Introduction

During the course of the past few years, it became evident that the purely algebraic concept of homological dimension was closely related to the set theoretic foundations of mathematics. The classical uses of various homological dimensions in ring theory were in the study of commutative noetherian rings and finitely generated modules over them—the commutative algebra arising from algebraic geometry. The outstanding result obtained by these methods was the theorem that a regular local ring is a unique factorization domain, a proof of which (due to I. Kaplansky) is in these notes, Chapter 1, §3 and Chapter 2, §4. However, once finiteness conditions such as noetherian rings or finitely generated modules were dropped, entirely different phenomena occurred. Collected here are some of these. For example, if **R** denotes the real numbers, the projective dimension of $\mathbf{R}(x_1, x_2, x_3)$ as an $\mathbf{R}[x_1, x_2, x_3]$–module is $2 \iff$ the continuum hypothesis holds. And if V is a countable dimensional vector space over **R**, the global dimension of $\mathrm{Hom}_{\mathbf{R}}(V, V) = k+1 \iff 2^{\aleph_0} = \aleph_k$. Using the same techniques for modules over small additive categories, B. Mitchell obtained similar results on the vanishing of $\varprojlim^{(k)}$. His attack is sketched here.

Because set theoretic manipulations obviously play an important role in obtaining such results, an appendix on elementary set theory is included. For those to whom the axiom of choice and cardinal and ordinal arithmetic are mysterious things they know about but still don't really understand, the appendix may not clear up the mystery but it will give the results necessary in a reasonably short space.

These notes were prepared for a series of ten lectures given at the American University June 20–25, 1971. The bulk of the lectures were on projective dimensions of "very large" modules as given in Chapter 2. Chapter 1 and the appendix were included for reference and in general only referred to in passing or in private conversations. The material on flat modules (Theorems 1.29 to 1.34) seemed to be referred to most frequently although other portions of these presumably familiar sections were of use to some people in attendance. Some, such as §3 of Chapter 1, were incorporated into the talks. Since there seemed to be a feeling that having basic results and definitions readily at hand was of value, the purely background listing of Chapter 1 and the Appendix were left in the final form of the notes.

Although the material in these notes is not new, there are several places where existing work has been simplified. For example, a commutative local nondomain of global dimension 3 is described without reference to analysis, and the dimension of a quotient field of a polynomial ring rather than a regular local ring is calculated. A derivation of Tor one step at a time without the usual derived functor machinery is included in Chapter

2, §2. The author is grateful to A. Zaks for pointing out this approach.

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