Annals of the University of Bucharest (mathematical series)
6 (LXIV) (2015), 211–236

On the maximal reduction of games

Monica Patriche

Communicated by Vasile Preda

Abstract - We study the conditions under which the iterated elimination of strictly dominated strategies is order independent and we identify a class of discontinuous games for which order does not matter. The considered discontinuities characterize some economic game models, where the payoff functions are neither semicontinuous, nor lower semicontinuous. Then, we approach the issue of mixed strategy dominance. Thus, we introduce several types of dominance relations and game reductions and we deduce their properties. We also establish new theorems concerning the existence and uniqueness of the maximal game reduction when the pure strategies are dominated by mixed strategies.

Key words and phrases : Game theory, strict dominance, iterated elimination, order independence, maximal reduction.

Mathematics Subject Classification (2010) **:** 91A10, 91A26.

1. Introduction

The problem of the rationalizable strategic behaviour of the players is crucial in noncooperative strategic situations. It was formulated by Pearce (see [17]), and the incorporation of the rationalizability in the theory of games allows some interpretations, subsequently largely developed. As being given by Bernheim (see [6]) and Pearce (see [17]), the definition of the rationalizable strategies of a strategic game makes use of the iterative processes of elimination of dominated strategies, considered otherwise 'undesirable'. The notion of order independence characterizes the situations when the result of this iterative process does not depend on the order of removal. The elimination of the 'undesirable' strategies leads to the solution concept named "the maximal reduction of a game". Many authors searched for classes of games, and defined dominance relations, which are meant to preserve the advantages of the order independence and to assure the existence of a unique, nonempty maximal reduction.

In this paper, we identify a class of discontinuous games for which order independence holds. The payoff functions are transfer weakly upper continuous in the sense of Tian and Zhou (see [23]). These authors defined the transfer upper continuity and proved generalizations of Weierstrass and the maximum theorem. The motivation of the research, which is based on these types of discontinuities, lies in the interest in economic game models where the payoff functions are neither upper semicontinuous, nor lower semicontinuous, and which verify the new assumptions.

It is well known that the discontinuities of the payoff functions of a strategic game may fail to ensure the existence of a unique and nonempty maximal reduction. In the analysis of this topic, Dufwenberg and Stegeman gave some examples (see [10]). Then, these authors focused on the study of the continuous case. They stated a result concerning a class of games for which the existence of a unique and nonempty maximal reduction is fulfilled. The properties satisfied by games, for which the iterated elimination for strictly dominated strategies (IESDS) preserves the set of Nash equilibria, are: the compactness of the strategy spaces and the continuity of payoff functions. The authors also proved that if, in addition, the payoff functions are upper semicontinuous in own strategies, then the order does not matter.

Now, the contribution of Dufwenberg and Stegeman in [10], which restricted the payoff functions to be continuous, is revisited. Since many useful situations are excluded by this restriction, the problem raised by them deserves a development.

After presenting the notions of games, parings, dominance and reduction, we introduce two types of discontinuous games: own transfer upper continuous and own transfer weakly upper continuous. For our main results, which treat the noncompact case, we need the existence of the maximum of the payoff functions, and we introduce other new conditions which may enssure it: the condition K refers to the dominance relation and the condition $\mathcal{M}$ refers to utilities. Under these assumptions, we firstly obtain a more general lemma, than Lemma 1 of Dufwenberg and Stegeman in [10]: a key lemma which will be used for stating the main theorem in Section 4. This section is designed to provide our main result concerning the existence and uniqueness of nonempty maximal reductions of discontinuous games. So, the continuity conditions on payoff functions, which describe the game model in [10], are weakened.

Our study can be compared to other treatments in similar settings. We make here a short review. We firstly mention the paper [11] by Gilboa, Kalai and Zemel, who provide conditions (including strict dominance), which guarantee the uniqueness of the reduced games. A different approach was made by Marx and Swinkels in [12], who defined nice weak dominance and proved that under this order relation, order does not matter. The contribution of Chen, Long and Luo (see [8]) is also important. They provided a new definition of IESDS, that proved to be order-independent. Apt's approach (2007) uses operators on complete lattice and their transfinite iterations. The monotonicity of the operators enssures the order independence of iterated eliminations. Apt's paper in [2] provides an analysis of different ways of iterated eliminations of strategies. The notions of dominance and rationalizability are involved by other two strategy elimination procedures explored by Apt in [1]. In order to study the problem of order independence, the author considers three reduction relations on games and belief structures.

What happens if, instead of a pure strategy dominance, we have a mixed strategy dominance? To put it another way, what happens in the case when a pure strategy is dominated by a mixed strategy? This is the second question we address in this paper. We develop our study by defining several new types of dominance relation, which refer to mixed strategies and by characterizing the relations among them. In this context, we can define new types of game reduction, and we also obtain relations among them. Then, we continue our study in a similar way to the one of the pure strategies. In the case of the mixed strategies, our main theorem states the existence of a unique maximal ($\mapsto^*$) reduction of an upper semicontinuous game. These results are new and have not been reported in literature. We use notions of measurability and especially some results of Robson in [19].

The paper is organized in the following way: Section 2 contains some preliminary definitions and results. Extensions of Dufwenberg-Stegeman Lemma are presented in Section 3. The main result is stated in Section 4 and the mixed strategy case is treated in Section 5. These are followed by concluding remarks, in the end. A list with the main notations used in the paper is added in Appendix.

