UNIVERSITATEA DIN BUCUREŞTI
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TEZĂ DE DOCTORAT

CURVES IN METRIC SPACES

REZUMAT

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REZUMAT

Teza de doctorat, intitulata **"CURVES IN METRIC SPACES"** ("**CURBE IN SPATII METRICE**" este structurata pe cinci capitole, dupa cum urmeaza:

Capitolul 1 contine motivatia pentru studiul subiectelor tezei si un rezumat detaliat al acesteia.

Capitolul 2, intitulat **"Spatii metrice. Curbe in geometria euclidiana"** prezinta notiuni fundamentale despre spatiile metrice si un scurt rezumat al teoriei curbelor in cazul euclidian.

Se defineste o curba parametrizata regulata si se dau mai multe exemple de astfel de curbe (dreapta, cercul, elicea circulara).

Apoi, reamintim pe scurt rezultate fundamentale ale teoriei locale a curbelor si, evident, reperul Frenet si ecuatiile lui Frenet .

Ca un exemplu, consideram elicea circulara parametrizata canonic (de catre elementul ei de arc)

$$\alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right),$$

unde $c = \sqrt{a^2 + b^2}$.

Se determina reperul Frenet asociat, curbura κ si torsiunea τ ale curbei α. Retinem ca atat curbura cat si torsiunea sunt constante.

Urmatoarele propozitii furnizeaza diferite caracterizari ale curbelor din punct de vedere local.

Propozitia 2.2.7. *Pentru orice curba parametrizata regulata α avem* $\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$.

Propozitia 2.2.8. *O curba stramba este o dreapta daca si numai daca are curbura identic nula.*

Propozitia 2.2.9. *O curba stramba este plana daca si numai daca are torsiunea identic nula. Singurele curbe plane care au curbura constanta nenula sunt cercurile.*

O elice generalizata este o curba stramba cu $\kappa \neq 0$ ai carei vectori tangenti fac un unghi constant cu o directie fixa. Elicele generalizate sunt caracterizate de urmatoarea teorema (Lancret):

Propozitia 2.2.10. *O curba stramba este elice generalizata daca si numai daca raportul τ/κ este constant.*

Principalul rezultat al teoriei curbelor este:

Teorema 2.2.12 (Teorema fundamentala a teoriei curbelor). *Doua curbe strambe C si C^* sunt congruente (adica, difera printr-o izometrie) daca si numai daca parametrizarile canonice $\alpha, \alpha^* : [0, L] \to \mathbf{R}^3$ au proprietatea* ca *$\kappa(s) = \kappa^*(s)$ si $\tau(s) = \tau^*(s)$, pentru orice $s \in [0, L]$.*

Teoream de mai sus raspunde la intrebarea: *"Cand cele doua curbe difera doar printr-o izometrie?"*

In finalul capitolului sunt prezentate rezultate fundamentale pentru curbe plane: teorema Hopf Umlaufsatz, teorema celor patru varfuri a lui Herglotz si inegalitatea izoperimetrica.

Ultima raspunde la o intrebare fundamentala de geometrie, si anume: fiind data o curba inchisa de lungime L, cand aceasta margineste aria maxima?

Capitolul 3 are titlul **"Curbe in spatiul Minkowski".**

Reamintim definitiile varietatilor pseudo-riemanniene , in particular spatiul semi-euclidian $\mathbf{R}^n_\gamma$.

Pentru orice $p \in \mathbf{R}^n$, exista un izomorfism liniar de la $\mathbf{R}^n$ la $T_p(\mathbf{R}^n)$ care duce v in $v_p = \sum v^i \partial_i$. Astfel, obtinem un tensor metric cu

$$\langle v_p, w_p \rangle = v \cdot w = \sum v^i w^i,$$

i.e., varietatea Riemanniana $\mathbf{R}^n$, numita *n-spatiul euclidian.*

Pentru un numar intreg γ, cu $0 \leq \gamma \leq n$, schimband primele γ semne plus de mai sus in minus obtinem un tensor metric

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\gamma} v^i w^i + \sum_{j=\gamma+1}^{n} v^j w^j.$$

Obtinem asadar *spatiul semi-euclidian* $\mathbf{R}^n_\gamma$ care se reduce la $\mathbf{R}^n$ daca $\gamma = 0$.

