic Models with AGARCH and TGARCH Errors

Theorem 14 *[132]Consider a doubly stochastic model of the following form with TGARCH(1,1) errors*

$$\begin{aligned} y_t &= (\phi + b_t)y_{t-1} + \varepsilon_t \\ b_{t+1} &= ab_t + (1+b_t)\upsilon_{t+1} \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t &= \omega + \alpha_{1,+}\varepsilon_{t-1}^{+} - \alpha_{1,-}\varepsilon_{t-1}^{-} + \beta_1\sigma_{t-j} \end{aligned}$$

Then we have following results.

1. $E(y_t^2) = \frac{\omega^2\sigma_z^2(1+a_1)(1-a^2-\sigma_\upsilon^2)}{(1-a_1)(1-a_2)(1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$

2. $E(y_t^4) = \dfrac{\begin{array}{c} 3\sigma_z^4\omega^4\frac{(1+3a_1+5a_2+3a_1a_2+3a_3+5a_1a_3+3a_2a_3+a_1a_2a_3)}{(1-a_1)(1-a_2)(1-a_3)} + \\ +\frac{6\omega^4\sigma_z^4(1+a_1)^2[\phi^2(1-a^2)+\sigma_\upsilon^2(1-\phi^2)]}{(1-a_1)^2(1-a_2)^2(1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)} \end{array}}{\begin{array}{c} 1-\phi^4-3\sigma_\upsilon^4[\frac{(-1-a^2-5\sigma_\upsilon^2+a^3+a^5-16a^3\sigma_\upsilon^2+3a\sigma_\upsilon^2-9a\sigma_\upsilon^4}{(-1+a^3+3a\sigma_\upsilon^2)(-1+a^2+\sigma_\upsilon^2)(-1+6a^2\sigma_\upsilon^2+a^4+3\sigma_\upsilon^4)}] - 6\phi^2[\frac{\sigma_\upsilon^2}{(1-a^2-\sigma_\upsilon^2)}] - \\ -4\phi 6a[\frac{\sigma_\upsilon^4}{(1-a^3-3a\sigma_\upsilon^2)(1-a^2-\sigma_\upsilon^2)}] \end{array}}$

3. *The Kurtosis of the process follows the definition* $K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2}$

Theorem 15 *[131]Consider the following doubly stochastic time series with conditions 1,2 and 3 as:*

$$\begin{aligned} y_t &= (\phi + b_t)y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \\ b_{t+1} &= ab_t + (1+b_t)\upsilon_{t+1} \end{aligned}$$

1. $\binom{\upsilon_t}{\varepsilon_t} \sim N\left(\binom{0}{0}, \begin{pmatrix} \sigma_\upsilon^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{pmatrix}\right)$

2. $1 - a^2 - \sigma_\upsilon^2 < 1$

3. $1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2<1$

where ε_t is an identically distributed independent sequence of variables with mean zero and variance σ_ε^2 then we have results for $E(y_t^2), E(y_t^4)$ and Kurtosis as follows.

1. $E(y_t^2)=\frac{\omega^2\sigma_\varepsilon^2(1+\theta^2)(1-a^2-\sigma_\upsilon^2)}{(1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$

2. $E(y_t^4)=\frac{3\sigma_\varepsilon^4(1+6\theta^2+\theta^4)+6\sigma_\varepsilon^2(1+\theta^2)[(\phi^2+E(b_t^2)]E(y_{t-1}^2)}{1-\phi^4-E(b_t^4)-6\phi^2E(b_t^2)-4\phi E(b_t^3)}$

3. *The Kurtosis of the process follows the definition* $K^{(y)}=\frac{E(y_t^4)}{[E(y_t^2)]^2}$

Theorem 16 [132]*Consider a doubly stochastic model of the following form with TGARCH(1,1) :*

$$\begin{aligned} y_t &= (\phi+b_t)y_{t-1}+\varepsilon_t+\theta\varepsilon_{t-1} \\ b_{t+1} &= ab_t+(1+b_t)\upsilon_{t+1} \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t &= \omega+\alpha_{1,+}\varepsilon_{t-1}^{+}-\alpha_{1,-}\varepsilon_{t-1}^{-}+\beta_1\sigma_{t-j} \end{aligned}$$

Then we have following results.

1. $E(y_t^2)=\frac{\omega^2\sigma_z^2(1+\theta^2)(1+a_1)(1-a^2-\sigma_\upsilon^2)}{(1-a_1)(1-a_2)(1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$

2. $$E(y_t^4)=\frac{\begin{array}{c}3\sigma_z^4\omega^4\frac{(1+3a_1+5a_2+3a_1a_2+3a_3+5a_1a_3+3a_2a_3+a_1a_2a_3)}{(1-a_1)(1-a_2)(1-a_3)}(1+6\theta^2+\theta^4)+\\ +\frac{6\omega^4\sigma_z^4(1+\theta^2)^2(1+a_1)^2[\phi^2(1-a^2)+\sigma_\upsilon^2(1-\phi^2)]}{(1-a_1)^2(1-a_2)^2(1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}\end{array}}{\begin{array}{c}1-\phi^4-3\sigma_\upsilon^4[\frac{(-1-a^2-5\sigma_\upsilon^2+a^3+a^5-16a^3\sigma_\upsilon^2+3a\sigma_\upsilon^2-9a\sigma_\upsilon^4}{(-1+a^3+3a\sigma_\upsilon^2)(-1+a^2+\sigma_\upsilon^2)(-1+6a^2\sigma_\upsilon^2+a^4+3\sigma_\upsilon^4)}]-6\phi^2[\frac{\sigma_\upsilon^2}{(1-a^2-\sigma_\upsilon^2)}]-\\ -4\phi 6a[\frac{\sigma_\upsilon^4}{(1-a^3-3a\sigma_\upsilon^2)(1-a^2-\sigma_\upsilon^2)}]\end{array}}$$

3. *The Kurtosis of the process follows from its definition* $K^{(y)}=\frac{E(y_t^4)}{[E(y_t^2)]^2}$

Theorem 17 [132]*Consider a doubly stochastic model of the following form with TGARCH(1,1) errors as:*

$$\begin{aligned} y_t &= (\phi+b_t+\Phi s_t)y_{t-1}+\varepsilon_t+\theta\varepsilon_{t-1} \\ b_{t+1} &= ab_t+(1+b_t)\upsilon_{t+1} \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t &= \omega+\alpha_{1,+}\varepsilon_{t-1}^{+}-\alpha_{1,-}\varepsilon_{t-1}^{-}+\beta_1\sigma_{t-j} \end{aligned}$$

Then we have following results.

