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# The Work of Gh. Vrănceanu on Nonholonomic Spaces

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Vrănceanu liked to say that his fame comes from his discovery of nonholonomic spaces. The Romanian "Minerva" encyclopedia, published in the years 1930-1940, mentions this achievement of the Romanian geometer.

1. Gh. Vrănceanu began his study on nonholonomic spaces under the guidance of the famous Italian mathematician T. Levi Civita in 1926, but it is after he met É. Cartan at the International Congress of Mathematicians, that took place in 1928 in Bologne, that Gh. Vrănceanu began a systematic research on that topic. The concept of a nonholonomic space came from Analytic Mechanics: a mechanical nonholonomic system $S$, consisting in a set of material points, is characterized by a Lagrange function, defining a Riemannian metric, and a system of constraints, consisting in a finite number of relations imposed to the positions and velocities of the material points.

The most familiar case arises when these constraints are expressed through linear equations in the velocities, which have the form

$$f^\alpha = \sum_{A=1}^{n} \omega_A^\alpha(x)\ \dot{x}^A = 0\ ,\ (\alpha = m+1, ..., n),$$

where $n$ is the dimension of the configurations space and $\omega_A^\alpha$ are differentiable functions depending on the Lagrange coordinates $x^A$.

We can suppose that the Lagrange function

$$L(x, \dot{x}) = \frac{1}{2} \sum_{A,B=1}^{n} g_{AB}(x)\ \dot{x}^A\ \dot{x}^B$$

has been written as a sum of $n$ squares of linear forms

$$L = \frac{1}{2} \sum_{A=1}^{n} (f^A)^2\ ,\ f^A = \sum_{B=1}^{n} \omega_B^A\ \dot{x}^B,$$

with the constraints $f^\alpha = 0\ ,\ (\alpha = m+1, ..., n)$. The ranges of the indexes will be as follows:

$$A, B, C = 1, ..., n\ ;\ i, j, k = 1, ..., m\ ;\ \alpha, \beta, \gamma = m+1, ..., n.$$

Under these conditions, the equations of motion of the system $S$ are due to Lagrange and can be written in any of the two following equivalent forms:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^A}\Big) = \frac{\partial L}{\partial x^A} + \sum_{\alpha=m+1}^{n} \lambda_\alpha\ \omega^\alpha_A\ ,\ f^\alpha = 0$$

$$\frac{d}{dt}\Big(\sum_{i=1}^{m} f^i\ \omega^i_A\Big) = \sum_{i=1}^{n} f^i\ \frac{\partial \omega^i_D}{\partial x^A}\ \dot{x}^D + \sum_{\alpha=m+1}^{n} \lambda_\alpha\ \omega^\alpha_A\ ,\ f^\alpha = 0,$$

where $\lambda_\alpha$ are functions to be determined.

Suppose that, in view of the constraints $f^\alpha = 0$, we have

$$\dot{x}^A = \sum_{i=1}^{m} \pi^A_i f^i. \tag{1}$$

Then

$$\sum_{A=1}^{n} \omega^i_A\ \pi^A_j = \delta^i_j\ ,\ \sum_{A=1}^{n} \omega^\alpha_A\ \pi^A_j = 0$$

and the equations of motion will be given by the system formed by the equations (1) and by

$$\dot{f}^i = \sum_{k,j=1}^{m} w^k_{ij} f^k f^j = -\sum_{k,j=1}^{m} \gamma^i_{kj} f^k f^j \tag{2}$$

where

$$w^i_{jk} = \sum_{A,B=1}^{n} \Big(\frac{\partial \omega^i_A}{\partial x^B} - \frac{\partial \omega^i_B}{\partial x^A}\Big)\pi^A_k \pi^B_j$$

$$\gamma^i_{jk} = \frac{1}{2}(w^i_{jk} + w^j_{ki} - w^k_{ij}).$$

Let us introduce the 1-forms, also named Pfaff forms:

$$\omega^A = \sum_{B=1}^{n} \omega^A_B\ dx^B.$$

The French mathematician J. Hadamard had pointed out that the equations of motion remain equivalent to themselves when one performs a transformation of the form

$$\omega'^i = \sum_{j=1}^{m} c^i_j\ \omega^j\ ,\ \omega'^\alpha = \sum_{\beta=m+1}^{n} c^\alpha_\beta\ \omega^\beta, \tag{3}$$

where $c^\alpha_\beta$ , $c^i_j$ are functions of the $x^A$ such that

$$\sum_{i=1}^{m} c^i_j c^i_k = \delta_{jk}\ . \tag{4}$$

The transformations of the form (3) were named by Cartan and Vrănceanu *separable transformations of Pfaff forms.*