2. Reductions of normal form games

The iterative process of elimination of the 'undesirable' strategies leads to the solution concept named "the maximal reduction of a game".

In order to define the 'undesirable' strategies, we need the notion of paring of a game, which induces a strict dominance relation on that game. By removing the dominated strategies, a maximal reduction of the game is obtained. The notion of order independence characterizes the situations when the result of this iterative process does not depend on the order of removal.

In Section 4, we will prove that, under new assumptions, the iterated elimination of strictly dominated strategies (IESDS) also produces an unique maximal reduction. The setting, we work in, refers to new classes of discontinuous games, the payoff functions being transfer upper continuous or transfer weakly upper continuous in the sense defined by Tian and Zhou (see [23]).

The normal form of n-person game is $G = (I, (G_i)_{i\in I}, (u_i)_{i\in I})$ where, for each $i \in I = \{1, 2, ..., n\}$, G_i is a nonempty set (the set of individual strategies of player i) and $u_i : \prod_{k\in I} G_k \to \mathbb{R}$ is the payoff function of i.

We will denote $G_{-i} = \prod_{j \in I \setminus \{i\}} G_j$ and $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \in G_{-i}$.

Remark 2.1. A list with the main notations used in the paper is added in Appendix, in order to make the reading easier.

A *paring* of G is a triple $H = (I, (H_i)_{i \in I}, (u'_i)_{i \in I})$, where $H_i \subseteq G_i$ and $u'_i = u_{i|\prod_{j \in I} H_j}$. A paring is nonempty if $H_i \neq \emptyset$ for each $i \in I$.

Let H be a paring of G. For each $i \in I$, the *strict dominance relation* $\overset{H}{\succ}_i$ on G_i is defined below.

Let $x_i, y_i \in G_i$. We say that y_i strictly dominates x_i (with respect to H_i), and we denote $y_i \overset{H}{\succ}_i x_i$, if $H_{-i} \neq \emptyset$ and $u_i(y_i, s_{-i}) > u_i(x_i, s_{-i})$, for each $s_{-i} \in H_{-i}$.

If $H = G$, we obtain the following strict dominance relation with respect to G, which will be used in order to define a reduction.

$y_i \overset{G}{\succ}_i x_i$, if $x_i, y_i \in G_i$, $G_{-i} \neq \emptyset$ and $u_i(y_i, s_{-i}) > u_i(x_i, s_{-i})$, for each $s_{-i} \in G_{-i}$.

Let us consider the parings K and H of G, such that $H_i \subseteq K_i$ for each $i \in I$.

$K \to H$ is called *a reduction,* if for each $i \in I$ and $x_i \in K_i \backslash H_i$, there exists $y_i \in K_i$, such that $y_i \overset{K}{\succ}_i x_i$.

The reduction $K \to H$ is called *fast,* if for each $i \in I$, and for each $x_i \in K_i$, which is dominated by an element $y_i \in K_i$ (that is, $y_i \overset{K}{\succ}_i x_i$), we have $x_i \notin H_i$. Thus, for each $i \in I$, the set H_i does not contain elements from K_i which are strictly dominated by elements from K_i. The paring H of G, obtained by this type of reduction, is unique.

An iterative process of reductions is expressed by the following notion. As it can be seen, a special notation is used.

The reduction $K \to^* H$ is defined by the existence of (finite or countable infinite) sequence of parings A^t of K, $t = 0, 1, 2...$, such that $A^0 = K$, $A^t \to A^{t+1}$, for each $t \geq 0$ and $H_i = \cap_t A_i^t$, for each $i \in I$.

The 'final' result of the iterative process of reductions is called the maximal reduction of the game.

H is said to be a *maximal* $(\to^*)$*-reduction* of the game G, if $G \to^* H$ and $H \to H'$, only for $H' = H$.

The aim of this paper is to find classes of discontinuous games for which: the order independence holds and the nonempty and maximal reduction of a game exists. In the next subsection, we will define two classes of games with payoff functions satisfying Tian and Zhou's weaker continuity assumptions (see [23]). In Section 4, we will prove that: under these assumptions, the problem we raised is solved.

The functions introduced by Tian and Zhou have the property of reaching their maximum on a compact set. We will use this property in order to prove the lemma which states the existence of the undominated elements of a game with respect to a strict dominance relation $\overset{H}{\succ}_i$ defined by a game reduction. We begin by providing here the concepts we need. We are following [23].

Let X be a subset of a topological space and let $f : X \to \mathbb{R} \cup \{-\infty\}$ be a function.

The function f is said to be *upper semicontinuous,* if for each point $x' \in X$, we have $\limsup_{x \to x'} f(x) \leq f(x')$, or equivalently, its epigraph $\text{epi}f = \{(x, a) \in X \times \mathbb{R} : f(x) \geq a\}$ is a closed subset of $X \times \mathbb{R}$.

We say that f is *transfer upper continuous* on X, if for points $x, y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of y such that $f(z) < f(x')$ for all $z \in \mathcal{N}(y)$.

The function f is said to be *transfer weakly upper continuous on* X if, for points $x, y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of y, such that $f(z) \leq f(x')$ for all $z \in \mathcal{N}(y)$.