1. $E(y_t^2)=\frac{\omega^2\sigma_z^2(1+\theta^2)(1+a_1)(1-a^2-\sigma_\upsilon^2)}{(1-a_1)(1-a_2)(1-a^2-2\sigma_\upsilon^2-\Phi^2+\Phi^2a^2+\Phi^2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$

2. $$E(y_t^4)=\frac{\begin{array}{c}3\sigma_z^4\omega^4(1+\theta^4+6\theta^2)\frac{(1+3a_1+5a_2+3a_1a_2+3a_3+5a_1a_3+3a_2a_3+a_1a_2a_3)}{(1-a_1)(1-a_2)(1-a_3)}+\\ +\frac{6\sigma_z^4\omega^4(1+\theta^2)^2(1+a_1)^2\left[(\phi^2+\Phi^2)[(1-a^2-\sigma_\upsilon^2)+\sigma_\upsilon^2\right]}{(1-a_1)^2(1-a_2)^2(1-a^2-2\sigma_\upsilon^2-\Phi^2+\Phi^2a^2+\Phi^2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}\end{array}}{\begin{array}{c}1-\phi^4-3\sigma_\upsilon^4[\frac{(-1-a^2-5\sigma_\upsilon^2+a^3+a^5-16a^3\sigma_\upsilon^2+3a\sigma_\upsilon^2-9a\sigma_\upsilon^4}{(-1+a^3+3a\sigma_\upsilon^2)(-1+a^2+\sigma_\upsilon^2)(-1+6a^2\sigma_\upsilon^2+a^4+3\sigma_\upsilon^4)}]-6(\phi^2+\Phi^2)[\frac{\sigma_\upsilon^2}{(1-a^2-2\sigma_\upsilon^2)}]-\\ -\Phi^4-6\phi^2\Phi^2\end{array}}$$

3. *The Kurtosis of the process follows from its definition* $K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2}$

Theorem 18 *[131]Consider the doubly stochastic volatility process with AGARCH(I)-(0,1) errors of the following form:*

$$\begin{aligned} y_t &= (\phi + b_t)y_{t-1} + \varepsilon_t \\ b_{t+1} &= ab_t + (1+b_t)\upsilon_{t+1} \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \omega + \alpha\left(\varepsilon_{t-1} + r\right)^2 \end{aligned}$$

Then we have following results

1. $E(y_t^2) = \frac{\sigma_z^2(\omega+\alpha r^2)(1-a^2-\sigma_\upsilon^2)}{[1-\alpha\sigma_z^2](1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$
2. $E(y_t^4) = \frac{3\sigma_z^4E\left(\sigma_t^4\right)+6\sigma_z^2E\left(\sigma_t^2\right)\left(E\left(b_t^2\right)+\phi^2\right)E\left(y_{t-1}^2\right)}{1-\phi^4-E\left(b_t^4\right)-6\phi E\left(b_t^2\right)--4\phi E\left(b_t^3\right)}$
3. *The Kurtosis of the process follows from its definition* $K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2}$

Theorem 19 *[131]Consider the doubly stochastic volatility process with AGARCH(I)-(0,1) errors of the following form:*

$$\begin{aligned} y_t &= (\phi + b_t)y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \\ b_{t+1} &= ab_t + (1+b_t)\upsilon_{t+1} \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \omega + \alpha\left(\varepsilon_{t-1} + r\right)^2 \end{aligned}$$

Then we have following results

1. $E(y_t^2) = \frac{\sigma_z^2(\omega+\alpha r^2)(1-a^2-\sigma_\upsilon^2)}{[1-(\alpha\sigma_z^2+\beta)](1-a^2-2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$
2. $E(y_t^4) = \frac{3\sigma_z^4E\left(\sigma_t^4\right)+6\sigma_z^2E\left(\sigma_t^2\right)\left(E\left(b_t^2\right)+\phi^2\right)E\left(y_{t-1}^2\right)}{1-\phi^4-E\left(b_t^4\right)-6\phi E\left(b_t^2\right)--4\phi E\left(b_t^3\right)}$
3. *The Kurtosis of the process follows from its definition* $K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2}$

Theorem 20 *[131]Consider the doubly stochastic volatility process with AGARCH(I)-(0,1) errors of the following form:*

$$\begin{aligned} y_t &= (\phi + b_t + \Phi s_t)y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \\ b_{t+1} &= ab_t + (1+b_t)\upsilon_{t+1} \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \omega + \alpha\left(\varepsilon_{t-1} + r\right)^2 \end{aligned}$$

Then we have following results:

1. $E(y_t^2) = \frac{\sigma_z^2(1+\theta^2)\left(\omega+\alpha r^2\right)(1-a^2-\sigma_\upsilon^2)}{(1-\alpha\sigma_z^2)(1-a^2-2\sigma_\upsilon^2-\Phi^2+\Phi^2a^2+\Phi^2\sigma_\upsilon^2-\phi^2+\phi^2a^2+\phi^2\sigma_\upsilon^2)}$
2. $E(y_t^4) = \frac{3\varepsilon_t^4(1+6\theta^2+\theta^4)+6\varepsilon_t^2(1+\theta^2)[(\phi^2+\Phi^2+E(b_t^2)]E(y_{t-1}^2)}{1-\phi^4-3E(b_t^4)-6(\phi^2+\Phi^2)E(b_t^2)-\Phi^4-6\phi^2\Phi^2}$
3. *The Kurtosis of the process follows from its definition* $K^{(y)} = \frac{E(y_t^4)}{[E(y_t^2)]^2}$

Chapter 4

Data Analysis

4.1 Model Selection

In this section we provide our numerical results for the best fitted GARCH selected models for the underlying indexes.

TABLE 4.1 Results for NL, AUS and SG Indexes Monthly (Weekly)							
Index	Model	Distribution	AIC	BIC	SIC	HQIC	L
NL	APARCH	SGED	-3.140	-2.947	-3.147	-3.062	259.68
	EGARCH	JSU	-3.091	-2.917	-3.020	-3.097	254.77
	(EGARCH)	SSTD	-4.4193	-4.3603	-4.4197	-4.3965	1538.0
	(APARCH)	JSU	-4.428	-4.362	-4.402	-4.428	1542.0
SG	IGARCH	SSTD	-3.871	-3.697	-3.877	-3.800	316.74
	IGARCH	JSU	-3.124	-2.989	-3.069	-3.128	255.40
AUS	EGARCH	SGED	-3.121	-2.986	-3.124	-3.066	255.13
	EGARCH	JSU	-3.856	-3.683	-3.786	-3.862	315.62

We use short notation for the models as APA for APARCH, EG is EGARCH, IG is IGARCH and SGED for skew generalized error distribution, JSU for Johnson reparametrized distribution and SSTD for skew student-t distribution[133].