On the other side, the equations (2) are very similar to the equations of the geodesic lines in a Riemannian manifold, written in the formalism of *Ricci Calculus.*

Inspired by these remarks, Gh. Vrănceanu defined a nonholonomic space [6] by considering the structure, denoted $V_n^m$ and consisting in a system of $n-m$ Pfaff equations

$$\omega^\alpha = 0 \ , \ (\alpha = m+1, ..., n)$$

and a quadratic differential form of rank $m$

$$ds^2 = \sum_{i=1}^m (\omega^i)^2,$$

such that the $n$ forms $\omega^A$ are linearly independent.

Vrănceanu's theory of nonholonomic spaces starts with the problem of finding necessary and sufficient conditions for the equivalence of two nonholonomic spaces $V_n^m \ , \ V_n^{'m}$ with respect to the *separable* group of transformations of Pfaff forms

$$\omega^{'i} = c_j^i \ \omega^j \ , \ \omega^{'\alpha} = c_\beta^\alpha \ \omega^\beta.$$

To solve this problem, Vrănceanu studied the linear connections which are invariantly associated with a given nonholonomic space.

It is this problem that was the starting point of a very interesting correspondence between Vrănceanu and Cartan. This correspondence referred mainly to the equivalence problem of two nonholonomic spaces and lead to a lot of important results, which are contained in many articles.

But it is important to point out that Cartan had already published in 1910 a large article on the continuous groups of transformations, depending on arbitrary parameters and functions [1].

On the other side, E. Goursat had published an important book titled "Leçons sur le problème de Pfaff". The works of Cartan and Vrănceanu rely significantly on this book, which had to be developed. For instance, the notion of the *class of a Pfaff system* had to be defined properly.

Most of the examples produced by Cartan and Vrănceanu concerned nonholonomic spaces of types $V_5^2 \ , \ V_5^3$ .

2. It is interesting to note that, while studying under the guidance of Cartan the problem of equivalence of two nonholonomic spaces, Vrănceanu was using Levi-Civita and Ricci's formalism of tensor calculus, almost ignoring *the formalism of exterior differential calculus* used by Cartan. This explains to some extent the isolation which characterized Vrănceanu's activity for a long period of time.

Let us explain the relation between the two formalisms. We will use Einstein's summation convention. Suppose we have a Riemannian metric

$$ds^2 = \sum_{i,j=1}^n g_{ij}(x) \ dx^i \ dx^j,$$

which has been written as a sum of squares of 1-forms

$$ds^2 = \sum_{A=1}^n (\omega^A)^2 \ , \ \omega^A = \omega_i^A(x) \ dx^i. \tag{5}$$

Let us introduce the dual objects

$$(\pi_A^i) = (\omega_i^A)^{-1} \ , \ \partial_i = \frac{\partial}{\partial x^i} \ , \ \pi_A = \sum_{i=1}^{n} \ \pi_A^i \partial_i$$

so that the differential $df$ of a function $f$ can be written as follows:

$$df = \pi_A(f) \ \omega^A.$$

When $C_{jk}^i$ are Christoffel's symbols, *the covariant derivatives* of the 1-forms $\omega^A$ are given by the formulas

$$\nabla_{\pi_B} \omega^A = \pi_B^j (\partial_j \ \omega_i^A - C_{ij}^k \ \omega_k^A) dx^i = \gamma_{CB}^A \ \ \omega^C,$$

where

$$\gamma_{CB}^A = \pi_C^i \ \pi_B^j \ (\partial_j \ \omega_i^A - C_{ij}^k \ \omega_k^A)$$

