PREFACE

The object of Modular Representation Theory is to investigate the relations between representations of a finite group G in characteristic zero and in positive characteristic p. This is done via the study of the group algebra $\mathcal{O}G$ of G over a complete discrete valuation ring $\mathcal{O}$ having a residue field k of characteristic p and a quotient field $\mathcal{K}$ of characteristic zero. The link between representations of G over $\mathcal{K}$ and over k is then established by the functors "reduction modulo p" $k \otimes_{\mathcal{O}} -$ and "extension of scalars" $\mathcal{K} \otimes_{\mathcal{O}} -$.

Chapter I is intended to be an up-to-date introduction to the subject. After introducing group algebras and twisted group algebras in the first section, we present the needed facts about algebras over fields and complete local rings: lifting idempotents, Morita equivalences, self-injective and symmetric algebras.

The particular way in which the group algebra is constructed leads to specific constructions and results such as the Frobenius reciprocity, Mackey's formula, Higman's criterion on relative projectivity, Green's theory of vertices and sources, and the Green correspondence.

The group algebra $\mathcal{O}G$ need not be indecomposable as an algebra in general. It turns out to be useful to decompose $\mathcal{O}G$ into a direct sum of indecomposable ideals and study the structure of each of these summands separately. Such a direct summand is called a block of $\mathcal{O}G$. We introduce in sections 1.12 and 1.13 some fundamental concepts from block theory, emphasizing that the typical objects that we have to deal with are pairs consisting of a finite group and a conjugacy class of primitive idempotents in some subalgebra of fixed points. Puig's concept of pointed groups is a systematic approach to block theory, and this way of looking at modular representation theory, gives rise to the notion of source algebras of blocks. The local structure of a block can be expressed in terms of Brauer pairs or pointed groups. Both the Green correspondence and the Brauer correspondence, described in section 1.11 as correspondences between isomorphism classes of modules can be rephrased as correspondences between pointed groups.

Dealing with twisted group algebras leads to the consideration of a generalization of p-permutation modules, the linear source modules introduced by R. Boltje. The Brauer construction works well when applied to such modules, which are involved in the definition of the splendid equivalence.

We also develop some tools of local representation theory of twisted group algebras. Such algebras occur naturally when we investigate relations between representations of normal subgroups and representations of the whole group. Dade has shown that a good framework to discuss this topic is that of group graded algebras.

We start Chapter II by formulating some of the most important conjectures in the field. Most of them come from old problems posed by R. Brauer, which led to the more recent conjectures of Alperin, Dade and Donovan. We also give a short account on what Puig considers to be the *fundamental problem in block theory* - the determination of the structure of the source algebras of blocks of finite groups with a given defect group.

There are many examples of very similar block algebras that are not Morita equivalent. This motivated the discussion of weaker types of equivalences. For instance, it happens frequently that the module category of a block algebra can be determined only up to projectives, that is, up to stable equivalences, and a difficult problem is to to reconstruct the algebra structure from its stable category.

We also give an introduction to the subject of derived categories and equivalences, and we emphasize their applications to the representation theory of groups. Section 2.3 deals with triangulated categories; in particular we study the homotopy category of complexes. Section 2.4 contains the definition of derived categories and in Section 2.5 we discuss tilting theory and the important results of J. Rickard.

Derived equivalences compatible with the local structure of blocks of group algebra were studied in full generality by L. Puig. Here we discuss the more explicit approach to splendid equivalences between arbitrary blocks of group algebras given by M. Linckelmann.

In the last three sections we study graded stable Morita equivalences and graded Rickard equivalences between blocks of twisted group algebras and their local structure. In section 2.7 we state generalized versions of Broué's abelian defect group conjecture and discuss their connections with Dade's conjectures, which come from the fact that Rickard equivalences induced by complexes of graded bimodules preserve the relevant Clifford theoretical invariants. As an application, we show, by generalizing Rouquier's construction of a splendid Rickard equivalence, that the extended form of Broué's conjecture and Dade's Inductive Conjecture holds for blocks with with cyclic defect groups.

This book grew out of lectures given by the author over several years. Most of the material cannot be found in a single source. For the material which is standard we refer the reader to the books of D. Benson and J. Thévenaz. Nevertheless, we include most of the proofs or sketches of proofs, and several proofs are left as exercises. The book should be accessible to students with a good background in general algebra and some basic knowledge in module theory, category theory and homological algebra.