TABLE 4.2 Parametric Estimations Results NL, SG and AUS Monthly (Weekly)											
Index	Model	Dist	μ	ω	α_1	β_1	γ_1	δ	Skew	AR-1	MA-1
NL	APA	SGED	-0.002	0.0	0.05	0.84	0.99	0.09	0.40	0.9	0.9
	EG	JSU	-0.000	-0.4	-0.33	0.92	0.13		-18.7	0.8	-0.7
	(EG)	SSTD	0.000	-0.5	-0.24	0.93	0.16		0.72	-0.8	0.8
	(APA)	JSU	0.000	0.0	0.11	0.87	1.00	0.62	-2.29	-0.8	0.8
SG	IG	SSTD	0.006	0.0	0.22	0.77			0.66	0.5	-0.4
	IG	JSU	0.006	0.0	0.22	0.77			-0.80	0.5	-0.4
AUS	EG	SGED	0.004	-0.7	-0.18	0.88	0.10		0.52	0.7	-0.6
	EG	JSU	0.003	-0.7	-0.19	0.89	0.07		-19.7	0.7	-0.6

4.1.1 The Nyblom Test Statistics

TABLE 4.3 Nyblom Statistics Results NL, SG and AUS Monthly (Weekly)											
Index	Model	Dist	μ	ω	α_1	β_1	γ_1	δ	Skew	AR-1	MA-1
NL	APA	SGED	1.05	0.16	0.16	0.16	0.16	0.16	0.27	1.6	1.7
	EG	JSU	0.17	0.15	0.06	0.14	0.65		0.45	0.09	0.11
	(EG)	SSTD	0.27	0.18	0.06	0.18	0.07		0.02	0.14	0.14
	(APA)	JSU	0.19	0.11	0.13	0.12	NA	0.11	0.02	0.14	0.16
SG	IG	SSTD	0.10	0.06	0.22	0.05			0.05	0.25	0.23
	IG	JSU	0.13	0.06	0.23				0.05	0.29	0.27
AUS	EG	SGED	0.08	0.24	0.17	0.22	0.06		0.22	0.20	0.27
	EG	JSU	0.11	0.25	0.15	0.23	0.07		0.31	0.18	0.27

Index	Model	Dist	A.C (J.S)			A.C (I.S)			J.S
			10%	5%	1%	10%	5%	1%	
NL	APA	SGED	2.29	2.54	3.05	0.35	0.47	0.75	4.4
	EG	JSU	2.1	2.32	2.82	0.35	0.47	0.75	1.03
	(EG)	SSTD	2.1	2.32	2.82	0.35	0.47	0.75	0.93
	(APA)	JSU	2.29	2.54	3.05	0.35	0.47	0.75	NA
SG	IG	SSTD	1.69	1.9	2.35	0.35	0.47	0.75	0.88
	IG	JSU	1.69	1.9	2.35	0.35	0.47	0.75	0.91
AUS	EG	SGED	2.1	2.32	2.82	0.35	0.47	0.75	1.42
	EG	JSU	2.1	2.32	2.82	0.35	0.47	0.75	1.01

4.1.2 Distribution of Simulated Parameters

TABLE 4.4 True vs Simulation Mean EGARCH (SGED) Parameter Distribution					
	μ	AR-1	MA-1	ω	α_1
T.C	0.0040000	0.71000	-0.63300	-0.75000	-0.18300
W-2000	0.0039572	0.68829	-0.60967	-0.75081	-0.18497
W-3000	0.0041476	0.69091	-0.61623	-0.73718	-0.18094
W-4000	0.0040695	0.68661	-0.60867	-0.76338	-0.18158
W-5000	0.0040683	0.69241	-0.61360	-0.76100	-0.18428
W-6000	0.0040396	0.70784	-0.63199	-0.74960	-0.18142

	β_1	γ_1	Skew	Shape
T.C	0.88800	0.10500	0.52400	2.0390
W-2000	0.88783	0.10236	0.51917	2.0366
W-3000	0.89002	0.10102	0.52286	2.0335
W-4000	0.88605	0.10444	0.52339	2.0266
W-5000	0.88645	0.10317	0.52229	2.0293
W-6000	0.88809	0.10512	0.52203	2.0326

TABLE 4.5 True vs Simulation Mean EGARCH (JSU) Parameter Distribution					
	μ	AR-1	MA-1	ω	α_1
T.C	0.030000	0.70000	-0.60000	-0.70000	-0.19000
W-2000	0.029809	0.68811	-0.58914	-0.73930	-0.19644
W-3000	0.029827	0.69376	-0.59494	-0.73098	-0.19529
W-4000	0.029933	0.69632	-0.59644	-0.72082	-0.19436
W-5000	0.029930	0.69762	-0.59781	-0.70821	-0.19210
W-6000	0.029919	0.69821	-0.59782	-0.70318	-0.19180

	β_1	γ_1	Skew	Shape
T.C	0.89000	0.070000	19.700	3.0100
W-2000	0.88381	0.063163	12.186	2.9261
W-3000	0.88510	0.066076	11.275	2.9253
W-4000	0.88669	0.067522	11.395	2.9278
W-5000	0.88868	0.067741	11.597	2.9494
W-6000	0.88942	0.068208	10.742	2.9469

4.1.3 GARCH Bootstrap Forecast

TABLE 4.6(a) Australia-EGARCH (SGED) Series Forecast						
	Min	q-25	Mean	q-75	Max	Forecast
t+1	-0.10496	-0.013616	0.005360	0.031588	0.061136	0.005335
t+2	-0.18354	-0.014950	0.003275	0.029843	0.083109	0.004969
t+3	-0.15659	-0.014547	0.003647	0.029316	0.076743	0.004708
t+4	-0.20544	-0.014769	0.004507	0.030786	0.080975	0.004523
t+5	-0.15899	-0.014632	0.004043	0.030524	0.093746	0.004392

TABLE 4.6(b) Australia-EGARCH (SGED) $\sigma-$ Forecast						
	Min	q-25	Mean	q-75	Max	Forecast
t+1	0.030701	0.030701	0.030701	0.030701	0.030701	0.030701
t+2	0.027746	0.028807	0.031345	0.032568	0.050103	0.031078
t+3	0.025524	0.027889	0.032027	0.034610	0.077433	0.031417
t+4	0.024239	0.027899	0.032845	0.035573	0.077865	0.031722
t+5	0.023149	0.027899	0.032845	0.036182	0.096704	0.031995

TABLE 4.7(a) Australia-EGARCH (JSU) Series Forecast						
	Min	q-25	Mean	q-75	Max	Forecast
t+1	-0.10604	-0.012826	0.004877	0.029199	0.061208	0.005456
t+2	-0.12329	-0.015332	0.003577	0.028795	0.074161	0.005036
t+3	-0.13957	-0.016986	0.002740	0.029779	0.077403	0.004731
t+4	-0.15453	-0.015119	0.003808	0.030384	0.078566	0.004510
t+5	-0.16100	-0.016734	0.003271	0.030902	0.086115	0.004350

TABLE 4.7(b) Australia-EGARCH (JSU) $\sigma-$ Forecast						
	Min	q-25	Mean	q-75	Max	Forecast
t+1	0.031578	0.031578	0.031578	0.031578	0.031578	0.031578
t+2	0.027834	0.029617	0.032166	0.033552	0.050118	0.031921
t+3	0.025096	0.028957	0.032800	0.035172	0.056148	0.032229
t+4	0.023325	0.029132	0.033446	0.036440	0.061437	0.032507
t+5	0.022963	0.029023	0.033787	0.037145	0.070204	0.032756