Pentru $n \geq 2$, $\mathbf{R}^n_1$ este numit n-spatiul Minkowski; daca $n = 4$, aceasta este cel mai simplu exemplu de spatiu-timp relativist.

In partea a doua a capitolului intitulata "Curbe in spatiul Minkowski $\mathbf{R}^3_1$" consideram spatiul $\mathbf{R}^3_1$ definit ca fiind $\mathbf{R}$-spatiul vectorial 3-dimensional care contine vectorii

$$\{(x_1, x_2, x_3) / x_1, x_2, x_3 \in \mathbf{R}\},$$

inzestrat cu produsul interior

$$\langle X, Y \rangle_1 = -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Acest spatiu se numeste 3-*spatiu Minkowski* sau 3-*spatiu Lorentz.*

Vectorii tangenti sunt definiti exact ca in cazul spatiului euclidian $\mathbf{R}^3$.

In cazul nostru distingem trei caractere cauzale ale unui vector X:

– *spatial,* daca $\langle X, X \rangle_1 > 0$ sau $X = 0$;

– *temporal* daca $\langle X, X \rangle_1 < 0$;

– *luminos* sau *izotrop* sau *nul*, daca $\langle X, Y \rangle_1 = 0$, dar $X \neq 0$.

Multimea tuturor vectorilor nuli din $\mathbf{R}^3_1$ formeaza conul luminos, descris in coordonate prin

$$\left\{(x_1, x_2, x_3) / x_1^2 = x_2^2 + x_3^2, x_1 \neq 0\right\}.$$

In $\mathbf{R}_1^3$ regulile de calcul raman aceleasi ca si in spatiul euclidian $\mathbf{R}^3$, astfel putem vorbi despre imersii sau curbe regulate la fel ca in cazul euclidian.

O curba regulata $c : \mathbf{I} \to \mathbf{R}_1^3$ se numeste:

– *spatiala*, daca $\langle c', c' \rangle_1 > 0$

– *temporala*, daca $\langle c', c' \rangle_1 < 0$

– *luminoasa* sau *izotropa* sau *nula*, daca $\langle c', c' \rangle_1 = 0$.

Se dau trei exemple de astfel de curbe: hiperbolele $x_1^2 = x_2^2 + 1, x_3 = 0$ (care este spatiala) si $x_1^2 = x_2^2 - 1, x_3 = 0$ (care este temporala), precum si dreapta $c(t) = (t, t, 0)$ care este nula, inclusa in conul luminos cu exceptia punctului $t = 0$.

Aceasta sectiune se incheie cu ecuatiile lui Frenet in spatiul Minkowski de dimensiune 3.
medskip

Teorema 3.6 (**Ecuatiile lui Frenet in spatiul Minkowski).** *Fie C o curba spatiala sau temporala care se presupune ca este parametrizata prin lungimea de arc si care satisface conditia$\langle c'', c'' \rangle_1 \neq 0$. Atunci aceasta curba induce un 3-reper Frenet*

$$e_1 = c', \; e_2 = \frac{c''}{\sqrt{|\langle c'', c'' \rangle_1|}}, \; e_3 = e_1 \times e_2$$

pentru care au loc urmatoarele ecuatii de tip Frenet:

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa\eta & 0 \\ -\kappa\varepsilon & 0 & -\tau\varepsilon\eta \\ 0 & -\tau\eta & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

unde $\varepsilon = \langle e_1, e_1 \rangle_1$ and $\eta = \langle e_2, e_2 \rangle_1$.

