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Some Aspects of the Didactic and Scientific Activities of Professor Gheorghe Vrănceanu

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1 Introduction

Today, we commemorate 25 years from the passing into the eternity of the great Romanian geometer, the academician Gheorghe Vrănceanu. Worldwide estimated as one of the founders of modern Geometry, professor Gheorghe Vrănceanu brought through his work an outstanding brilliancy to the fame of Romanian mathematics.

Born in 1900 in the Valea Hogei village (belonging to the Doagele commune from the Vaslui district), professor Gheorghe Vrănceanu conserved all his life the charme of unequalled story teller in a sweet Moldavian accent.

After graduating from the Faculty of Mathematics in Iaşi (1922), professor Gheorghe Vrănceanu obtained in 1924 a distinguished Ph.D. in Rome, under the guidance of Levi-Civita, famous for his research in Geometry and Mechanics. The thesis evaluation committee, with 11 members, was chaired by Vito Volterra and awarded him (the maximum of) 110 points.

Influenced by Levi-Civita's ideas, Gheorghe Vrănceanu discovered in 1926 the notion of non-holonomic space, which will bring him an early celebrity.

In the academic year 1927-1928, he obtained a Rockefeller grant to travel in USA and to study in Harvard and Princeton universities. With this opportunity, he met the outstanding American geometers G. Birkhoff and O. Veblen, linking with them a longstanding friendship.

Between 1921 and 1970, Gheorghe Vrănceanu belong to the teaching staff of the universities of Iassy (1921-1929), Cernautzi (1929- 1939) and Bucharest (1939-1970).

Like Gheorghe Tzitzeica, after the second World War Gheorghe Vrănceanu was Dean of the faculty, in a period of severe poverty and of risky political situation. With tact and perseverance, Gheorghe Vrănceanu made all his best to help the faculty members to pursue a normal activity. We mention here the care of Gheorghe Vrănceanu for organizing and completing the library of our faculty.

Professor Gheorghe Vrănceanu gave courses on Analysis, Analytical Geometry, Mechanics, Projective Geometry, Differential Geometry, General Relativity and

Global Differential Geometry. As teacher, his lectures were models of erudition and clarity, following only the pure line of polishing the mathematical reasoning. Concerning his teaching method, we quote academician Solomon Marcus ([4]):

He was our teacher in 1945-1946. He spoke freely, naturally, and disliked the stage-like discourse, with special effects. It was no difference between his teaching method and his current language. Consistent with this approach, he looked to avoid artefacts, to develop the natural line of ideas. In contrast with the lectures of the kind: "definition - axiom - theorem - proof - corrolaries", our great teacher, in his lectures on Projective Geometry he gave us in 1945-1946, was beginning with a problem, whose importance he was arguing and then was looking for the answer, through natural but not always successful explorations. He was not using any written notes. The theorem was crystallizing through bringing together the in-road facts; it was not the starting but the ending point.

In our opinion, this characterization of professor Gheorghe Vrănceanu catches the true essence of his personality.

The textbooks of professor Gheorghe Vrănceanu and his four volume monography (translated in French and German) supported, for many years, the geometrical training of students and researchers. Concerning his monography on Differential Geometry, Kostake Teleman writes ([21],[5]): *The lectures on Differential Geometry of Gheorghe Vrănceanu assert themselves, from the beginning, through the approach of central problems in Geometry, which led to important, deep results and with many possibilities of generalization. The book exposes especially the results obtained by the author, many becoming clasical ones, concerning the theory of transformations groups and the spaces with connection. There is no other monography on geometry to contain so varied problems. A mathematician who loves geometry cannot but to admire the professor Gheorghe Vrănceanu's manner to expose all these problems, deepening all the important aspects and revealing the links between those chapters of geometry.*

The mathematical work of Gheorghe Vrănceanu, containing more than 300 papers, is characterized by the depth of the notions he introduced and by the importance of the solved problems. The scientific research of Gh. Vrănceanu was directed through: non-holonomic spaces, differential calculus of congruences, analytical mechanics, geometrization of partial differential equations of second order, non-holonomic unitary theory, spaces with conformal connection, partial projective spaces, Lie groups, global geometry, motion groups of spaces with affine connection, spaces with constant connection, harmonic tensors, Riemannian spaces with constant connection, the curvature of a differentiable manifold, Riemannian spaces in geodesic correspondence, the embedding of Riemannian spaces in Euclidean spaces, submanifolds of the sphere, spaces with nonlinear connection, the geometrization of mechanical systems.