Chapter 5

Entropy Measures and Stock Options

5.1 Maximum Entropy Problem and Risk-Neutral Density

Theorem 21 *[66] If the prior information set is $I = \{þ, S, P, \sigma^2\}$then risk-neutral density $f(\ S_T)$ solves the following maximum-entropy problem*

$$\max_g \ -E^g\left[\log g(Y_T)\right] \tag{P−1}$$

subject to

$$\begin{aligned} E^g\left[I_{\{Y_T>0\}}\right] &= 1 && \textit{(C-1)} \\ E^g\ [Y_T] &= \frac{S}{P} && \textit{(C-2)} \\ E^g\ \left[Y_T^2\right] &= \sigma^2 + \left(\frac{S}{P}\right)^2 && \textit{(C-3)} \end{aligned}$$

where $g(S_T) > 0$ on the price space þ. Then the unique solution of the optimization problem P-1 with constranits C-1, C-2 and C-3 is given by:

$$f(S_T) = \frac{\exp\left(\lambda_1 S_T + \lambda_2 S_T^2\right)}{\int_þ \exp\left(\lambda_1 S_T + \lambda_2 S_T^2\right) dS_T}$$

where λ_1, λ_2 are chosen so that $f(S_T)$ satisfies the price and variance constraints and $f(S_T) > 0$ on the price space þ.

Theorem 22 *Consider the maximum entropy problem for the case of weighted entropy, $u > 0$ with constraints C-1, C-2 and C-3 of theorem 52,then we look for the solution of the following problem:*

$$\max_g \ -E^g\left[u(Y_T)\log g(Y_T)\right]$$

Then the unique solution of the optimization problem is given by:

$$f(S_T) = \frac{\exp\left(\frac{\lambda_0+\lambda_1 S_T+\lambda_2 S_T^2}{u(S_T)}\right)}{\int \exp\left(\frac{\lambda_0+\lambda_1 S_T+\lambda_2 S_T^2}{u(S_T)}\right)}$$

whereλ_0, λ_1, λ_2 are chosen so that $f(S_T)$ satisfies the price and variance constraints.

Theorem 23 *Consider the case of Tsallis entropy maximization problem with the constraints of the previous theorem, and we look for the solution of risk neutral denstiy $f(S_T)$ of following problem:*

$$\max_g \ -E^g\left[\log_{\{q\}} g(Y_T)\right]$$

Then the unique solution of the optimization problem is given by:

$$f(S_T) = \left(\frac{1 + q\lambda_0 + q\left(\lambda_1 S_T + \lambda_2 S_T^2\right)}{q+1}\right)^{\frac{1}{q}}$$

where$\lambda_0,\ \lambda_1, \lambda_2$ are chosen so that $f(S_T)$ satisfies the price and variance constraints.

Theorem 24 *Consider the case of Weighted Tsallis entropy maximixation problem ,$u > 0$,with the constraints C-1, C-2 and C-3 of the previous theorem, and we look for the solution of risk neutral density $f(S_T)$ of following problem as:*

$$\max_g \ -E^g\left[u(Y_T)\log_{\{q\}} g(Y_T)\right]$$

Then the unique solution of the optimization problem is given by:

$$f(S_T) = \left(\frac{u\left(S_T\right) + q\lambda_0 + q\left(\lambda_1 S_T + \lambda_2 S_T^2\right)}{u(S_T)\left(q+1\right)}\right)^{\frac{1}{q}}$$

where $\lambda_0,\ \lambda_1, \lambda_2$ are choosen so that $f(S_T)$ satisfies the price and variance constraints.

Theorem 25 *Consider the case of Kaniadakis entropy maximization problem with the constraints C-1, C-2 and C-3 of the previous theorem, and we look for the solution of risk neutral denstiy $f(S_T)$ of following problem as:*

$$\max_g \ -E^g\left[\log_{\{k\}} g\left(Y_T\right)\right]$$

Then the unique solution of the optimization problem is given by:

$$f(S_T) = \left(\frac{k\left(\lambda_0 + \lambda_1 S_T + \lambda_2 S_T^2\right) + \sqrt{k^2\left(\lambda_0 + \lambda_1 S_T + \lambda_2 S_T^2 - 1\right) + 1}}{k+1}\right)^{\frac{1}{k}}$$

where $\lambda_0,\ \lambda_1, \lambda_2$ are choosen so that $f(S_T)$ satisfies the price and variance constraints.

Theorem 26 *Consider the case of Weighted Kaniadakis entropy, $u > 0$ with the constraints C-1, C-2 and C-3 of the previous theorem, and we look for the solution of risk neutral denstiy $f(S_T)$ of following problem:*

$$\max_g \ -E^g\left[u\left(Y_T\right)\log_{\{k\}} g\left(Y_T\right)\right]$$

Then the unique solution of the optimization problem is given by:

$$f(S_T) = \left(\frac{k\left(\lambda_0 + \lambda_1 S_T + \lambda_2 S_T^2\right) + \sqrt{k^2\left(\lambda_0 + \lambda_1 S_T + \lambda_2 S_T^2 - \left(u\left(S_T\right)\right)^2\right) + \left(u\left(S_T\right)\right)^2}}{\left(k+1\right)u(S_T)}\right)^{\frac{1}{k}}$$

where $\lambda_0,\ \lambda_1, \lambda_2$ are choosen so that $f(S_T)$ satisfies the price and variance constraints.

5.2 Pricing European Call and Put Options

Theorem 27 *[Sheraz] If time to expiry is T , a call option pays $\max(0, S_T - K)$ and put option pays $\max(0, K - S_T)$, where K is the strike price. Then using the linear pricing rule and the risk neutral density $g(S_T)$, the price of European Call and Put are given by:*

$$\begin{aligned} Call &= S - PK\left[G(K) - g(K) + g(0) + 1\right] + P\int_0^K G\left(S_T\right) dS_T \\ Put &= P\int_0^K G\left(S_T\right) dS_T \end{aligned}$$

Chapter 6

Statistical Heterogeneity

6.1 Gini's Index and Risk Neutral Density of Maximum Entropy

6.1.1 Shannon's Entropy Problem.

Theorem 28 *We consider the risk-neutral density $g(S_T)$ which solves the following maximization problem P-2 subject to conditions C-1, C-2 and C-4 as:*

$$\max_{g} \; -E^g\left[\log g\left(Y_T\right)\right] \tag{P−2}$$

subject to

$$\begin{aligned} E^g\left[I_{\{Y_T>0\}}\right] &= 1 && \textit{(C-1)} \\ E^g\left[Y_T\right] &= \frac{S}{P} && \textit{(C-2)} \\ E^g\left[\left|Y_T-\frac{S}{P}\right|^r\right] &= \delta \;\; , r>0 && \textit{(C-4)} \end{aligned}$$

Then we have risk neutral density as follows:

$$g(S_T)=\exp\left[-1-\lambda_1-\lambda_2 S_T-\lambda_3\left|S_T-\frac{S}{P}\right|^r\right]$$

where $\lambda_1,\lambda_2,\lambda_3$ are to be determined by using given constraints.