Ultima parte a acestui capitol este un rezultat original al autorului.

Am determinat relatiile dintre curbura si torsiunea unei curbe pentru care normalele principale coincid cu binormalele unei alte curbe din spatiul Minkowski $\mathbf{E}_1^3$.

Mai precis, am demonstrat cu urmatorul rezultat.

Teorema 3.7. *Fie c si c^* doua curbe, astfel incat normalele lui c coincid cu binormalele lui c^*. Notam prin κ si τ curbura si torsiunea curbei c. Atunci:*

I. *Daca c este o curba spatiala, rezulta:*

1. $\kappa = \alpha\left(\kappa^2 - \tau^2\right)$ *daca c" este spatiala;*

2. $\kappa = -\alpha\left(\kappa^2 + \tau^2\right)$ *daca c" ste temporala unde* $\alpha \neq 0$*;*

3. *Daca c" este nula singurele curbe care indeplinesc conditia din enunt sunt dreptele.*

II. *Daca c este o curba temporala, atunci* $\kappa = \alpha\left(-\kappa^2 + \tau^2\right)$.

III. *Daca c este o curba nula, atunci c este o dreapta, pentru $\kappa = 0$, sau $\tau = -\frac{1}{\alpha}$ pentru $\kappa = 1$.*

Demonstratia acestei teoreme este impartita in trei cazuri, conform celor trei caractere cauzale al curbei.

Acest rezultat face subiectul articolului [49] *A new type of curves in the Minkowski* 3*-space*, prezentat sub forma unui poster la Conferinta Internationala "Riemannian Geometry and Applications–RIGA 2011".

Capitolul 4 se numeste **"Campuri de versori de-a lungul unei curbe".**

Geometria campurilor de versori de-a lungul unei curbe este o generalizare a teoriei uzuale a curbelor, in care campul de versori nu este altceva decat campul de versori tangenti la curba data.

Campurile de versori de-a lungul unei curbe au fost studiate in cazul euclidian trei dimensional de Radu Miron, in [77] *Geometria configuratiilor Myller*, recent imbunatatita si tradusa in limba engleza [78].

In prima parte a capitolului autorul introduce familii de directii orientate si reperul Frenet al unei familii (C, ξ_1), intr-un punct $P\left(r\left(s\right)\right)$ din spatiul euclidian. Apoi se stabilesc formulele Frenet corespunzatoare familiei (C, ξ_1).

Prima sectiune se incheie cu patru teoreme remarcabile ale geometriei familiilor de directii din spatiul euclidian trei dimensional.

Teorema 4.3. *Unei familii de directii (C, ξ_1) regulata de clasa C^n putem sa-i asociem in orice punct P un unic reper Frenet $R\left(P; \xi_1, \xi_2, \xi_3\right)$ si un unic sistem de invarianti $K_1(s), K_2(s), a_1(s), a_2(s), a_3(s)$, unde $a_1^2 + a_2^2 + a_3^2 = 1$ care satisfac ecuatiile fundamentale.*

Teorema 4.4 (Teorema fundamentala a geometriei familiilor de directii). *Fie $K_1(s), K_2(s), a_1(s), a_2(s), a_3(s)$ functii continue pe intervalul $[s_1, s_2]$ satisfacand conditiile $K_1(s) > 0$ si $a_1^2 + a_2^2 + a_3^2 = 1$. Consideram, de asemenea, un punct $P_0\left(r_0\right)$ si un triplet ordonat, ortonormat, pozitiv orientat $\left(\xi_1^0, \xi_2^0, \xi_3^0\right)$. In aceste conditii, exista o unica familie de directii (C, ξ_1) care verifica relatiile $r(s_0) = r_0, \xi_k\left(s_0\right) = \xi_k, s_0 \in \left(s_1, s_2\right), (k = 1, 2, 3)$, unde s este parametrul canonic pe curba C si care admite ca invarianti functiile $K_1(s), K_2(s), a_1(s), a_2(s), a_3(s)$ date.*

