Preface

Integration and measure theory at the introductory level are generally not topics that excite mathematicians, and the reader may well wonder why anyone would choose to write another book on this subject. In the case of the present book the answer is very easy to give. I felt that I had a more interesting way of presenting part of the theory. I wanted to place integration theory ahead of measure theory and to adopt a functional analytic point of view.

One can teach integration and measure theory by starting with measure theory and then developing integration theory in terms of that, or vice versa. The approach that begins with measure theory has always struck me as being unnatural, whereas the other approach is easy to motivate and more natural and elegant. For example, starting with measure theory requires that one abandon the familiar Riemann integral, which is what is to be generalized, and then later working hard to reconcile it and the new Lebesgue integral. And it is necessary to begin by defining σ-algebras and measures, and early in a course this is not easy to motivate and difficult to illustrate with good examples. On the other hand, starting with integration theory means that one can use the Riemann integral to construct the Lebesgue integral and then use the Lebesgue integral to give examples of σ-algebras and measures. Moreover, putting the integrals ahead of the measures encourages one to think in terms of linear functionals and fosters a more elegant functional analytic outlook.

I must admit that, after having taken two courses on integration and measure theory as a student, I had no desire to ever teach the subject. One of the courses began with measure theory and was uninspiring because all of the theorems were expected and there were few, if any, interesting proofs, ideas, or examples. The other course began with integration theory and, at least to me, was much more interesting.

Having said this, why would I, 25 years later, teach a course and then, later, write a book on integration and measure theory? Well, I can no longer quite recall why I agreed to teach the course. But once I started thinking about what and how I wanted to teach and reflected upon my own experiences as a student, I decided not to teach a course that took measure theory as its starting point. I wanted to begin with a discussion of why it is necessary to replace the Riemann integral with either the Stieltjes or the Lebesgue integral. I wanted to give a fairly complete account of the Stieltjes integral. I wanted to develop the Lebesgue integral from the Riemann integral along the lines of the Daniell procedure. And, finally, I wanted to view as many things as possible from a functional analytic perspective. The present book is the outgrowth of notes that I wrote first for myself and then later distributed to students and, finally, enlarged sufficiently to serve as a textbook.

At our university we teach an introductory course on integration and measure theory which consists of 36 lectures. It is taken by fourth-year mathematics students, beginning graduate students in mathematics, and graduate students from economics and from some of the engineering departments. In teaching this course from one of several preliminary versions of this book, I have covered most of the material in Chapters 1, 3, 7–10, and 13–15 and parts of that in Chapters 4, 11, 12, and 16. Instructors who teach at an institution in which courses consist of more that 36 lectures can include proportionately more material.

A prospective reader who is not persuaded of the advantages of beginning the study of integration and measure theory with integration theory can still make use of this book, albeit at a certain cost. Such a reader should begin with Chapters 9–11 and then proceed to Chapter 13 and the rest of Part V. He or she might, however, have some difficulty with the motivation in Sections 13.1 and 13.2. More seriously, such a reader would have to omit Section 13.4 and would not see a definition of the Lebesgue–Stieltjes integral, σ-algebra, or measure.

This book is divided into six parts, and each of these parts is itself further subdivided into chapters. The first part deals with the reasons for studying integration theory. After all, if the Riemann integral were adequate for the needs of mathematicians and other scientists, there would be no need to study its generalizations due to Stieltjes and Lebesgue. I think that it is important for students to understand why the Riemann integral is not adequate, and the two chapters in this part discuss this from two points of view, namely, analysis and probability theory. Both analysis and probability theory need a more robust integral, the Lebesgue integral, because of problems with integrability of functions and limits of integrals. In addition, probability theory needs a more versatile integral, the Stieltjes integral, in order to calculate expectations and other quantities from the distribution function of a random variable.

The second part contains a systematic and fairly complete presentation of the Riemann–Stieltjes integral. Chapters 3 and 6 contain the basic facts and some more esoteric ones, respectively, about this integral. Chapter 4 gives a characterization of those functions that are Riemann–Stieltjes integrable. While this characterization itself will not be used in this book, the preliminary discussion of null sets and the Cantor set and function will be encountered in the later chapters. Chapter 5 is an application of the Riemann–Stieltjes integral to a problem which, in some sense, belongs to functional analysis, namely, that of describing and/or characterizing the continuous linear functionals on the space $C[a, b]$ of all continuous functions on an interval $[a, b]$. The possibility of such a description in terms of the Riemann–Stieltjes integral shows that this integral must play important role in analysis in addition to its usefulness in probability theory.