6.1.2 Gini's Maximization Problems

Theorem 29 *Consider the following problem P-3 for:*

$$G(S_T)=E^g\left[I_{\{Y_T>S_T\}}\right]$$

Then entropy maximization problem is equivalent to the convex optimization problem:

$$\min \int_0^\infty \bar{G}(Y_T)^2\, dY_T \tag{P−3}$$

subject to C-1, C-2 and C-4 as given the previous theorem then the solution is :

$$g(S_T)=\frac{\lambda_3}{2}r(r-1)\left|S_T-\frac{S}{P}\right|^r$$

Theorem 30 *Consider the following Entropy-Gini-Maximizarion problem P-4 :*

$$\max \ E^g \ [\ln g\left(Y_T\right)] \tag{P−4}$$

subject to

$$\begin{aligned} E^g\left[I_{\{Y_T>0\}}\right] &= 1 && \textit{(C-1)} \\ E^g\ [Y_T\ ] &= \frac{S}{P} && \textit{(C-2)} \\ \int_0^\infty G\left(Y_T^2\right) dY_T &= \delta \ \ , r>0 && \textit{(C-5)} \end{aligned}$$

Then G satisfies the following differential equation and c is a constant of integration:

$$G'(S_T) = c - \lambda_2 G(S_T) - \lambda_3 G(S_T)^2$$

The solution of this differential equation is as follows:

$$G(S_T) = \frac{1}{c_3 \exp\left(-c_1 S_T\right) + (1-c_3)}$$

where c_1 and c_3 are positive valued parameters, $c_1 = \frac{\psi(c_3)}{\frac{S}{P}}$, $\psi\left(c_3\right) = \frac{\ln(c_3)}{c_3-1}$, $c_1 = -\lambda_2, c_3 = -\lambda$

Theorem 31 *Consider the following Entropy-Gini-Maximizarion problem P-5 :*

$$\max \ \int_0^\infty G(Y_T)^2 \ dY_T \tag{P−5}$$

subject to

$$\begin{aligned} E^g\left[I_{\{Y_T>0\}}\right] &= 1 && \textit{(C-1)} \\ E^g\ [Y_T\ ] &= \frac{S}{P} && \textit{(C-2)} \\ E^g\left[\ln g\left(Y_T\right)\right] &= \xi && \textit{(C-6)} \end{aligned}$$

Then solution satisfies the following differential equation:

$$G'(S_T) = \frac{c}{\lambda_3} - \frac{\lambda_2}{\lambda_3} G(S_T) - \frac{1}{\lambda_3} G(S_T)^2$$

Theorem 32 *Consider the optimization problem P-6 :*

$$\max D(g) \tag{P-6}$$

subject to

$$\begin{aligned} E^g\left[I_{\{-\infty<Y_T<\infty\}}\right] &= 1 && \textit{(C-1)} \\ E^g\ [Y_T\ ] &= \frac{S}{P} && \textit{(C-2)} \\ E^g\left[\left|Y_T - \frac{S}{P}\right|^r\right] &= \delta \ \ , r>0 && \textit{(C-4)} \end{aligned}$$

Then we have

$$g\left(S_T\right) = -r\left(r-1\right)\lambda_3 \left|S_T - \frac{S}{P}\right|^{r-2}, r>2$$

6.1.3 Shannon's Entropy and Gini's Index

Theorem 33 *Consider the optimization problem P-7 :*

$$\min E^g \left[\ln g\left(Y_T\right)\right] \tag{P-7}$$

subject to

$$\begin{aligned} E^g\left[I_{\{-\infty<Y_T<\infty\}}\right] &= 1 && \textit{(C-1)} \\ E^g\left[Y_T\right] &= \frac{S}{P} && \textit{(C-2)} \\ D\left(g\right) &= \gamma && \textit{(C-3)} \end{aligned}$$

Then solution satisfies the Riccati equation :

$$G_*'\left(S_T\right) = \frac{c}{2} - \frac{2\lambda_2+\lambda_3}{2}G_*(S_T) + \frac{\lambda_3}{2}G_*(S_T)^2$$

Theorem 34 *Consider the following optimization problem:*

$$\max D(g) \tag{P-8}$$

subject to

$$\begin{aligned} E^g\left[I_{\{-\infty<Y_T<\infty\}}\right] &= 1 && \textit{(C-1)} \\ E^g\left[Y_T\right] &= \frac{S}{P} && \textit{(C-2)} \\ E^g\left[\ln g\left(Y_T\right)\right] &= \xi && \textit{(C-6)} \end{aligned}$$

Then G_* is the solution of following differential equation:

$$y'\left(S_T\right) = \frac{c}{2\lambda_3} - \frac{1+2\lambda_2}{2\lambda_3}G_>(S_T) + \frac{1}{2\lambda_3}G_>(S_T)^2$$

Chapter 7

Renyi Entropy Problems for Risk Neutral Densities

7.1 Renyi Entropy for Risk Neutral Density Problems

Theorem 35 *Consider the case of following Renyi -entropy maximization problem:*

$$\max I_r^R \left[g\left(S_T \right) \right] = \frac{1}{1-r} \ln E^g \left[g^{r-1}(Y_T) \right]$$

subject to

$$\begin{aligned} E^g \left[I_{\{-\infty < Y_T < \infty\}} \right] &= 1 && \textit{(C–1)} \\ E^g \left[\varphi_i \left(Y_T \right) \right] &= c_i \ , \ i = 1, ..n && \textit{(C–2)} \end{aligned}$$

where $\varphi : p \rightarrow \mathbb{R}$ *and* $c_1, c_2 ... c_n$ *are given real values and* p *is state space of real line* $\mathbb{R}$. *Then the soultion of problem is:*

$$g\left(S_T \right) = \frac{\left[1 - \frac{1-r}{r} \left(\sum_{i=1}^{n} \beta_i \left(c_i - \varphi_i \left(S_T \right) \right) \right) \right]^{\frac{1}{r-1}}}{\int_{-\infty}^{\infty} \left[1 - \frac{1-r}{r} \left(\sum_{i=1}^{n} \beta_i \left(c_i - \varphi_i \left(S_T \right) \right) \right) \right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1, \beta_2 ... \beta_n$ *are to be determined by given constraints.*