Teorema 4.5. *O familie de directii (C, ξ_1) este determinata de invariantii sai modulo o izometrie a spatiului $\mathbf{E}^3$.*

Teorema 4.6. *Fie $K_1(s) > 0, K_2(s)$ functii continue de lungimea de arc s a curbei $C\left(r = r\left(s\right)\right), s \in \left[s_1, s_2\right]$, regulata de clasa cel putin 1 si un triplet ordonat, ortonormat, pozitiv orientat $\left(\xi_1^0, \xi_2^0, \xi_3^0\right)$ asociat unui punct P_0 de pe curba C. Atunci, exista o familie de directii (ξ_1) definita pe curba C care admite drept curbura si torsiune functiile $K_1(s), K_2(s)$ si satisface $\xi_k\left(s_0\right) = \xi_k^0, s_0 \in \left(s_1, s_2\right), (k = 1, 2, 3)$.*

In urmatorul sub-capitol, autorul a extins formulele lui Frenet la campuri de versori in spatii Lorentz de dimensiuni $n = 3$ si respectiv $n = 4$.

Este evident faptul ca formule similare de tip Frenet au loc pentru orice dimensiune n. In cazul spatiului Lorentz trei dimensional, am studiat separat cele trei caractere cauzale ale lui $\xi = \xi_1$ si am obtinut formulele lui Frenet in fiecare caz. Am demonstrat urmatoarea teorema fundamentala:

Teorema 4.7. *Daca sunt date functiile* $K_1(s) > 0, K_2(s), a_1(s), a_2(s), a_3(s)$, $-a_1^2+a_2^2+a_3^2 = \pm 1$, *de clasa* C^∞ *cu* $s \in [a,b]$, *atunci exista o curba* $C : [a,b] \to \mathbf{E}_1^3$ *parametrizata prin lungimea de arc si un camp de versori* $\xi(s)$, $s \in [a,b]$, *pentru care curbura, torsiunea si functiile* $a_i(s)$ *sunt* $K_1(s), K_2(s)$ *si* $a_i(s)$. *Orice doua astfel de campuri de versori* (C,ξ) *difera printr-o izometrie Lorentz.*

Urmatoarea sectiune se ocupa cu studiul campurilor de versori intr-un spatiu Lorentz de dimensiune patru. In conformitate cu caracterele cauzale ale lui $\xi = \xi_1$, consideram diferite cazuri. Am obtinut, de asemenea, urmatoarea teorema de caracterizare:

Teorema 4.8 (Teorema fundamentala). *Fie* $a_1, a_2, a_3, K_1, K_2, K_3 : [0,L] \to \mathbf{R}$ *functii diferentiabile reale de clasa* C^k, $k \geq 1$. *Atunci, exista campuri de versori* (C,ξ) *pentru care* K_1, K_2, K_3 *sunt curburile si* a_1, a_2, a_3 *sunt componentele lui* ξ *in reperul Frenet asociat campurilor de versori* ξ.

Aceste rezultate se gasesc in articolul autorului [48] "Versor fields along a curve in a four dimensional Lorentz space".

Capitolul 5, intitulat **"Suprafete produs tensorial a doua curbe pseudo-euclidiene",** este dedicat studiului geometriei suprafetelor produs tensorial a doua curbe pseudo-euclidiene. In prima parte sunt reamintite notiuni fundamentale asupra subvarietatilor pseudo-riemanniene folosind monografia bine cunoscuta a lui B. O'Neill: conexiunea Levi-Civita, formulele Gauss si Weingarten, a doua forma fundamentala a unei subvarietati M si campul vectorial curbura medie H.

Istoria studiului produsului tensorial a doua subvarietati ca subiect de cercetare este schitata si documentata cu multe referinte si rezultate in [75]. La baza studiului produsului tensorial a doua subvarietati sta notiunea de a doua imersie standard f a unei unei sfere m-dimensionale.

