Preface

The aim of this book is to present in a systematic manner a collection of relatively recent concepts and results that could be *summarized* in „theoretical algorithms" for solving, in the framework of Dynamic Programming, not only simple „class-room" examples but also more complicated (even some „real-life") *optimal control problems;* in the later case, the computer results (mainly on numerical integration of certain „constrained" differential systems) should be accompanied and supported by solid theoretical arguments to decide the optimality and the real nature of the solution.

In particular, the content of this book is intended to fill the (unfortunately) ever increasing gap between the tremendous development of the „theory" of optimal control problems and, on the other side, the rather „heuristical procedures" used lately by many authors to „solve" (very often using *just* „computer simulations" and very little or no theory at all) particular examples of optimal control problems.

In view of the vast experience accumulated along the years in Calculus of Variations and Optimal Control (e.g., Berkovitz (1974), Bliss (1925), (1946), Boltyansky (1964,1968), Bolza (1946), Cesari (1983), Fleming and Rishel (1975), Hestenes (1966), Lee and Markus (1967), Tonelli (1923), Young (1969), etc.), a reasonable way of using the latest developments of Dynamic Programming to solve concrete problems seems to be the following:

- *first,* compute (describe, characterize, etc.) a „feasible" selection of (possibly optimal, for instance, „extremal") trajectories which, using a generic term, may be called a „**generalized field of extremals"**;

- *secondly,* compute (describe, characterize, etc.) the corresponding **value function** of the field of extremals (i.e. the *value of the cost-functional along each trajectory of the field*) and use a suitable „verification theorem" to decide the optimality of the chosen trajectories.

In more precise terms, this approach is described in the introductory Chapter 1 which contains also detailed *formulations and classifications* of optimal control problems, several illustrative *examples*, some *basic properties of the value functions* and an ,,*outline*" *of the Dynamic Programming Method.*

In this setting, the first main ,,theoretical challenge" of Dynamic Programming consists in *finding verifiable sufficient conditions for the monotonicity along admissible trajectories of a given real function with scarce regularity properties.*

In order to obtain in Ch.3 an ,,almost comprehensive" list of such results, that are applicable to very large classes of (autonomous) optimal control problems, we present in Ch.2 the following necessary ,,tools" from Nonsmooth Analysis:

– extensions of some Calculus concepts and results on open sets and on differentiable submanifolds of $\mathbb{R}^n$ (Section 2.1), to stratified sets and mappings (Section 2.2) and to arbitrary sets and mappings (Section 2.3) where the classical derivative is replaced by ,,generalized derivatives" defined, mainly, by the *contingent and by the quasitangent cones;*

– the proofs in Section 2.4 of some very general *monotonicity theorems for real functions*, that go far beyond the classical Lebesgue's theorem for *AC* functions or the so called ,,Corollary of the Zygmund's Lemma" for continuous functions;

– some ,,*upper estimates*" in Section 2.5 of certain *sets of generalized tangent directions to the trajectories of a differential inclusion;*

– the proofs in Section 2.6 of certain general results concerning *invariant sets and monotonic functions along solutions of autonomous differential inclusions* which, in particular, ,,produce" two very general ,,verification theorems" for semicontinuous value functions;

– several estimates in Section 2.7 of the generalized (,,extreme contingent") derivatives of some types of *marginal functions.*

Using these ,,auxiliary" results, we prove in Sections 3.1–3.3 a rather large number of ,,verification theorems" (for autonomous optimal control problems) containing *sufficient optimality conditions for generalized fields of extremals* whose value functions have different types of regularity properties and satisfy corresponding differential inequalities; the multitude of verification theorems is justified by the necessity of careful choices of the ,,best" differential inequality for each type of regularity property (i.e., Lipschitzianity, continuity, semicontinuity) of the value function, since this is rarely found in explicit form and the differential inequalities are difficult to ,,check" in particular examples.

In Section 3.4 we obtain in the same way several results concerning the *set-valued optimal feedback control* in the framework of Dynamic Programming.

The second main ,,theoretical challenge" of Dynamic Programming seems to consist in *finding the most efficient ways of computing (describing, characterizing, etc.) generalized fields of extremals* to which the verification theorems in Ch.3 could easily be applied.

Chapter 4 contains in the first two sections some extensions of the classical results concerning *smooth Hamiltonian and Characteristic flows* and *Cauchy's Method of Characteristics* for autonomous Hamilton-Jacobi equations; aimed as ,,tools" to be used in a ,,piecewise" manner, in the next sections, these concepts and results suggest the introduction in Sections 4.3, 4.4, of two concepts (of ,,*stratified*" type and, respectively, of ,,contingent" type) of *generalized characteristic flows* that may be associated to an optimal control problem; next, these objects are used in a ,,finite-dimensional" minimization problem to define simultaneously both, a *generalized field of extremals and its value function.* The main advantage of this procedure is that the value function satisfies already certain differential inequalities that are very close to the ones needed in the verification theorems in Ch.3.

In Section 4.5 we analyze the apparently more natural way of describing a ,,proper" field of extremals using Pontryagin's Minimum Principle (**PMP**) as a *necessary optimality condition*; however, besides the restrictive hypotheses under which **PMP** is proved, in this case the differential inequalities needed in the verification theorems seem more difficult to check ,,directly" due to the absence of the basic properties of the characteristic flows in Sections 4.1–4.4; this shortcoming may be avoided by the proof of the fact that the ,,extremal pairs" satisfying **PMP** are solutions of a certain ,,Pontryagin-type Hamiltonian inclusion" which is contained in the more general one in Sections 4.3, 4.4.

The theoretical results in Ch.3 and Ch.4 will then be summarized in Ch.5 in the form of several ,,theoretical Dynamic Programming Algorithms" whose efficiency will be illustrated in Ch.6 on some significant examples of Calculus of Variations and Optimal Control problems.

In particular, we present complete theoretically justified solutions to the classical problems of the Brachistochrone and, respectively, to the Euler-Plateau problem on minimal surfaces of revolution; perhaps it is interesting to note here that the later problem has been studied on some 35 pages in Bliss(1925), using not only the whole ,,arsenal" of the classical theory of Calculus of Variations but also a large number of intuitive-geometric and ad-hoc arguments.

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