Theorem 36 *Consider the case of following weighted- Renyi- entropy maximization problem:*

$$\max I_r^R \left[g\left(S_T \right) \right] = \frac{1}{1-r} \ln E^g \left[u\left(Y_T \right) g^{r-1}(Y_T) \right]$$

subject to

$$\begin{aligned} E^g \left[I_{\{-\infty < Y_T < \infty\}} \right] &= 1 && \textit{(C–1)} \\ E^g \left[u\left(Y_T \right) \varphi_i \left(Y_T \right) \right] &= c_i \ , \ i = 1, ..n && \textit{(C–2)} \end{aligned}$$

where $\varphi : \not{p} \longmapsto \mathbb{R}$ *and* $c_1, c_2 ... c_n$ *are given real values and* $\not{p}$ *is state space of real line* $\mathbb{R}$*. Then the solution is:*

$$g\left(S_T\right) = \frac{\left[\frac{1-\frac{1-r}{r}\left(\sum_{i=1}^{n}\beta_i(c_i-\varphi_i(S_T))\right)}{u(S_T)}\right]^{\frac{1}{r-1}}}{\int_{-\infty}^{\infty}\left[\frac{1-\frac{1-r}{r}\left(\sum_{i=1}^{n}\beta_i(c_i-\varphi_i(S_T))\right)}{u(S_T)}\right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1, \beta_2 ... \beta_n$ *are to be determined by given constraints.*

Theorem 37 *Consider the case of following Renyi-Entropy maximization problem:*

$$\max I_r^R\left[g\left(S_T\right)\right] = \frac{\ln E^g\left[u\left(Y_T\right)\ g^{r-1}(Y_T)\right] - \ln E^g\left[Y_T\right]}{1-r}$$

subject to C–1 and C–2 of Theorem 60, then thesoluti on of problem is:

$$g\left(S_T\right) = \frac{\left[\frac{1}{rE^g[u(S_T)]} - \frac{\frac{1-r}{r}\left(\sum_{i=1}^{n}\beta_i(c_i-\varphi_i(S_T))\right)}{u(S_T)}\right]^{\frac{1}{r-1}}}{\int_{-\infty}^{\infty}\left[\frac{1-\frac{1-r}{r}\left(\sum_{i=1}^{n}\beta_i(c_i-\varphi_i(S_T))\right)}{u(S_T)}\right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1, \beta_2 ... \beta_n$ *are to be determined by given constraints.*

Theorem 38 *Consider the case of following Renyi-entropy maximization problem:*

$$\max I_r^R\left[g\left(S_T\right)\right] = E^g\left[u\left(Y_T\right)\right] + \frac{1}{1-r}\left[\ln E^g\left[u\left(Y_T\right)\ g^{r-1}(Y_T)\right] - \ln E^g\left[u\left(Y_T\right)\right]\right]$$

subject to C–1 and C–2 of Theorem 60, then the solution of problem is:

$$g\left(S_T\right) = \frac{\left[\frac{1}{r\left(1+\sum_{i=1}^{n}\beta_i+\lambda\right)} - \frac{1-r}{r} + \frac{1-r}{r}\frac{\left(\lambda+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)}{u(S_T)}\right]^{\frac{1}{r-1}}}{\int_{-\infty}^{\infty}\left[\frac{1}{r\left(1+\sum_{i=1}^{n}\beta_i+\lambda\right)} - \frac{1-r}{r} + \frac{1-r}{r}\frac{\left(\lambda+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)}{u(S_T)}\right]^{\frac{1}{r-1}} dS_T}$$

where $\beta_1, \beta_2 ... \beta_n$ *are to be determined by given constraints.*

Theorem 39 *Consider the case of following Weighted-Tsallis -Entropy maximizationProblem*

$$\max I_q^{T,u}\left[g\left(S_T\right)\right]=\frac{1}{1-q}E^g\left[u\left(Y_T\right)\quad\left[g^{q-1}(Y_T)-\ 1\right]\right]$$

subject to C–1 and C–2 of Theorem 60, thenthe solution of problem is:

$$g\left(S_T\right)=\left[\frac{1}{q}+\frac{1-q}{q}\frac{\lambda+\sum_{i=1}^{n}\beta_i\varphi_i\left(S_T\right)}{u\left(S_T\right)}\right]^{\frac{1}{q-1}}$$

where $\beta_1,\beta_2...\beta_n$ are to be determined by given constraints.

Theorem 40 *Consider the case of following Weighted utility-Tsallis Entropy maximization Problem:*

$$\max I_q^{T,u}\left[g\left(S_T\right)\right]=E^g\left[u\left(Y_T\right)\right]+\frac{1}{1-q}E^g\left[u\left(Y_T\right)\quad\left[g^{q-1}(Y_T)-\ 1\right]\right]$$

subject to C–1 and C–2 of Theorem 60, then the solution of problem is:

$$g\left(S_T\right)=\left[1-\frac{1-q}{q}\frac{\lambda+\sum_{i=1}^{n}\beta_i\varphi_i\left(S_T\right)}{u\left(S_T\right)}\right]^{\frac{1}{q-1}}$$

Theorem 41 *Consider the case of following Weighted-Kanidakis -Entropy maximization Problem:*

$$\max I_k^{K,u}\left[g\left(S_T\right)\right]=-E^g\left[u\left(Y_T\right)\frac{g^k(Y_T)-g^{-k}(Y_T)}{2k}\right]$$

subject to C–1 and C–2 of Theorem 60, then the solution of problem is:

$$g\left(S_T\right)=\left[\frac{-2\left(\lambda+\sum_{i=1}^{n}\beta_i\varphi_i\left(S_T\right)\right)+\sqrt{\left[k\left(\lambda+\sum_{i=1}^{n}\beta_i\varphi_i\left(S_T\right)\right)\right]^2+\left(u\left(S_T\right)\right)^2\left(1-k^2\right)}}{u\left(S_T\right)\left(1+k\right)}\right]^{\frac{1}{k}}$$

Theorem 42 *Consider the case of following Weighted utility-Kanidakis Entropy maximization Problem*

$$\max I_k^{K,u}\left[g\left(S_T\right)\right]=E^g\left[u(Y_T)\right]-E^g\left[u\left(Y_T\right)\frac{g^k(Y_T)-g^{-k}(Y_T)}{2k}\right]$$

subject to C–1 and C–2 of Theorem 60, then the solution of problem is:

$$g\left(S_T\right)=\left[\frac{-2\left(u\left(S_T\right)-\gamma\right)+\sqrt{\left[k\left(u\left(S_T\right)-\gamma\right)\right]^2+\left(u\left(S_T\right)\right)^2\left(1-k^2\right)}}{u\left(S_T\right)\left(1+k\right)}\right]^{\frac{1}{k}}$$

where $\lambda-\sum_{i=1}^{n}\beta_i\varphi_i\left(S_T\right)=\gamma$

7.2 Pricing European Call and Put Options

Theorem 43 *If time to expiry is T , a call option pays $\max(0, S_T - K)$ and put option pays $\max(0, K - S_T)$, Where K is the strike price. Then using the linear pricing rule and the risk neutral density $g(S_T)$, the price of European Call and Put are given by:*

$$\begin{aligned} Call &= S - PK\left[G(K) - g(K) + g(0) + 1\right] + P\int_0^K G\left(S_T\right) dS_T \\ Put &= P\int_0^K G\left(S_T\right) dS_T \end{aligned}$$