De asemenea, vrem sa mentionam ca reprezentarea patratica a unei subvarietati M intr-un spatiu euclidian, introdusa de B.Y.Chen si I.Dimitric, este un exemplu natural de imersie produs tensorial.

Pornind de la aceasta notiune de produs tensorial a doua imersii ale unei varietati diferentiabile, produsul tensorial a doua imersii ale doua varietati diferentiabile diferite a fost studiat de diferiti autori: F.Decruyenaere, F.Dillen, W.Goemans, I. Mihai, B. Rouxel, L. Verstraelen, L. Vrancken, etc.

Pentru doua varietati diferentiabile M si N si imersiile

$$f : M \to \mathbf{E}^m, p \to (f_1(p), ..., f_m(p))$$

si

$$h : N \to \mathbf{E}^n, q \to (h_1(q), ..., h_n(q)),$$

suma directa este definita prin

$$f \oplus h : M \times N \to \mathbf{E}^{m+n},$$

$$(p,q) \mapsto (f(p), h(q)) = (f_1(p), ..., f_m(p), h_1(q), ..., h_n(q)),$$

iar aplicatia produs tensorial prin

$$f \otimes h : M \times N \to \mathbf{E}^{mn},$$

$$(p,q) \mapsto f(p) \otimes h(q) = (f_1(p)\, h_1(q), ..., f_1(p)\, h_n(q), ..., f_m(p)\, h_n(q))$$

S-a demonstrat ca, in anumite conditii, produsul tensorial a doua imersii ale doua varietati diferentiabile realizeaza o imersie a varietatii produs.

Exemple interesante de imersii produs tensorial sunt suprafetele de rotatie ale lui Vranceanu, parametrizate prin

$$x(u,v) = (r(u)\cos u \cos v, r(u)\cos u \sin v, r(u)\sin u \cos v, r(u)\sin u \sin v)$$

sau, exprimate ca produs tensorial, prin

$$x(u,v) = (r(u)\cos u, r(u)\sin u) \otimes (\cos v, \sin v).$$

Mai precis, aceste imersii produs tensorial sunt exemple specifice de produse tensoriale a doua curbe, adica suprafete produs tensorial. In cateva articole au fost studiate diverse conditii pentru curbura si alte caracterizari ale suprafetelor produs tensorial. Se cunosc diverse rezultate pentru suprafetele produs tensorial a doua curbe plane. De exemplu, in [71] au fost studiate suprafetele produs tensorial minimale, total reale, complexe, oblice si pseudo-ombilicale a doua curbe plane euclidiene. O clasificare a suprafetelor produs tensorial minimale, total reale si pseudo-minimale a doua curbe plane lorentziene este data in [74]. Suprafetele produs tensorial minimale si pseudo-minimale dintre o curba plana Lorentz si o curba plana euclidiana sunt prezentate in [73]. In [2] sunt clasificate suprafetele produs tensorial minimale a doua curbe plane semi-euclidiene. O abordare diferita in examinarea suprafetelor produs tensorial a doua curbe plane euclidiene este data in [72]. Se arata ca proprietatile conicei lui Kommerell caracterizeaza suprafete produs tensorial speciale in $\mathbf{E}^4$.

Recent, suprafetele produs tensorial minimale a doua curbe pseudo-euclidiene arbitrare au fost clasificate de W. Goemans, I. Van de Woestyne and L. Vrancken [37], generalizand rezultatele particulare mentionate mai sus. Reamintim aceasta teorema de clasificare in sectiunea 5.2.

In urmatoarea sectiune sunt prezentate unele rezultate din geometria suprafetelor produs tensorial a doua curbe plane euclidiene, bazate pe lucrarile geometrilor I. Mihai, R. Rosca, B. Rouxel, I. Van de Woestyne, L. Verstaelen, J. Walrave, etc..