This simple enumeration of the research directions approached and in which he brought remarkable contributions, gives us only a partial image of the true dimensions of Gheorghe Vrănceanu's mathematical work.

It is extremely difficult to analyse the whole scientific work of academician

Gheorghe Vrănceanu. Therefore, we will limit ourserves to expose briefly two of his contributions: *the spaces of constant connection* and *the Riemannian spaces in geodesic correspondence.*

2 Spaces with constant affine connection

Academician Gheorghe Vrănceanu says a space with an affine connection A_n is with constant connection if there exists a system of coordinates $x^1, ..., x^n$, in which the connection components Γ^i_{jk} are constant.

With respect to an affine coordinate transformation

$$(*) \qquad x'^i = a^i_j x^j + a^i$$

(where a^i_j and a^i are constant and $det(a^i_j) \neq 0$), the connection components transform themselfs following the rule

$$\Gamma'^i_{rs} a^r_j a^s_k = \Gamma^s_{jk} a^i_s$$

so Γ^i_{jk} behaves, with respect to these transformations, like a type (1,2)-tensor. It follows that a space with constant connection A_n is determined, with respect to preferred changes of coordinates (*), by a constant type (1,2)-tensor.

The spaces A_n with constant connection were called ([14]) *Vrănceanu spaces.* The following theorem was proved in [30]:

A space A_n with constant affine connection is a Vrănceanu space if and only if A_n admits a simply transitive and abelian motions group. The affine connection of A_n becomes constant in the coordinates with respect to whom this group is the translations group.

Studying the spaces A_n with constant affine connection, Gh. Vrănceanu was concerned with the (global) equivalence of such locally Euclidean space with the Euclidean space E_n. He found sufficient conditions in order A_n be globally equivalent with E_n.

Around 1960, during a discussion between academician Gh. Vrănceanu and academician Grigore Moisil, a natural method was indicated ([28]) (and subsequently developped by Gheorghe Vrănceanu in [29]) in order to associate, to each real finite algebra a space with constant affine connection (the structure constants of the algebra being the connection coefficients). Conversely, the coefficients of every constant affine connection may be considered the structure constants, in a certain basis, of a real finite algebra. In this manner, one may establish a correspondence between certain algebraic and certain geometric properties. For example, a Vrănceanu space A_n is locally euclidean if and only if the corresponding algebra is associative and commutative.

Another important problem concerning the Vrănceanu spaces is their classification. For $n = 2$, this problem was solved by academician Gheorghe Vrănceanu in [30]. For $n = 3$ and in case the contracted connection vanishes, the problem was solved by P. Mocanu ([7]). The respective classification problem may be expressed in an algebraic form, via the following result ([23]: *A Vrănceanu space* A_n *is equivalent, in the large, with the Euclidean space if and only if its associated algebra is commutative, associative and nilpotent.*

K. Teleman gave ([23]) a method which allows the determination, by recurrence with respect to the dimension, of all the finite dimensional, commutative and associative algebras, hence of all the locally Euclidean Vrănceanu spaces A_n. The theorem of Teleman proves the following are equivalent:

(i) the determination of finite dimensional, commutative and associative algebras;

(ii) the determination of linear, abelian, locally transitive groups acting on the affine complex space;

(iii) the determination of linear, abelian, locally transitive groups acting on the projective complex space;

(iv) the determination of locally Euclidean Vrănceanu spaces;

(v) the determination of spaces with affine connection which are projectively Euclidean and have constant associated Thomas connection.

The results obtained by Gheorghe Vrănceanu in the study of spaces with constant connection were quoted and used in Romania and abroad ([14]- [18], [23]-[25]).

The ideas introduced by Gheorghe Vrănceanu in the study of spaces with constant connection proved very useful in defining and studying the deformation algebra of two linear connections, domain with many written papers and interesting results ([10]).

3 Riemannian spaces in geodesic correspondence

The well-known theorem of Beltrami asserts ([30]): *if a Riemannian space* V_n *admits a geodesic representation on a* $\overline{V}_n$ *space with constant curvature, then* V_n *is also with constant curvature.*

Siniukov ([31]) proved: *if a Riemannian space* V_n *(*$n \geq 3$*) admits a non-trivial geodesic representation on a symmetric (Cartan)* $\overline{V}_n$ *space, then* V_n *and* $\overline{V}_n$ *are spaces with constant curvature.*

Gheorghe Vrănceanu gave a simpler proof of the theorem of Siniukov, by first proving the following result ([31]):