are the Ricci *rotation coefficients.* These coefficients have the properties

$$\gamma_{CB}^A + \gamma_{AB}^C = 0 \ , \ \gamma_{CB}^A - \gamma_{BC}^A = w_{BC}^A \ ,$$

where the coefficients

$$w_{BC}^A = \pi_C^i \ \pi_B^j (\partial_j \ \omega_i^A - \partial_i \ \omega_j^A)$$

appear in the formulas giving *the exterior derivatives*, frequently used by Cartan:

$$d\omega^A = \frac{1}{2} w_{BC}^A \ \omega^B \wedge \omega^C.$$

As a consequence, we get

$$\gamma_{BC}^A = \frac{1}{2}(w_{BC}^A + w_{CA}^B - w_{AB}^C).$$

Note that when we introduce the 1-forms

$$\gamma_B^A = \gamma_{BC}^A \ \omega^C,$$

we get *Cartan's structure equations*:

$$d\omega^A = -\gamma_B^A \wedge \omega^B$$

$$d\gamma_B^A + \gamma_C^A \wedge \gamma_B^C = \frac{1}{2} \gamma_{BCD}^A \ \omega^C \wedge \omega^D$$

containing the *curvature coefficients*

$$\gamma_{BCD}^A = -\pi_D(\gamma_{BC}^A) + \pi_C(\gamma_{BD}^A) + \gamma_{BF}^A \ w_{CD}^F + \gamma_{FC}^A \ \gamma_{BD}^F - \gamma_{FD}^A \ \gamma_{BC}^F \ .$$

When we perform a linear transformation of 1-forms

$$\tilde{\omega}^A = c_B^A \ \omega^B, \tag{6}$$

the transformation laws of the coefficients $\gamma^A_{BC}$ and $w^A_{BC}$ look as follows:

$$\pi_A(c^{B'}_B) + c^{B'}_D \gamma^D_{BA} = \tilde{\gamma}^{B'}_{D'A'}\ c^{D'}_B\ c^{A'}_A$$

$$\pi_B(c^{A'}_A) - \pi_A(c^{A'}_B) + c^{A'}_D\ w^D_{BA} = \tilde{w}^{A'}_{B'D'}\ c^{B'}_B\ c^{D'}_A\ .$$

Suppose the transformation (6) has

$$c^\alpha_i = 0\ ,\ (1 \le i \le m < \alpha \le n). \tag{7}$$

Then the transformation laws above give

$$\pi_i(c^{j'}_j) + c^{j'}_k\ \gamma^k_{ji} + c^{j'}_\alpha\ \gamma^\alpha_{ji} = \tilde{\gamma}^{j'}_{ki'}\ c^k_j\ c^{i'}_i$$

$$c^\beta_\lambda\ \gamma^\lambda_{ik} = \tilde{\gamma}^\beta_{lj}\ c^l_i\ c^j_k\ ,\ c^\beta_\lambda\ \gamma^\lambda_{i\ \alpha} = \tilde{\gamma}^\beta_{j\ \mu}\ c^j_i\ c^\mu_\alpha + \tilde{\gamma}^\beta_{jk}\ c^j_i\ c^k_\alpha$$

$$c^\beta_\lambda\ w^\lambda_{ik} = \tilde{w}^\beta_{lj}\ c^l_i\ c^j_k\ ,\ c^\beta_\lambda\ w^\lambda_{i\ \alpha} = \tilde{w}^\beta_{j\mu}\ c^j_i\ c^\mu_\alpha + \tilde{w}^\beta_{jk}\ c^j_i\ c^k_\alpha.$$

Supposing that no linear combination of the Pfaff forms $\omega^\alpha$ has vanishing exterior derivative modulo these forms, excepting the combination with zero coefficients, Cartan tried to prove, at the Bologna Congress, that it is possible, in an invariant way, to get, besides the relations (7), the relations

$$c^i_\alpha = 0. \tag{8}$$

At the beginning of his research, Vrănceanu was looking for linear connections, associated with two *complementary* Pfaff systems; such a structure is invariant under a group (6) with $c^i_\alpha = c^\alpha_i = 0$.