Ultimele sectiuni contin studii originale ale autorului.

In prima parte sunt investigate suprafetele produs tensorial pseudo-minimale a doua curbe pseudo-euclidiene. Deoarece cazul general este foarte complicat, consideram produsul tensorial dintre o curba in spatiul Lorentz $\mathbf{E}_1^3$ si o curba in planul euclidian $\mathbf{E}^2$. Am obtinut ecuatiile diferentiale pentru astfel de curbe al caror produs tensorial este pseudo-minimal si am evidentiat cazuri particulare importante. Aceste ecuatii fiind foarte dificile nu au putut fi rezolvate pana in prezent. Totusi, am dat un exemplu de suprafata produs tensorial non-minimala pseudo-minimala $c_1 \otimes c_2$, unde

$$c_1(s) = ke^{lt}(ch\ s, 0, sh\ s), k \in \mathbf{R}^*, l > 1,$$

$$c_2(t) = ae^{\sqrt{l^2-1}t}(\cos t, \sin t), a > 0,$$

adica c_1 este o spirala logaritmica lorentziana plana si respectiv c_2 este o spirala logaritmica.

In ultima sectiune am determinat suprafetele produs tensorial plate a doua curbe pseudo-euclidiene arbitrare. In [38] autorii au clasificat suprafetele produs tensorial plate a doua curbe pseudo-euclidiene plane.

Utilizand formula lui Brioschi ([70], [38]), curbura Gauss a unei suprafete pseudo-riemanniene parametrizate prin $f(s,t)$ este data de

$$K(s,t) = \frac{1}{(EG-F^2)^2} \times$$

$$\left\{ \begin{vmatrix} -\frac{1}{2}E_{tt} + F_{st} - \frac{1}{2}G_{ss} & \frac{1}{2}E_s & -\frac{1}{2}E_t + F_s \\ F_t - \frac{1}{2}G_s & E & F \\ \frac{1}{2}G_t & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_t & \frac{1}{2}G_s \\ \frac{1}{2}E_t & E & F \\ \frac{1}{2}G_s & F & G \end{vmatrix} \right\},$$

unde $E(s,t)$, $F(s,t)$ si $G(s,t)$ sunt componentele metricii induse pe suprafata.

Atunci platitudinea unei suprafete produs tensorial a doua curbe pseudo-euclidiene c_1 si c_2 este echivalenta cu

$$\begin{aligned} &g_1(\dot{c_1}, \dot{c_1}) \left\{ g_1(c_1, \dot{c_1})^2 - g_1(c_1, c_1) g_1(\dot{c_1}, \dot{c_1}) \right\} \\ &g_2(\dot{c_2}, \dot{c_2}) \left\{ g_2(c_2, \dot{c_2})^2 - g_2(c_2, c_2) g_2(\dot{c_2}, \dot{c_2}) \right\} = \\ &= g_1(c_1, c_1) \left\{ g_1(\dot{c_1}, \dot{c_1}) g_1(c_1, \ddot{c_1}) - g_1(c_1, \dot{c_1}) g_1(\dot{c_1}, \ddot{c_1}) \right\} \\ &g_2(c_2, c_2) \left\{ g_2(\dot{c_2}, \dot{c_2}) g_2(c_2, \ddot{c_2}) - g_2(c_2, \dot{c_2}) g_2(\dot{c_2}, \ddot{c_2}) \right\}. \end{aligned} \tag{5.5}$$

Mentionam doua cazuri speciale de suprafete produs tensorial plate.

i) Presupunem ca una dintre curbele c_1 sau c_2 este o dreapta care trece prin origine. Atunci suprafata produs tensorial $f(s,t) = c_1(s) \otimes c_2(t)$ are curbura Gauss nula si, prin urmare, este plata.

ii) Presupunem ca una dintre curbele c_1 sau c_2 este o curba nula. De exemplu, daca curba c_1 satisface $g_1(\dot{c_1}, \dot{c_1}) = 0$, atunci $g_1(\dot{c_1}, \ddot{c_1}) = 0$; deci ecuatia (5.5) este satisfacuta. Asadar, suprafata produs tensorial $f(s,t) = c_1(s) \otimes c_2(t)$ este plata daca una dintre curbele c_1 sau c_2 este o curba nula.