Now, we are defining two types of games. Our main result will state the existence and uniqueness of nonempty maximal reductions for these games.

Definition 2.1. *The game G is called*

(i) *own-transfer upper continuous, if $u_i(\cdot, s_{-i})$ is transfer upper continuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$;*

(ii) *own-transfer weakly upper continuous, if $u_i(\cdot, s_{-i})$ is transfer weakly upper continuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$.*

3. Existence of undominated strategies

The lemmata established in this section are the tools used in order to prove the main theorem in Section 4. They refer to the existence of the nondominated strategies of the own-transfer weakly upper continuous games, defined in the previous subsection.

For our results, which treat the noncompact case, we need the existence of the maximum of the payoff functions, and thus we introduce other new conditions which may ensure it: the condition K, which refers to the dominance relation, and the condition $\mathcal{M}$, which refers to utilities. Under the condition K, we firstly obtain Lemma 3.1.

Let us consider the game G, such that: for each $i \in I$, the set G_i is a Hausdorff topological space and $\prod_{i \in I} G_i$ is endowed with the product topology.

We start by introducing the following definition. It asserts the existence of a compact set of a given type, included in the strategy set of each agent.

Definition 3.1. *Let us consider the paring H of the game G and let $i \in I$. The relation $\overset{H}{\succeq}_i$ has the property K, if for each $y_i \in G_i$, there exists $z_i^0 \in G_i$ with $z_i^0 \overset{H}{\succeq}_i y_i$, such that $\{z_i \in G_i : z_i \overset{H}{\succeq}_i z_i^0\}$ is compact.*

Now, we are stating Lemma 3.1, which extends the Dufwenberg-Stegeman Lemma in [10], by relaxing the continuity assumption on the payoff functions of the game. Our main result, stated in Section 4, concerning the existence and the uniqueness of a maximal reduction, relies on this lemma.

Lemma 3.1. *Let us assume that $G \to^* H$, for an own-transfer weakly upper continuous game G. Let $i \in I$ and $x_i, y_i \in G_i$, such that $y_i \overset{H}{\succ}_i x_i$. If $\overset{H}{\succ}_i$ has the property K, then, there exists $z_i^* \in H_i$, such that $z_i \overset{H}{\not\succ}_i z_i^* \overset{H}{\succ}_i x_i$, for each $z_i \in G_i$.*

In order to prove Lemma 3.1, we need the next necessary condition for a function to have a maximum on a set. Theorem 3.1 generalizes the Weierstrass theorem.

Theorem 3.1. (see [23]) *Let X be a compact subset of a topological space and let $f : X \to \mathbb{R} \cup \{-\infty\}$ be a function. Then f reaches its maximum on X, if and only if f is transfer weakly upper continuous on X.*

Proof of Lemma 3.1. Since $G \to^* H$, according to the definition of $\to^*$ (see Section 2), there exists a sequence of parings A^t, $t = 0, 1, 2...$, such that $A^0 = G$, $A^t \to A^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t A_i^t$, for each $i \in I$.

Let $i \in I$ and $x_i, y_i \in G_i$, as in the statement of Lemma 3.1. The nonemptiness of H_{-i} follows from $y_i \overset{H}{\succ}_i x_i$ and the definition of the dominance relation. Hence, the set Z_i is well defined, where

$Z_i := \{z_i \in G_i : u_i(z_i, s_{-i}) \geq u_i(y_i, s_{-i})$ for each $s_{-i} \in H_{-i}\}$.

According to the property K of $\overset{H}{\succeq}_i$, there exists $z_i^0 \in G_i$, such that $z_i^0 \overset{H}{\succeq}_i y_i$ and $U_i := \{z_i \in G_i : z_i \overset{H}{\succeq}_i z_i^0\} \subset Z_i$ is compact.

Now, let us consider a fixed element $s_{-i}^* \in H_{-i}$ and let us define

$f_i : U_i \to \mathbb{R}$ by $f_i(z_i) = u_i(z_i, s_{-i}^*)$ for each $z_i \in U_i$.

Being transfer weakly upper continuous on U_i, f_i reaches its maximum in $z_i^* \in U_i \subset Z_i$. We note that $z_i^* \in Z_i$ and $y_i \overset{H}{\succ}_i x_i$ imply $z_i^* \overset{H}{\succ}_i x_i$. If $z_i \overset{H}{\succ}_i z_i^*$ for some $z_i \in G_i$, then $u_i(z_i, s_{-i}) > u_i(z_i^*, s_{-i})$ for each $s_{-i} \in H_{-i}$, implying that $z_i \in U_i$ and $f_i(z_i) > f_i(z_i^*)$, which is a contradiction. This allows us to deduce that $z_i \overset{H}{\not\succ}_i z_i^*$ for each $z_i \in G_i$. Further, we exploit the fact that $H_{-i} \subseteq A_{-i}^t$ for each $t \geq 0$ and we obtain $z_i \overset{A^t}{\not\succ}_i z_i^*$ for each $z_i \in G_i$ for each $t \geq 0$. Consequently, $z_i^* \in A_i^t$ for each $t \geq 0$, which implies that $z_i^* \in H_i$. $\square$

A direct application of the above lemma is given in the following example.