Chapter 8

Risk Neutral Densities with New Approaches

8.1 Ubriaco Entropy Measure and Risk-Neutral Densities

8.1.1 New Approach-The Discrete Case

Theorem 44 *Consider the following problem with Ubriaco entropy measure*

$$\max H_U(p) = \sum_i p_i \left(\ln \frac{1}{p_i} \right)^d$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} p_i &= 1 \\ \sum_{i=1}^{n} \varphi_{i,} p_i &= a_i \end{aligned}$$

where $i = 1, ...n$, $, d > 0$, $p_i \geq 0$ *and* $\varphi_i \in \mathbb{R}$ *and* $a_1, a_2..a_n$ *are given real values. Then the solution of the above problem is:*

$$p_i = (\psi')^{-1} \left(\lambda + \sum_{i=1}^{n} \beta_i \varphi_i \right)$$

Theorem 45 *For a weighted entropy with* $u_i > 0 \ \forall \ i$ *then we write the new problem :*

$$\max H_U(p) = \sum_i u_i p_i \left(\ln \frac{1}{p_i} \right)^d$$

subject to constranits

$$\begin{aligned} \sum_{i=1}^{n} p_i &= 1 \\ \sum_{i=1}^{n} \varphi_{i,} p_i &= a_i \end{aligned}$$

where $i = 1, ...n$, $, d > 0$, $p_i \geq 0$ *and* $\varphi_i \in \mathbb{R}$ *and* $a_1, a_2..a_n$ *are given real values. Then the solution of the above problem is:*

$$p_i = (\psi')^{-1} \left(\frac{\lambda + \sum_{i=1}^{n} \beta_i \varphi_i}{u_i} \right)$$

where $\lambda, \beta_1, \beta_2, ..\beta_n$ *are Lagrange multipliers and can be determined using given constraints.*

8.1.2 Main Results with Ubriaco Entropy Measure

Theorem 46 *Consider the the following maximum-entropy problem for the case of Ubriaco entropy measure :*

$$\max H_U(g(S_T)) = E^g \left[\left(\ln \left(\frac{1}{g(Y_T)} \right) \right)^d \right]$$

subject to

$$E^g \left[I_{\{Y_T > 0\}} \right] = 1 \qquad \text{(C-1)}$$

$$E^g [Y_T] = \frac{S}{P} \qquad \text{(C-2)}$$

$$E^g \left[Y_T^2 \right] = \sigma^2 + \left(\frac{S}{P} \right)^2 = \gamma \qquad \text{(C-3)}$$

Then the risk neutral density $g(S_T)$ *is:*

$$g(S_T) = (\psi')^{-1} \left(\lambda + \beta_1 S_T + \beta_2 S_T^2 \right)$$

where $\lambda, \beta_1, \beta_2$ *are to be determined using C-1, C-2 and C-3.*

Remark 47 *Similarly we can write solution of the above problem for the case of Weighted-Ubriaco entropy ,*$u(S_T) > 0$ *as follows:*

$$g(S_T) = (\psi')^{-1} \left(\frac{\lambda + \beta_1 S_T + \beta_2 S_T^2}{u(S_T)} \right)$$

Theorem 48 *Consider the case of Ubriaco entropy maximization problem :*

$$\max H_U(g(S_T)) = E^g \left[\left(\ln \left(\frac{1}{g(Y_T)} \right) \right)^d \right]$$

subject to

$$E^g \left[I_{\{Y_T > 0\}} \right] = 1 \qquad \text{(C-1)}$$

$$E^g [Y_T] = \frac{S}{P} \qquad \text{(C-2)}$$

$$E^g \left[\left| Y_T - \frac{S}{P} \right|^r \right] = \delta \quad , r > 0 \qquad \text{(C-3)}$$

Then the risk-neutral density $g(S_T)$ as follows:

$$g(S_T) = (\psi')^{-1}\left(\lambda + \beta_1 S_T + \beta_2 \left|S_T - \frac{S}{P}\right|^r\right)$$

where $\lambda, \beta_1, \beta_2$ are to be determined using C-1,C-2 and C-3.

Theorem 49 *Consider the case of following Ubriaco entropy problem:*

$$\max H_U(g(S_T)) = E^g\left[\left(\ln\left(\frac{1}{g(Y_T)}\right)\right)^d\right]$$

subject to

$$\begin{aligned} E^g\left[I_{\{-\infty<Y_T<\infty\}}\right] &= 1 && \text{(C–1)} \\ E^g\left[\varphi_i(Y_T)\right] &= c_i \ , \ i = 1, ..n && \text{(C–2)} \end{aligned}$$

where $\varphi_i \in \mathbb{R}$ and $c_1, c_2, ...c_n$ are real given values. Then we have solution:

$$g(S_T) = (\psi')^{-1}(\lambda + \beta_1 \varphi_i(S_T))$$

where $\lambda, \beta_1, \beta_2$ are to be determined using C-1,C-2 and C-3

Theorem 50 *Consider the case of Ubriaco entropy maximization problem:*

$$\max H_U(g(S_T)) = E^g\left[\left(\ln\left(\frac{1}{g(Y_T)}\right)\right)^d\right]$$

subject to

$$\begin{aligned} E^g\left[I_{\{Y_T>0\}}\right] &= 1 && \text{(C-1)} \\ E^g\left[Y_T\right] &= S_0 e^{rT} && \text{(C-2)} \\ E^g\left[(S_T - K_0)^+\right] &= C_0 e^{rT} && \text{(C-3)} \end{aligned}$$

where K_0 is the strike price, T is the time to expiry and r is risk-free interest rate. Then the risk-neutral density $g(S_T)$ is:

$$g(S_T) = (\psi')^{-1}\left(\lambda + \beta_1 S_T + \beta_2 (S_T - K_0)^+\right)$$

where $\lambda, \beta_1, \beta_2$ are to be determined using C-1,C-2 and C-3

8.2 Shafee Entropy Measure and Risk-Neutral Densities

Theorem 51 *Consider the following entropy maximization problem:*

$$\max -E^g\left[g(Y_T)^{a-1} \ln g(Y_T)\right], a > 0$$

subject to

$$\begin{aligned} E^g\left[I_{\{-\infty<Y_T<\infty\}}\right] &= 1 && \text{(C–1)} \\ E^g\left[\varphi_i(Y_T)\right] &= c_i \ , \ i = 1, ..n && \text{(C–2)} \end{aligned}$$

where $\varphi_i \in \mathbb{R}$ *and* $c_1, c_2, ...c_n$ *are real given values Then the solution is :*

$$g(S_T) = \left[\frac{aW\left(\frac{\gamma(1-a)}{a} e^{-\left(\frac{1-a}{a}\right)}\right)}{\left(\alpha + \sum_{i=1}^{n} \beta_i \varphi_i \left(S_T\right)\right)(1-a)} \right]^{\frac{1}{1-a}}$$

where $\alpha, \beta_{1,} \beta_2 ... \beta_n$ *are Lagrange multipliers and W is a Lambert function.*