Vom considera cazul general, adica c_1(respectiv c_2) este o curba in spatiul pseudo-euclidian $\mathbf{E}^m_\mu$ $\left(\text{respectiv } \mathbf{E}^n_\gamma\right)$. Ecuatia (5.5) se poate scrie sub forma

$$\frac{g_1(\dot{c_1}, \dot{c_1}) \left\{ g_1(c_1, \dot{c_1})^2 - g_1(c_1, c_1) g_1(\dot{c_1}, \dot{c_1}) \right\}}{g_1(c_1, c_1) \left\{ g_1(\dot{c_1}, \dot{c_1}) g_1(c_1, \ddot{c_1}) - g_1(c_1, \dot{c_1}) g_1(\dot{c_1}, \ddot{c_1}) \right\}} = \tag{5.6}$$

$$\frac{g_2(c_2,c_2)\left\{g_2(\dot{c_2},\dot{c_2})g_2(c_2,\ddot{c_2})-g_2(c_2,\dot{c_2})g_2(\dot{c_2},\ddot{c_2})\right\}}{g_2(\dot{c_2},\dot{c_2})\left\{g_2(c_2,\dot{c_2})^2-g_2(c_2,c_2)g_2(\dot{c_2},\dot{c_2})\right\}}.$$

Membrul stang depinde numai de c_1 si membrul drept depinde numai de c_2. Rezulta ca ambii membri sunt egali cu o constanta, notata cu k, adica

$$\frac{g_1(\dot{c_1},\dot{c_1})\left\{g_1(c_1,\dot{c_1})^2-g_1(c_1,c_1)g_1(\dot{c_1},\dot{c_1})\right\}}{g_1(c_1,c_1)\left\{g_1(\dot{c_1},\dot{c_1})g_1(c_1,\ddot{c_1})-g_1(c_1,\dot{c_1})g_1(\dot{c_1},\ddot{c_1})\right\}}=k, \tag{5.7}$$

$$\frac{g_2(c_2,c_2)\left\{g_2(\dot{c_2},\dot{c_2})g_2(c_2,\ddot{c_2})-g_2(c_2,\dot{c_2})g_2(\dot{c_2},\ddot{c_2})\right\}}{g_2(\dot{c_2},\dot{c_2})\left\{g_2(c_2,\dot{c_2})^2-g_2(c_2,c_2)g_2(\dot{c_2},\dot{c_2})\right\}}=k. \tag{5.8}$$

Notam $\langle c_i, c_i\rangle = r_i$, $i=1,2$.

Vom rezolva ecuatiile de mai sus in raport cu caracterele cauzale ale curbelor c_1 si respectiv c_2.

I. i) c_1 este spatiala, adica $g_1(\dot{c_1},\dot{c_1})=1$.

Pentru rezolvarea ecuatiei (5.7) definim o functie p_1 prin

$$p(y)=\int_{y_0}^{y}\frac{dy}{\sqrt{4y+cy^{\frac{1}{k}}}},\ c\in\mathbf{R}.$$

Atunci ecuatia (5.7) este satisfacuta daca si numai daca r_1 este inversa functiei $p_1(y)=x$.

ii) c_1 este temporala, adica $g_1(c_1,c_1)=-1$.