Let V_n *and* $\overline{V}_n$ *be two Riemannian spaces in non-trivial geodesic correspondence, (*$n \geq 3$*). Suppose* V_n *is an Einstein space and* $\overline{V}_n$ *is a symmetric (Cartan) space. Then* V_n *and* $\overline{V}_n$ *are spaces with constant curvature.*

In [31], Gheorghe Vrănceanu generalizes the Levi-Civita's results on the geodesic correspondence of the Riemannian spaces. He determined all the Riemannian spaces V_n which admit a non-trivial geodesic correspondence on another Riemannian space $\overline{V}_n$ and proved that these spaces may belong to n families, following the property of the correspondence vector to have non-vanishing components on the orthogonal congruences, common to the spaces V_n and $\overline{V}_n$.

Moreover, Gheorghe Vrănceanu proves:

if V_n and $\overline{V}_n$ are in a trivial geodesic correspondence (and V_n is irreducible), then the metrics of V_n and $\overline{V}_n$ are homothetic.

The ideas of Vrănceanu on Riemannian spaces in geodesic correspondence are being used by many Romanian and foreign geometers. In the last three decades, by the Vrănceanu's method, were studied the geodesic correspondence of two Riemannian spaces, with different geometric conditions imposed to the curvature, to the concircular or to the conharmonic tensors, respectively.

We mention that P. Venzi wrote a Ph.D. thesis following the results of Gheorghe Vrănceanu on the geodesic correspondence of Riemannian spaces ([26]). Another Ph.D. thesis was writen in Kazan University, in 1993, by A. Aminova ([1]).

The method of Vrănceanu in the study of the Riemannian spaces in geodesic correspondence proved useful also in the study of subgeodesic correspondence, which was developped in Romania during the last 30 years([11]).

4 Conclusions

The results established by professor Gheorghe Vrănceanu entered for a long time in the international scientific heritage. They influenced, directly or indirectly, many Romanian or foreign geometers, being sometimes starting points for new researches made by: Thomas, Wagner, K. Yano, Walker, Nomizu, Kobayashi, Egorov, Rasevsky, Petrov, Blascke, Helgason, E. Cartan. The scientific work of professor Gheorghe Vrănceanu influenced, during half a century, the geometrical research worldwide.

Gheorghe Vrănceanu imposed himself as one of the great modern geometers, through the vastity of approached problems, their diversity, the ingenious solutions, the fecund methods, the novelty of the spirit and the permanent effort to promote the modern ideas.

In our country, many researchers proved their enthousiasm for mathematics, devoting their activity in the domains of geometry initiated by Gh. Vrănceanu. We mention: K. Teleman, Şt. Petrescu, R. Blum, N. N. Mihăileanu, C. Simionescu, I. Teodorescu, A. Popovici, A. Dobrescu, P. Mocanu, Th. Hangan, T. Postelnicu, V. Dumitraş, P. Stavre, M. Stoka, G.G. Vrănceanu, M. Ţarină, G. Mărgulescu, V. Vodnev, A. Hristev, I. Popovici, D. Smaranda, R. Iordănescu, A. Turtoi, S. Ianuş, Al. Mihai, M. Dediu, V. Boju, L. Nicolescu.

The students of professor Gheorghe Vrănceanu found new and various ways of study in the most important domains of Differential Geometry. Their results proved the fertility of Gheorghe Vrănceanu's ideas.

Many Romanian researchers used, in their work, papers of academician Gheorghe Vrănceanu. We mention: M. Haimovici, G.T. Gheorghiu, A. Haimovici, O.

Gheorghiu, R. Miron, C. Udrişte, R. Roşca, Gh. Gheorghiev, T. Mihăilescu, M. Anastasiei.

Romanian geometers, all over the world, bring their homage to the me-mory of professor Gheorghe Vrănceanu, who formed many teachers and contributed in an essential way to the development of the modern school of geometry and, implicitely, to the growth of the Romanian scientific fame.

If Tzitzeica, Myller, Mayer contributed to the foundation of Kleinian geometries, Gheorghe Vrănceanu remains the founder of the new directions, of global Differential Geometry in Romanian mathematics. His contributions, together with those of the geometry school he created in Bucharest, and with the contribution of his spiritual students in Iassy, Craiova, Cluj-Napoca and Timisoara, made a resolute step toward the worldwide appreciation of the Romanian geometry.

The remarkable scientific research and the topical of the approached problems by professor Gheoghe Vrănceanu, lead to the conclusion that he may be considered the very founder of modern Differential Geometry in Romania.

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