Example 3.1. Let $I = \{1,2\}$, $G_1 = G_2 = [0,2]$. For $i, j \in I$, $i \neq j$, let $u_i : G_i \times G_j \to \mathbb{R}$ be defined by

$$u_i(x,y) = \begin{cases} 1, & \text{if } \quad x = 0; \\ 2, & \text{if } \quad x \in (0,1); \\ x+1, & \text{if } x \in [1,2]. \end{cases}$$

Let $H_1 = H_2 = [0,1]$.

We notice that, for each $i, j \in \{1,2\}$, $y \in G_j$, $u_i(.,y)$ is transfer weakly upper continuous on $[0,2]$ and $u_i(.,y)$ is not upper semicontinuous at $x = 0$.

We will prove that $\overset{H}{\succeq}_i$ has the property K, $i = 1, 2$.

Firstly, we consider $i = 1$.

If $y = 0$, there exists $z_0 = 0$, such that $U(0) = \{z \in [0,2] : u_1(z,s) \geq u_1(0,s)$, for each $s \in H_2\} = [0,2]$ is a compact set.

If $y \in (0,1)$, there exists $z_0 = \frac{3}{2}$, such that $U(\frac{3}{2}) = \{z \in [0,2] : u_1(z,s) \geq u_1(\frac{3}{2},s)$, for each $s \in H_2\} = [\frac{3}{2},1]$ is a compact set.

If $y \in [1,2]$, there exists $z_0 = y$, such that $U(z_0) = \{z \in [0,2] : u_1(z,s) \geq u_1(z_0,s)$, for each $s \in H_2\} = [y,2]$ is a compact set.

We have that for any $x, y \in [0,2]$, such that $y \overset{H}{\succ}_1 x$, there exists $z^* \in [0,2]$ such that $z^* \overset{H}{\succ}_1 x$ and $z \overset{H}{\not\succ}_1 z^* \overset{H}{\succ}_1 x$ for each $z \in H_1$.

The same argument can be done if $i = 2$.

If $H = G$, we obtain the following corollary.

Corollary 3.1. *Let us assume that G is an own-transfer weakly upper continuous game. Let $i \in I$ and $x_i, y_i \in G_i$, such that $y_i \overset{G}{\succ}_i x_i$. If $\overset{G}{\succ}_i$ has the property K, then, there exists $z_i^* \in G_i$, such that $z_i \overset{G}{\not\succ}_i z_i^* \overset{G}{\succ}_i x_i$ for each $z_i \in G_i$.*

We recall that the game G is called *compact,* if G_i is compact for each $i \in I$.

If in the last corollary, the game G is own-transfer weakly upper continuous and compact, we obtain the following result.

Corollary 3.2. *Let us assume that G is a compact, own-transfer weakly upper continuous game. Let $i \in I$ and $x_i, y_i \in G_i$, such that $y_i \overset{G}{\succ}_i x_i$. Then, there exists $z_i^* \in G_i$ such that $z_i \overset{G}{\not\succ}_i z_i^* \overset{G}{\succ}_i x_i$ for each $z_i \in G_i$.*

If G is compact and own-upper semicontinuous, as a consequence of Lemma 3.1, we obtain Lemma in [10]. Firstly, we recall that the game G is called *own-upper semicontinuous,* if $u_i(\cdot, s_{-i})$ is upper semicontinuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$; *continuous,* if $u_i(\cdot, s_{-i})$ is continuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$.

Corollary 3.3. (Lemma in [10, pag. 2012]) *Let us assume that $G \rightarrow^* H$ for a compact and own-upper semicontinuous game G. Let $i \in I$ and $x_i, y_i \in G_i$ such that $y_i \overset{H}{\succ}_i x_i$. Then, there exists $z_i^* \in H_i$, such that $z_i \overset{H}{\not\succ}_i z_i^* \overset{H}{\succ}_i x_i$ for each $z_i \in G_i$.*

We focus on developing the study of the noncompact games in a different way. To do this, we further define the property $\mathcal{M}$ for a function u. It states that the maximum of the function u is not reached in a certain set.

Definition 3.2. *Let X be a subset of a topological space. The function $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ has the property $\mathcal{M}$ on X if for each $y \in X$, the existence of a point $x \in cl\{z \in X : u(z) \geq u(y)\} \setminus \{z \in X : u(z) \geq u(y)\}$ implies the existence of $x' \in X$, such that $u(x') > u(x)$.*

We provide an example of transfer weakly upper continuous function which verifies the property $\mathcal{M}$.

Example 3.2. $u : [0, 1] \rightarrow \mathbb{R}$, $u(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number,} \\ 0, & \text{otherwise.} \end{cases}$

In order to obtain other extension of Dufwenberg-Stegeman Lemma in [10], we consider noncompact games with payoff functions having the property $\mathcal{M}$. The following result is stated.

Lemma 3.2. *Let us assume that $G \rightarrow^* H$ for a compact and own-transfer weakly upper continuous game G. Let $i \in I$ and $x_i, y_i \in G_i$, such that $y_i \overset{H}{\succ}_i x_i$. If for each $s_{-i} \in H_{-i}$, the function $u_i(\cdot, s_{-i})$ has the property $\mathcal{M}$, then, there exists $z_i^* \in H_i$ such that $z_i \overset{H}{\not\succ}_i z_i^* \overset{H}{\succ}_i x_i$ for each $z_i \in G_i$.*

Proof. Since $G \rightarrow^* H$, according to the definition of $\rightarrow^*$ (please see the Section 2), there exists a sequence of parings A^t, $t = 0, 1, 2...$ such that $A^0 = G$, $A^t \rightarrow A^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t A_i^t$, for each $i \in I$.