In acest caz, ecuatia (5.7) este echivalenta cu faptul ca r_1 este inversa functiei $p_2(y)=x$, unde

$$p_2(y)=\int_{y_0}^{y}\frac{dy}{\sqrt{4y+cy^{-\frac{1}{k}}}},\ c\in\mathbf{R}.$$

Analog, in raport cu caracterul cauzal al curbei c_2 avem urmatoarele cazuri:

II. i) c_2 este spatiala, adica $g_2(\dot{c_2},\dot{c_2})=1$.

Ecuatia (5.8) este echivalenta cu faptul ca r_2 este inversa functiei $p_3(y)=x$, cu

$$p_3(y)=\int_{y_0}^{y}\frac{dy}{\sqrt{4y+cy^{k}}},\ c\in\mathbf{R}.$$

ii) c_2 este temporala, adica $g_2(\dot{c_2},\dot{c_2})=-1$.

In acest caz, solutia lui (5.8) este data de r_2 este inversa functiei $p_4(y)=x$, unde

$$p_4(y)=\int_{y_0}^{y}\frac{dy}{\sqrt{4y+cy^{-k}}},\ c\in\mathbf{R}.$$

La sfarsitul tezei, vom da o bibliografie detaliata care cuprinde atat referinte de baza cat si unele noi din geometria curbelor in spatii pseudo-Riemanniene.

De asemenea, trei rezultate originale ale autorului sunt mentionate: primul a fost publicat intr-o revista internationala, a doua lucrare a fost publicata in Proceedings-ul Conferintei Internationale *"Riemannian Geometry and Applications – RIGA 2011"* (bazata pe un poster pe care l-am prezentat la aceasta conferinta), iar al treilea este un preprint, care va fi prezentat pentru publicare foarte curand. Bibliografia include 115 de referinte.

Rezumand, am demonstrat urmatoarea teorema de clasificare.

Teorema 5.13 [70]. *Fie $c_1 : \mathbf{R} \to \mathbf{E}^m_\mu$ si $c_2 : \mathbf{R} \to \mathbf{E}^n_\nu$ doua curbe pseudo-euclidiene. Atunci suprafata produs tensorial $c_1 \otimes c_2 : \mathbf{R}^2 \to \mathbf{E}^{mn}_\rho$, unde $\rho = \mu(n-\nu) + \nu(m-\mu$, este plata daca si numai daca:*

i) *Una din curbele c_1 sau c_2 este o dreapta ce trece prin origine.*

ii) *Una din curbele c_1 sau c_2 este o curba nula.*

iii)

a) *c_1 este o curba spatiala si $r_1 = g_1(c_1, c_1)$ este inversa functiei*

$$p_1(y) = \int_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{\frac{1}{k}}}}, \ c \in \mathbf{R},$$

sau

c_1 este o curba temporala si $r_1 = g_1(c_1, c_1)$ este inversa functiei

$$p_2(y) = \int_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{-\frac{1}{k}}}}, \ c \in \mathbf{R},$$

si

b) *c_2 este o curba spatiala si $r_2 = g_2(c_2, c_2)$ este inversa functiei*

$$p_3(y) = \int_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{k}}}, \ c \in \mathbf{R},$$

sau

c_2 este o curba temporala si $r_2 = g_2(c_2, c_2)$ este inversa functiei

$$p_4(y) = \int_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{-k}}}, \ c \in \mathbf{R}.$$

Observam ca platitudinea depinde numai de functiile $r_1(s)$ si $r_2(t)$ si nu depinde de toate componentele curbelor $c_1(s)$ si $c_2(t)$.

Teza se incheie cu o bibliografie detaliata care cuprinde atat referinte fundamentale, cat si referinte recente din geometria curbelor in spatii pseudo-Riemanniene.

In particular, sunt incluse trei articole originale ale autorului: primul articol [48] a fost publicat intr-o revista internationala, a doua lucrare a fost publicata in Proceedings-ul Conferintei Internationale *"Riemannian Geometry and Applications – RIGA* 2011" (bazata pe un poster pe care l-am prezentat la aceasta conferinta), iar al treilea articol este un preprint, care va fi trimis spre publicare foarte curand. Bibliografia include 115 referinte.

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