In a letter dated May 9, 1932, he wrote to Vrănceanu acknowledging that an example given by Vranceanu contradicted this statement, so that he was, generally speaking, wrong.

Vranceanu had shown, at the same Congress, this:

**Proposition**. *Given a metric of the form (5) and a separable group (6-8), it is possible to get an invariant connection having*

$$\gamma^i_{\beta A} = \gamma^i_{A\beta} = \gamma^\alpha_{Ai} = \gamma^\alpha_{iA} = 0.$$

In a letter dated April 27, 1934, Cartan pointed out that, as far as concerns Vrănceanu's problem, it is important to obtain just one linear connection, that is canonically associated to a given nonholonomic mechanical system.

3. One gets nonholonomic spaces when one considers a second order *elliptic* PDE

$$F(z, x, p, q) = 0,$$

with $n$ independent variables $x^1, ..., x^n$ and just one unknown function $z = f(x^1, ..., x^n)$. Using the notations

$$p_i = \frac{\partial z}{\partial x^i}\ ,\ q_{ij} = \frac{\partial^2 z}{\partial x^i \partial x^j}\ ,\ F^{ij} = \frac{\partial F}{\partial q_{ij}}\ ,\ (F_{ij}) = (F^{ij})^{-1},$$

ellipticity means that the matrix $(F_{ij})$ is positive definite.

Under these conditions, one gets a metric of rank $n$:

$$ds^2 = \sum_{i,j=1}^{n} F_{ij} dx^i dx^j$$

with coefficients $F_{ij}$ depending on all the variables $x^i,\ z,\ p_i,\ q_{ij} = q_{ji}$ . The solutions of the equation $F = 0$ will be those integral manifolds of the Pfaff system

$$dz = \sum_{i=1}^{n} p_i\ dx^i\ ,\ dp_i = \sum_{j=1}^{n} q_{ij}\ dx^j,$$

which are contained in the hypersurface defined by the equation $F = 0$.

4. In the year 1969, Vrănceanu organized a Symposium dedicated to the centenary of É. Cartan's birthday.

We mention that this commemoration provided the participation of prominent mathematicians such as G. de Rham, A. Lichnerowicz, J. Koszul and M. Kuranishi, who offered deep analyses on Cartan's work. Their contributions are contained in a volume which was published by the Romanian Academy, in 1975 [5].

The same volume also contains a short contribution by Vrănceanu, in which the Romanian geometer asserts that his work on nonholonomic spaces is due to the influence of Cartan.

The volume [5] contains a set of letters written by Cartan to Vrănceanu. These letters reflect the immense influence that Cartan had on Vrănceanu's work concerning the nonholonomic spaces.

From his side, Vrănceanu produced an example which allowed Cartan to make precise one of his results exposed at the Bologne Congress on nonholonomic mechanical systems. In the same time, it was Cartan who invented a lot of interesting examples, that allowed Vrănceanu to develop his theory on nonholonomic spaces.

5. The author of these lines obtained an interesting example of a nonholonomic space, showing that the complex and quaternionic projective spaces, endowed with the Fubini-Study metrics, can be viewed as nonholonomic spaces in the spheres $S^{2p+1}$, $S^{4p+3}$ [3]. This result extends to a large class of homogeneous Riemann spaces discovered by É. Cartan and named *symmetric spaces*.

6. As far as concerns the general concept of nonholonomic space in the sense of Vrănceanu, we mention that, in a more modern language, it is referred to as a structure consisting in two complementary distributions on a differentiable manifold [4]. We also mention that this concept is studied in a fundamental article by H. Guggenheimer and D. Spencer from the viewpoint of the theory of *pseudo-groups of transformations*.

7. Differential Geometry is traditionally connected to Mathematical Physics, especially to Analytic Mechanics and Relativity Theory. Riemann himself suggested that the spaces that he introduced should be used to explain the gravitational phenomena. This idea was successfully developed many years after by A. Einstein, who used the notions of *pseudo-Riemannian manifolds* and *Levi-Civita parallelism*.

At the same time, Einstein posed the problem of finding a geometric model capable to explain both the electro-magnetic and gravitational phenomena, using a unitary framework. Einstein himself offered the first essay of such an unitary theory, in which the electro-magnetic and the gravitational fields were represented by skew-symmetric, respectively by symmetric tensor fields of type $(0,2)$, on differential manifolds of dimension four.