Let $i \in I$ and $x_i, y_i \in G_i$ as in the statement of Lemma 3.2. The nonemptiness of H_{-i} follows from $y_i \overset{H}{\succ}_i x_i$ and the definition of the dominance relation. For each $s_{-i} \in H_{-i}$, let us define the set

$Z_i(s_{-i}) := \{z_i \in G_i : u_i(z_i, s_{-i}) \geq u_i(y_i, s_{-i})\}$ and let

$Z_i := \cap_{s_{-i} \in H_{-i}} \text{cl}Z(s_{-i})$.

We notice that Z_i is compact. Now, let us define

$f_i : Z_i \rightarrow \mathbb{R}$ by $f_i(z_i) = u_i(z_i, s_{-i}^*)$ for each $z_i \in Z_i$, where $s_{-i}^* \in H_{-i}$ is fixed.

Being transfer weakly upper continuous on Z_i, f_i attains its maximum in $z_i^* \in Z_i$. For each $s_{-i} \in H_{-i}$, the function $u_i(\cdot, s_{-i})$ has the property $\mathcal{M}$, then we can assert that $z_i^* \in Z_i(s_{-i})$ for each $s_{-i} \in H_{-i}$. Since $y_i \overset{H}{\succ}_i x_i$, it is obviuously that $z_i^* \overset{H}{\succ}_i x_i$. If $z_i \overset{H}{\succ}_i z_i^*$ for some $z_i \in G_i$, then $u_i(z_i, s_{-i}) > u_i(z_i^*, s_{-i})$ for each $s_{-i} \in H_{-i}$, implying that $z_i \in Z_i$ and $f_i(z_i) > f_i(z_i^*)$, contradiction. Therefore, $z_i \overset{H}{\not\succ}_i z_i^*$ for each $z_i \in G_i$, and $H_{-i} \subseteq A_{-i}^t$ for each $t \geq 0$ leads us to $z_i \overset{A^t}{\not\succ}_i z_i^*$ for each $z_i \in G_i$ for each $t \geq 0$. This last fact implies $z_i^* \in A_i^t$ for each $t \geq 0$. Consequently, $z_i^* \in H_i$. □

If $H = G$, we obtain the following corollary.

Corollary 3.4. *Let us assume that G is a compact and own-transfer weakly upper continuous game G. Let $i \in I$ and $x_i, y_i \in G_i$, such that $y_i \overset{G}{\succ}_i x_i$. If for each $s_{-i} \in G_{-i}$, the function $u_i(\cdot, s_{-i})$ has the property $\mathcal{M}$, then, there exists $z_i^* \in G_i$ such that $z_i \overset{G}{\not\succ}_i z_i^* \overset{G}{\succ}_i x_i$ for each $z_i \in G_i$.*

4. Maximal reductions; existence and uniqueness

This section is designed to prove our main result concerning the existence and the uniqueness of nonempty maximal reductions of discontinuous games. We extend Theorem 1 in [10] by weakening the continuity conditions on payoff functions which describe the game model.

4.1. Assumptions

We start by making a new assumption concerning the payoff functions of the game G.

To this end, let us consider the game $G = (I, (G_i)_{i\in I}, (u_i)_{i\in I})$ and for each $i \in I$, let us denote

$F_i(x_i, s_{-i}) = \{s_i \in G_i : u_i(x_i, s_{-i}) \leq u_i(s_i, s_{-i})\}$ for each $x_i \in G_i$ and $s_{-i} \in G_{-i}$.

For each $i \in I$ and $x_i \in G_i$, let us define the sets $Z_i(x_i)$, $P_i(x_i)$ and $P_{-i}(x_i)$ as follows:

$Z_i(x_i) = \{(s_i, s_{-i}) \in \prod_{k\in I} G_k : u_i(x_i, s_{-i}) \leq u_i(s_i, s_{-i})\}$,

$P_i(x_i) = \mathrm{pr}_i Z_i(x_i)$ and $P_{-i}(x_i) = \mathrm{pr}_{-i} Z^i(x_i)$.

We will suppose that, for each $i \in I$, the projection $P_{-i}(x_i)$ does not depend on $x_i \in G_i$. Intuitively, we can understand this condition in the following way: for each agent i, the set of the strategies taken by the other agents such that the election improves the value of the payoff function for agent i is an invariant. This property will be used in order to prove our main theorem. It makes the content of the following definition.

Definition 4.1. *Let $i \in I$. We say that the function $u_i : \prod_{k\in I} G_k \to \mathbb{R}$ has the intersection property with respect to the i^{th} variable, if there exists $S_{-i} \subset G_{-i}$ such that, for each $x_i \in G_i$, $P_{-i}(x_i) = S_{-i}$ and $P_i(x_i) = \cap_{s_{-i}\in S_{-i}} F_i(x_i, s_{-i})$.*

We provide an example of payoff function for which the intersection property is fulfilled.

