INTRODUCTION

An algebraic function y of a complex variable x is a function which satisfies an equation of the form $F(x, y) = 0$, where F is a polynomial with complex coefficients; i.e., y is a root of an algebraic equation whose coefficients are rational functions of x. This very definition exhibits a strong similarity between the notions of algebraic function and algebraic number, the rational functions of x playing a role similar to that played by the rational numbers. On the other hand, the equation $F(x, y) = 0$ may be construed to represent a curve in a plane in which x and y are the coordinates, and this establishes an intimate link between the theory of algebraic functions of one variable and algebraic geometry.

Whoever wants to give an exposition of the theory of algebraic functions of one variable is more or less bound to lay more emphasis either on the algebraico-arithmetic aspect of this branch of mathematics or on its geometric aspect. Both points of view are acceptable and have been in fact held by various mathematicians. The algebraic attitude was first distinctly asserted in the paper *Theorie der algebraischen Funktionen einer Veränderlichen*, by R. Dedekind and H. Weber (Journ. für Math., 92, 1882, pp. 181–290), and inspires the book *Theorie der algebraischen Funktionen einer Variabeln*, by Hensel and Landsberg (Leipzig, 1902). The geometric approach was followed by Max Noether, Clebsch, Gordan, and, after them, by the geometers of the Italian school (cf. in particular the book *Lezione di Geometria algebrica*, by F. Severi, Padova, 1908). Whichever method is adopted, the main results to be established are of course essentially the same; but this common material is made to reflect a different light when treated by differently minded mathematicians. Familiar as we are with the idea that the pair "observed fact—observer" is probably a more real being than the inert fact or theorem by itself, we shall not neglect the diversity of these various angles under which a theory may be photographed. Such a neglect should be particularly avoided in the case of the theory of algebraic functions, as either mode of approach seems liable to provoke strong emotional reactions in mathematical minds, ranging from devout enthusiasm to unconditional rejection. However, this does not mean that the ideal should consist in a mixture or synthesis of the two attitudes in the writing of any one book: the only result of trying to obtain two interesting photographs of the same object on the same plate is a blurred and dull image. Thus, without attacking in any way the validity *per se* of the geometric approach, we have not tried to hide our partiality to the algebraic attitude, which has been ours in writing this book.

The main difference between the present treatment of the theory and the one to be found in Dedekind-Weber or in Hensel-Landsberg lies in the fact that the constants of the fields of algebraic functions to be considered are not necessarily the complex numbers, but the elements of a completely arbitrary field. There are several reasons which make such a generalization necessary. First, the analogy between algebraic functions and algebraic numbers becomes even closer if one considers algebraic functions over finite fields of constants. In that case, on the one hand class field theory has been extended to the case of fields of functions, and, on the other hand, the transcendental theory (zeta function, L-series) may also be generalized (cf. the paper of F. K. Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p*, Math. Zeits., 33, 1931). Moreover, A. Weil has succeeded in proving the Riemann hypothesis for fields of algebraic functions over finite fields, thereby throwing an entirely new light on the classical, i.e., number theoretic, case (cf. the book of A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*, Paris, Hermann, 1948; this book contains an exposition of the theory from a geometric point of view, although this point of view is rather different from that of the Italian geometers). Secondly, if S is an algebraic surface, and R the field of rational functions on S, then R is a field of algebraic functions of one variable over $K\langle x\rangle$, where K is the basic field and x a non constant element of R. E. Picard, among others, has very successfully used the method of investigation of S which amounts to studying the relationship between R and various fields of the form $K\langle x\rangle$ (cf. E. Picard and G. Simart, *Théorie des fonctions algébriques de deux variables independantes*, Paris, Gauthier-Villars, 1897). Now, even when K is the field of complex numbers, $K\langle x\rangle$ is not algebraically closed, which makes it necessary to have a theory of fields of algebraic functions of one variable over fields which are not algebraically closed.

The theory of algebraic functions of one variable over non algebraically closed fields of arbitrary characteristic has been first developed by H. Hasse, who defined for these fields the notion of a differential (H. Hasse, *Theorie der Differentiale in algebraischen Funktionenkörpern mit volkommenen Konstantenkörper*, Journ. für Math., 172, 1934, pp. 55–64), and by F. K. Schmidt, who proved the Riemann-Roch theorem (F. K. Schmidt, *Zur arithmetischen Theorie der algebraischen Funktionen*, I, Math. Zeits., 41, 1936, p. 415). In this book, we have used the definition of differentials and the proof of the Riemann-Roch theorem which were given by A. Weil (A. Weil, *Zur algebraischen Theorie der algebraischen Funktionen*, Journ. für Math., 179, 1938, pp. 129–133).

As for contents, we have included only the elementary part of the theory, leaving out the more advanced parts such as class field theory or the theory of correspondences. However, we have been guided by the desire of furnishing a suitable base of knowledge for the study of these more advanced chapters. This is why we have placed much emphasis on the theory of extensions of fields of algebraic functions of one variable, and in particular of those extensions which are obtained by adjoining new constants, which may even be transcendental over the field of constants of the original field of functions. That the consideration of such extensions is desirable is evidenced by the paper of M. Deuring, *Arithmetische Theorie der Korrespondenzen algebraischen Funktionenkörper*, Journ. für Math., 177, 1937. The theory of differentials of the second kind has been given only in the case where the field under consideration is of characteristic 0. The reason for this restriction is that it is not yet clear what the "good" definition of the notion should be in the general case: should one demand only that the residues be all zero, or should one insist that the differential may be approximated as closely as one wants at any given place by exact differentials (or a suitable generalization of these)? Here is a net of problems which, it seems, would deserve some original research. The last chapter of the book is concerned with the theory of fields of algebraic functions of one variable over the field of complex numbers and their Riemann surfaces. The scissor and glue method of approach to the idea of a Riemann surface has been replaced by a more abstract definition, inspired by the one given by H. Weyl in his book on Riemann surfaces, which does not necessitate the artificial selecting of a particular generation of the field by means of an independant variable and a function of this variable. We have also avoided the cumbersome decomposition of the Riemann surface into triangles, this by making use of the singular homology theory, as developed by S. Eilenberg.

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