8.2.1 New Approach

Theorem 52 *Consider the case of Shafee entropy problem:*

$$\max -E^g \left[g\left(Y_T\right)^{a-1} \ln g\left(Y_T\right) \right], a > 0$$

subject to

$$\begin{aligned} E^g \left[I_{\{-\infty < Y_T < \infty\}} \right] &= 1 && \text{(C–1)} \\ E^g \left[\varphi_i \left(Y_T\right) \right] &= c_i \ , \ i = 1, ..n && \text{(C–2)} \end{aligned}$$

where $\varphi_i \in \mathbb{R}$ *and* $c_1, c_2, ...c_n$ *are real given values. Then the solution is:*

$$g(S_T) = \left(\psi'\right)^{-1} \left(\alpha + \sum_{i=1}^{n} \beta_i \varphi_i(S_T) \right)$$

Theorem 53 *Consider the case of Shafee -entropy problem:*

$$\max -E^g \left[g\left(Y_T\right)^{a-1} \ln g\left(Y_T\right) \right], a > 0$$

subject to

$$\begin{aligned} E^g \left[I_{\{Y_T > 0\}} \right] &= 1 && \text{(C-1)} \\ E^g \ \left[Y_T \right] &= \frac{S}{P} && \text{(C-2)} \\ E^g \ \left[Y_T^2 \right] &= \sigma^2 + \left(\frac{S}{P} \right)^2 = \gamma && \text{(C-3)} \end{aligned}$$

Then the risk-neutral densities can be written using Lambert function and new approach as follows respectively:

$$\begin{aligned} g(S_T) &= \left[\frac{aW\left(\frac{\gamma(1-a)}{a} e^{-\left(\frac{1-a}{a}\right)}\right)}{\gamma\left(1-a\right)} \right]^{\frac{1}{1-a}} \\ g(S_T) &= \left(\psi'\right)^{-1} \left(\lambda + \beta_1 S_T + \beta_2 S_T^2 \right) \end{aligned}$$

where $\lambda, \beta_1, \beta_2$ *are to be determined usingC-1 and C-2 and* $\gamma = \lambda + \beta_1 S_T + \beta_2 S_T^2$

Theorem 54 *Consider the case of Weighted-Shafeef entropy problem:*

$$\max -E^{g}\left[u\left(Y_{T}\right)g\left(Y_{T}\right)^{a-1}\ln g\left(Y_{T}\right)\right],q>0$$

subject to

$$\begin{aligned} E^{g}\left[I_{\{-\infty<Y_{T}<\infty\}}\right] &= 1 && \textit{(C–1)} \\ E^{g}\left[u\left(Y_{T}\right)\varphi_{i}\left(Y_{T}\right)\right] &= c_{i}\ ,\ i=1,..n && \textit{(C–2)} \end{aligned}$$

where $\varphi_{i}\in\mathbb{R}$ *and* $c_{1},c_{2},...c_{n}$ *are real given values. Then the solution is:*

$$g(S_{T})=\left[\frac{au\left(S_{T}\right)W\left(\frac{\gamma(1-a)}{au(S_{T})}e^{-\left(\frac{1-a}{a}\right)}\right)}{\gamma\left(1-a\right)}\right]^{\frac{1}{1-a}}$$

where $\gamma=\alpha+\sum_{i=1}^{n}\beta_{i}u\left(S_{T}\right)\varphi_{i}\left(S_{T}\right)$ *and* $\beta_{1},\beta_{2},...\beta_{n}$ *are Lagrange multipliers and to be determined using C-1, C-2.*

Chapter 9

Semi-Markov Regime Switching Interest Rate Models

9.1 Measure selection: Tsallis entropy

Proposition 55 *In the one-period model, the minimal entropy martingale measure is given by*

$$\begin{aligned} p_0^j &= \frac{(1-d_{Y_0})\,\pi_0^j}{(u_{Y_0}-d_{Y_0})\sum_{j=1}^{m}\pi_0^j}, \\ q_0^j &= \frac{(1-u_{Y_0})\,k_0^j}{(d_{Y_0}-u_{Y_0})\sum_{j=1}^{m}k_0^j} \end{aligned}$$

Theorem 56 *[137]The minimal Tsallis entropy martingale measure is given by:*

$$\begin{aligned} p_t^j &= \frac{(1-d_{Y_t})\,\pi_t^j}{(u_{Y_t}-d_{Y_t})\sum_{j=1}^{m}\pi_t^j} \\ q_t^j &= \frac{(1-u_{Y_t})\,k_t^j}{(d_{Y_t}-u_{Y_t})\sum_{j=1}^{m}k_t^j} \end{aligned}$$

9.2 Measure selection: Kaniadakis entropy

Theorem 57 *[137]The minimal Kaniadakis entropy martingale measure is given by*

$$\begin{aligned} p_t^j &= \frac{(1-d_{Y_t})\,\pi_t^j}{(u_{Y_t}-d_{Y_t})\sum_{j=1}^{m}\pi_t^j} \\ q_t^j &= \frac{(1-u_{Y_t})\,k_t^j}{(d_{Y_t}-u_{Y_t})\sum_{j=1}^{m}k_t^j}. \end{aligned}$$

9.3 Risk-Neutral Probabilities

Theorem 58 *Consider the following problem where $\mathbb{P}$and $\mathbb{Q}$ are two probability measures:*

$$S(\mathbb{P},\mathbb{Q}) = \mathbb{E}^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{v}\ln(\frac{d\mathbb{Q}}{d\mathbb{P}}),\ \ if\mathbb{Q}\ll\mathbb{P}, v>0$$

subject to

$$\begin{aligned}\sum_{j=1}^{m}\left(p_0^j+q_0^j\right) &= 1\\ u_{_{Y_0}}\sum_{j=1}^{m}p_0^j + d_{_{Y_o}}\sum_{j=1}^{m}q_0^j &= 1\end{aligned}$$

where u is an up movement and d is a down movement in the binomial option pricing model and Y_o is a semi-markov process. Also we define:

$$\begin{aligned}\psi_1(x) &= x\ln\frac{x}{\pi_0^j}\\ \psi_2(x) &= x\ln\frac{x}{K_0^j}\end{aligned}$$

Then we have the following results:

$$\begin{aligned}p_0^j &= \left(\psi_1'\right)^{-1}\left(-\lambda-\gamma u_{Y_o}\right)\\ q_0^j &= \left(\psi_2'\right)^{-1}\left(-\lambda-\gamma d_{Y_o}\right)\end{aligned}$$

and

$$p_0^j = \pi_0^j\left[\frac{vW\left(\left(\frac{1-v}{\alpha v}\right)\left(\pi_0^j\right)^{1-v}\cdot e^{\left(\frac{1-v}{v}\right)}\right)}{\alpha(1-v)\left(\pi_0^j\right)^{1-v}}\right]^{\frac{1}{1-v}}$$

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