Though Einstein's problem remains just a dream, it gave rise to a lot of articles and books written by a large number of mathematicians and physicists, who developed interesting new mathematical and physical theories.

Gh. Vrănceanu was among the first authors who contributed to Einstein's problem.

Before Vrănceanu, H. Weyl had produced an unitary theory, in which the gravitational field was represented by a pseudo-Riemannian metric of Lorentz type, defined on a four-manifold $M$, while the electro-magnetic vector potential was represented by a differentiable 1-form

$$\Phi = dl - \varphi\ l\ ,\ \varphi = \sum_{i=1}^{4} \varphi_i(x)\ dx^i$$

on $M \times \mathbb{R}$, where $x^i$ are local coordinates on $M$ and $l$ is a coordinate on $\mathbb{R}$ representing *length.* In this formula, the 1-form $\varphi$ represents the electro-magnetic vector-potential and $dl$ represents a change of lengths due to the electro-magnetic field, whose strength is represented by $d\varphi$. Weyl's theory has been refuted by Einstein, but it became the corner stone of modern Electro-magnetism, when a small but significant modification was introduced. Namely the pair $(M \times \mathbb{R}\ ,\ \Phi)$ has been replaced by the pair

$$\Big(M \times \mathbb{C}\ ,\ \psi = dz - \sqrt{-1}\ \varphi\ z\Big).$$

This pair represents, in modern Differential Geometry, *a linear unitary connection in a complex unitary line fibre bundle* over $M$. The curvature of this connection is proportional to the tensor representing the strength of the electro-magnetic field.

Weyl's model is the origin of modern *gauge theories* used in Differential Topology and also in Mathematical Physics.

Vrănceanu's *unitary theory* consists in a nonholonomic space $V_5^4$ defined by a pair $(M \times \mathbb{R}\ ,\ \omega^5)$, where $M$ is a four manifold and

$$\omega^5 = dx^5 - \textstyle\sum_{i=1}^4\ \varphi_i(x^1, x^2, x^3, x^4)\ dx^i.$$

It was É. Cartan who invented an inspired, general geometric differential structure [2], which was and continues to be frequently used, not only in Theoretical Physics, but also in Differential Geometry and Differential Topology. Cartan's general structure consists in *an infinitesimal connection in a differentiable fibre bundle.*

According to Cartan's definition, an infinitesimal connection in a differentiable fibre bundle $p : E \to M$ is a so called *horizontal distribution* $H$ on $E$, that is complementary to *the vertical distribution* $V$, formed by the vectors tangent to the fibres $p^{-1}(x)\ ,\ x \in M$.

The horizontal distribution determines a vector fibre bundle on $M$, which is isomorphic to the tangent bundle of $M$. Therefore, when $M$ is endowed with a Riemannian structure, we can lift the metric of $M$ to $H$ and then we get a nonholonomic space $V_n^m$ in Vrănceanu's sense, where $n$ is the dimension of $E$ and $m$ is the dimension of $M$.

Linear connections, projective connections and conformal connections are particular cases of infinitesimal connections.

Gh. Vrănceanu recognized the great influence played by É. Cartan on his own work, by dedicating his monumental Treatise on Differential Geometry, published in France in 1967, to the great French mathematician.

Vrănceanu has the merit of making known in our country a large part of the work of É. Cartan; moreover, he enriched Cartan's work with many original contributions to many important chapters of Differential Geometry, such as geometric equivalence theory, Riemannian geometry, theory of Lie groups and Lie algebras, transformation groups and homogeneous spaces, theory of linear, projective and conformal connections, space forms, the geometric theory of PDE, holonomy groups, etc.

A final remark is perhaps worthwhile to be made. Following the tradition respected until the end of World War II, Vrănceanu was writing all formulas using local coordinates. Most of the formulas obtained by Vrănceanu are complicated enough and are not easy to be read by modern scholars and students. A review of the results obtained by the Romanian mathematician, using the modern, invariant and global formalism, is necessary and would be very useful.

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