Example 4.1. Let $G_1 = G_2 = [0, 1]$ and let $u_1 : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$u_1(x,y) = \begin{cases} 1 + x + y, & \text{if } x \in [0,1] \cap Q; \\ x, & \text{if } \; x \in [0,1] \cap (\mathbb{R}\backslash Q). \end{cases}$$

We notice that, for each $y \in [0, 1]$, the function $u_1(\cdot, y)$ is not upper semicontinuous, but it is transfer upper continuous since, for a neighborhood $\mathcal{N} \subset [0, 1]$, we may choose any x' rational such that $\sup\{x : x \in \mathcal{N}\} < x\prime \leq 1$.

We prove that u_1 fulfills the intersection property with respect to the first variable.

We have that

$$Z_1(x) = \begin{cases} \{[x,1] \cap Q\} \times [0,1], & \text{if } \; x \in [0,1] \cap Q; \\ \{[0,1] \cap Q\} \times [0,1] \cup \{[x,1] \cap \mathbb{R}\backslash Q\} \times [0,1], & \text{if } x \in [0,1] \cap (\mathbb{R}\backslash Q), \end{cases}$$

$$P_1(x) = \begin{cases} [x,1] \cap Q, & \text{if } \; x \in [0,1] \cap Q; \\ \{[0,x] \cap Q\} \cup [x,1], & \text{if } x \in [0,1] \cap (\mathbb{R}\backslash Q), \end{cases}$$

$P_2(x) = [0, 1]$ for each $x \in [0, 1]$ and

$$F_1(x, s_2) = \begin{cases} [x,1] \cap Q, & \text{if } \; x \in [0,1] \cap Q; \\ \{[0,x] \cap Q\} \cup [x,1] \cap \mathbb{R}\backslash Q, & \text{if } x \in [0,1] \cap (\mathbb{R}\backslash Q), \end{cases} \quad \text{for each}$$

$s_2 \in [0, 1]$.

It follows that $P_1(x) = \cap_{s_2\in[0,1]} F_1(x, s_2)$ for each $x \in [0, 1]$.

Now, we can consider the following type of game, in which every agent has a payoff function with the property defined above.

Definition 4.2. *The game G has the intersection property, if for each $i \in I$, the function $u_i : \prod_{k\in I} G_k \to \mathbb{R}$ has the intersection property with respect to the i^{th} variable.*

Before stating Theorem 4.1, we introduce the following type of game reduction.

Definition 4.3. *The reduction $G \to^{**} H$ is defined by the existence of (finite or countable infinite) sequence of parings A^t of G, $t = 0, 1, 2...$, such that $A^0 = G$, $A^t \to A^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t A_i^t$ for each $i \in I$ and by the consistency with the continuity of the payoff functions, which means that for each $i \in I$, the payoff function u_i maintains the same continuity property on each set $\prod_{i\in I} A_i^t$, $t = 0, 1, 2...$, as it has on $\prod_{i\in I} G_i$.*

4.2. The existence of the unique maximal reduction

The aim of this subsection is to prove the existence of the unique maximal reduction for the new classes of games considered in this paper. For it, we use the lemmata stated in Section 3. Our proof also works under the new hypotheses introduced by us. The properties K and $\mathcal{M}$ replace the compactness of the game. The payoff functions are supposed to be own-transfer upper continuous. A new property is asked for the game: the intersection property, which refers to invariant sets of elections of the players. So, we needed to introduce new restrictions, when we weakened the assumptions concerning the continuity and the compactness of the game.

Theorem 4.1. *a) Let G be an own-transfer weakly upper continuous game, which also has the intersection property. Assume that for every $G \to H$ and for each $i \in I$, $\overset{H}{\succ}_i$ has the property K. Then, any nonempty maximal reduction $G \to^{**} M$ is the unique maximal reduction.*

b) Let G be an own-transfer upper continuous game which also has the intersection property. Assume that for every $G \to H$ and for each $i \in I$, $\overset{H}{\succ}_i$ has the property K. Then, G has a nonempty compact own-transfer upper continuous maximal ($\to^{**}$) reduction M and this reduction is unique.

In order to prove the theorem above, we need the following definitions and the next lemma, which characterizes the correspondences with transfer closed-values (see [23]).

A correspondence $F : X \to 2^Y$ is said to be *transfer closed-valued on* X if for every $x \in X$, $y \notin F(x)$ implies that there exists $x' \in X$ such that $y \notin \mathrm{cl}F(x')$. It is clear that, for any function $f : X \to \mathbb{R} \cup \{-\infty\}$, the correspondence $F : X \to 2^X$ defined by $F(x) = \{y \in X : f(y) \geq f(x)\}$ for all $x \in X$ is *transfer closed-valued on* X if and only if f is transfer upper continuous on X.

Lemma 4.1. (see [23]) *Let X and Y be two topological spaces, and let $F : X \to 2^Y$ be a correspondence. Then, $\cap_{x \in X} clF(x) = \cap_{x \in X} F(x)$, if and only if F is transfer closed-valued on X.*

Proof of Theorem 4.1. a) The proof follows the same line as Theorem 1 of Dufwenberg and Stegeman in [10]. For the sake of completeness, we add the proof here.

Let M and M' be maximal ($\to^{**}$)-reductions of G. Let us assume that M is nonempty and that $G \to^{**} M'$ is defined by the finite or infinite sequence of parings A^t, $t \geq 0$.