The third part contains the beginning of the study of the Lebesgue–Stieltjes integral. This integral is obtained by enlarging the domain of the Riemann–Stieltjes integral (i.e., the set of integrable functions) and extending the definition of the integral in a two-stage process. The first stage involves monotone limits and is carried out in Chapter 7, while the second uses a sandwiching procedure and is described in Chapter 8. In addition, Chapter 8 discusses the linearity and lattice properties of the Lebesgue–Stieltjes integral as well as the three convergence theorems. These convergence theorems ensure that the Lebesgue–Stieltjes integral does not suffer from the same criticisms as the Riemann integral vis-à-vis integrability and limits of integrals. Finally, Chapter 8 includes a comparison of the improper Riemann integral and the Lebesgue integral, with the conclusion being that the latter is both an extension of and far superior to the former.

The fourth part deals with measures and measurable functions. Chapter 9 introduces σ-algebras (which will be the domains of measures), Chapter 10 introduces the measurable functions, and Chapter 11 introduces the measures themselves. This material must be studied at this time because the further properties of the Lebesgue–Stieltjes integral can only be developed in terms of measures and measurable functions; this is, in fact, exactly what is done in part of Chapter 12. While some parts of Chapter 12 deal with the Lebesgue–Stieltjes measures themselves, this chapter can, and should, be viewed as the logical continuation of the study of the Lebesgue–Stieltjes integral begun in Part III.

The fifth part is by far the longest of the six parts. In Chapter 13 the so-called abstract Lebesgue integral is associated with a measure, and the properties of this integral are deduced from those of the measure. This is quite different from the way the Lebesgue-Stieltjes integral was introduced in Chapter 8 although the by-now familiar properties of the Lebesgue–Stieltjes integral are used to motivate the definition of the abstract Lebesgue integral. Chapter 14 studies the L^p-spaces in more detail than is customarily done at this stage in the development of the theory, and the reader would be well-advised to omit Sections 14.7–14.10 on a first reading. The material in Chapters 15–21 is standard, although the presentation here differs from many others in two respects. The discussion of signed measures is more complete than usual, and the convergence of sequences of functions is postponed until quite late.

In Chapter 22 the positive and the continuous linear functionals on two spaces of continuous functions on a locally compact Hausdorff space are studied. These functionals are characterized and/or described in terms of measures, and there are two ways of proving the relevant theorems. One is to use the linear functional itself to define a measure and then show that the functional is just integration with respect to this measure. The other is to regard the linear functional as being analogous to the Riemann–Stieltjes integral and then extend the domain of the functional along the lines of what was done in Chapters 7, 8, and 12 to produce, first, an analogue of the Lebesgue integral and, second, a measure that represents the functional. Needless to say, it is the second of these approaches which is carried out in this chapter. In fact, some of the definitions and arguments in

Chapters 7, 8, and 12 were given in such a manner that they could be reused in Chapter 22.

Finally, Chapters 23 and 24 treat two topics that are not usually included in books at this level. One of these is Hausdorff measures and dimension and the other is Lorentz spaces. Hausdorff measures and dimension are coming more and more to the forefront with the current interest in dynamical systems and fractals, and it seems only reasonable to include an introduction to them in a book on integration and measure theory. And Lorentz spaces are making their presence felt in real and harmonic analysis, and, again, it seems only reasonable to include an introduction in a book such as the present one.

The sixth and final part consists of two appendices. The first one, Appendix A, contains a review of a number of definitions and results from real variables as well as a brief discussion of the axiom of choice, ordinal numbers, and cardinality. The axiom of choice is needed in a number of proofs in measure theory, and students should be aware of this while ordinal numbers are used to construct some interesting examples of σ-algebras and measures. Appendix B contains a number of ideas and results from functional analysis that are needed in the book itself. This information is intended for the convenience of both the author and the reader and is by no means a systematic exposition of functional analysis.

Almost all of the chapters end with a fairly large number of exercises. It is important that students do a considerable number of them because (and this is a well-worn truism) the only way to learn mathematics is to do mathematics. There are a few easy exercises and a few difficult ones (which have hints), but the majority are, I believe, of a reasonable level of difficulty. The exercises serve a number of different purposes: Some ask the reader to prove a proposition or theorem which is stated but not proven in the text or (more commonly) to supply one step of a proof given in the text; some will be referred to later in proofs or discussions, although most will not; some contain interesting examples; and some give extensions or alternative proofs of the theorems in the text.

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