Let us suppose, on the contrary, that there exists $i \in I$ such that $M_i \subsetneq M'_i$. Then, there exists $T > 0$, such that $M_i \subsetneq A^t_i$ for each $t > T$. Let T_0 be the largest T, such that $A^{T_0}_i$ is well defined and $M_j \subseteq A^{T_0}_j$ for each $j \in I$.

Let us consider $x_i \in M_i \backslash A_i^{T_0+1}$. Then, $x_i \in A_i^{T_0} \backslash A_i^{T_0+1}$, which implies that there exists $y_i \in A_i^{T_0}$ such that $y_i \overset{A^{T_0}}{\succ_i} x_i$. Since $\emptyset \neq M_j \subseteq A_j^{T_0}$ for each $j \in I$, we can assert that $y_i \overset{M}{\succ_i} x_i$. According to Lemma 3.1, there exists $z_i^* \in M_i$ such that $z_i^* \overset{M}{\succ_i} x_i$, which contradicts the fact that M is a maximal $(\rightarrow^{**})$-reduction. Consequently, $M_i \subseteq M_i'$ for each $i \in I$, and, therefore, M' is nonempty.

By a similar reasoning, we can prove that $M_i' \subseteq M_i$ for each $i \in I$, and then, $M = M'$.

b) Recall that M is the result of an iterative process of reductions. In order to prove that M is nonempty and compact, we will firstly show that for each paring H of G with the property that $G \rightarrow H$ fast, it is true that H is nonempty and compact.

To this aim, let $G \rightarrow H$ fast and let $i \in I$ be such that $H_i \neq G_i$ (we consider only the nontrivial case). We note that there exist $x_i', y_i' \in G_i$ such that $y_i' \overset{G}{\succ_i} x_i'$. The nonemptiness of H_i can be easily proved by applying Corollary 3.1.

Further, we will show that H_i is the intersection of closed subsets of G_i. This will imply the compactness of H_i.

Let us define, for each $x_i \in H_i$,
$Z_i(x_i) = \{(s_i, s_{-i}) \in \prod_{k \in I} G_k : u_i(x_i, s_{-i}) \leq u_i(s_i, s_{-i})\}$, $P_i(x_i) = \mathrm{pr}_i Z_i(x_i)$ and $P_{-i}(x_i) = \mathrm{pr}_{-i} Z_i(x_i) = S_{-i}$.

Obviously, each $P_i(x_i)$ is nonempty, since $x_i \in P_i(x_i)$.

We claim that $H_i = \cap_{x_i \in H_i} P_i(x_i)$. To see this, let us choose an arbitrary element z_i of G_i. For each $x_i \in H_i$, if $z_i \notin P_i(x_i)$, then $u_i(x_i, s_{-i}) > u_i(z_i, s_{-i})$ for each $s_{-i} \in G_{-i}$, that is, $x_i \overset{G}{\succ_i} z_i$. The reduction $G \rightarrow H$ is fast and it follows that $z_i \notin H_i$. Thus, $H_i \subseteq \cap_{x_i \in H_i} P_i(x)$. To obtain the reverse inclusion, notice that if $z_i \notin H_i$, there exists $x_i \in G_i$ such that $x_i \overset{G}{\succ_i} z_i$ and Corollary 3.1 guarantees that there exists $x_i^* \in G_i$ with the property that $x_i^* \overset{G}{\succ_i} z_i$. Clearly, $z_i \notin P_i(x_i^*)$ and therefore, $z_i \notin \cap_{x_i \in H_i} P_i(x_i)$. We conclude that $\cap_{x_i \in H_i} P_i(x_i) \subseteq H_i$ and the equality $H_i = \cap_{x_i \in H_i} P_i(x_i)$ follows from the above assertions. Thus, the claim is shown.

Now, let us define

$F_i(x_i, s_{-i}) = \{s_i \in G_i : u_i(x_i, s_{-i}) \leq u_i(s_i, s_{-i})\}$ for each $x_i \in H_i$ and $s_{-i} \in G_{-i}$.

With this notation, $P_i(x_i) = \cap_{s_{-i} \in S_{-i}} F_i(x_i, s_{-i})$ for each $x_i \in H_i$. Since we have the reduction $G \rightarrow^{**} H$, the function $u_i(., s_{-i})$ is transfer upper continuous on H_i for s_{-i} fixed, and we can apply Lemma 3.1 to assert that $\cap_{x_i \in H_i} F_i(x_i, s_{-i}) = \cap_{x_i \in H_i} \mathrm{cl} F_i(x_i, s_{-i})$.

Therefore,

$H_i = \cap_{x_i \in H_i} P_i(x_i) = \cap_{x_i \in H_i} \cap_{s_{-i} \in S_{-i}} F_i(x_i, s_{-i}) =$

$= \cap_{s_{-i} \in S_{-i}} \cap_{x_i \in H_i} F_i(x_i, s_{-i}) = \cap_{s_{-i} \in S_{-i}} \cap_{x_i \in H_i} \mathrm{cl} F_i(x_i, s_{-i})$.

The closedness of H_i follows straightforward from the above statement. The compactness of G_i ensures that H_i is also compact.

Consequently, we proved that H is compact and nonempty.

Finally, let us consider $C(t)$ $t = 0, 1, \ldots$ the unique sequence of subgames of G, such that $C(0) = G$ and $C(t) \to C(t+1)$ is fast for each $t \geq 0$. By induction, the set $C(t)$ is compact and nonempty for each $t \geq 0$. Therefore, the paring $M = (I, (M_i)_{i \in I}, (u_{i|\prod_{i \in I} M_i})_{i \in I})$ is nonempty, compact and own-transfer upper continuous, where $M_i = \cap_{t \geq 0}(C(t))_i$ for each $i \in I$.

Now, we show that M is a maximal $(\to^{**})$-reduction of G. To do this, let us focus on any player i and let us consider $x_i, y_i \in M_i$. Let

$X_{-i}(t) := \{s_{-i} \in (C(t))_{-i} : u_i(y_i, s_{-i}) \leq u_i(x_i, s_{-i})\}$.

We claim that $X_{-i}(t) \neq \emptyset$. If not, for each $s_{-i} \in (C(t))_{-i}$, it follows that $u_i(y_i, s_{-i}) > u_i(x_i, s_{-i})$, so that $y_i \overset{C(t)}{\succ_i} x_i$, contradicting $x_i \in M_i$. Notice that $(C(t))_{-i}$ is compact and $\cap_{t \geq 0}(C(t))_{-i}$ is nonempty and compact.

Let us define

$X'_{-i} = \{s_{-i} \in \cap_{t \geq 0}(C(t))_{-i} : u_i(y_i, s_{-i}) \leq u_i(x_i, s_{-i})\}$
$\quad = \{s_{-i} \in M_{-i} : u_i(y_i, s_{-i}) \leq u_i(x_i, s_{-i})\}$.

Since M_{-i} is nonempty, it is not difficult to see that X'_{-i} is also nonempty. Therefore, $y_i \overset{M}{\not\succ_i} x_i$, and this proves that M is maximal. □

As a particular case of Theorem 4.1, we obtain Theorem 1 in [10].

Corollary 4.1. ([10, Theorem 1]) *a) If a game G is compact and own-upper semicontinuous, then, any nonempty maximal ($\to^*$) reduction of G is the unique maximal ($\to^*$) reduction of G.*

b) If a game G is compact and continuous, then, G has a unique maximal ($\to^*$) reduction; furthermore, M is nonempty, compact and continuous.

By applying Lemma 3.2, we obtain the following result. Its proof is similar to the proof of Theorem 4.1.

Theorem 4.2. *a) Let G be a compact and own-transfer weakly upper continuous game which has also the intersection property, such that for each $i \in I$, the payoff function u_i has the property $\mathcal{M}$. Then, any nonempty maximal reduction $G \to^{**} M$ is the unique maximal reduction.*

b) If G is a compact, own-transfer upper continuous game such that for each $i \in I$, the payoff function u_i has the property $\mathcal{M}$, then it has a nonempty compact own-transfer upper continuous maximal $(\to^{**})$ reduction M. The reduction M is unique.

Following Morgan and Scalzo (see [13]), if X is a topological space, we say that $f : X \to \mathbb{R}$ is *upper pseudocontinuous at* $z_0 \in X$ if $f(z_0) < f(z)$

implies $\limsup_{y\to z_0} f(y) < f(z)$. The class of upper pseudocontinuous functions is strictly included in the class of transfer upper continuous functions introduced by Tian and Zhou in [23].

We define the following class of games.

Definition 4.4. *The game G is called own-upper pseudocontinuous if $u_i(\cdot, s_{-i})$ is upper pseudocontinuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$.*

The next corollary can be stated.

Corollary 4.2. *The results established in this section, concerning the existence and the nonemptineness of a maximal reduction, also maintain for the class of the own-upper pseudocontinuous games.*

5. The Mixed Strategies Case

In this section, we approach the issue of mixed strategy dominance. We study several new types of dominance relations and game reductions, our work being an important extension of Dufwenberg and Stegeman's research. The two mentioned authors just distinguished between the case in which a pure strategy is dominated by a pure strategy and the case in which it is dominated by a mixed strategy and they simply considered the mixed extensions of finite games.

We find conditions under which Dufwenberg and Stegeman Lemma remains valid in the case of game reductions and then, we prove the existence and uniqueness of maximal reductions, when considering mixed strategies.

We consider $I = \{1, 2, ..., n\}$ and the game $G = (I, (G_i)_{i\in I}, (u_i)_{i\in I})$. We assume that for each $i \in I$, G_i is a compact subset of a metric space and $u_i(., s_{-i}) : G_i \to \mathbb{R}$ is upper semicontinuous for each $s_{-i} \in G_i$. Each $u_i(., s_{-i})$ is measurable since it is upper semicontinuous and since it is also bounded, it is integrable. We denote by $\Delta(G_i)$ the set of probability measures on the family of Borel sets of G_i. $\Delta(G_i)$ will be equipped with the weak topology. We recall that if $\{\mu_n\}_{n\geq 1}$, μ_n belong to $\Delta(G_i)$, then, "μ_n weakly converges to μ", written $\mu_n \overset{w}{\to} \mu$ (see [7, p. 7]) if and only if $\int_{G_i} f d\mu_n \to \int_{G_i} f d\mu$ for all $f : G_i \to \mathbb{R}$, f continuous. This topology is consistent with the Prohorov metric.

A mixed strategy for player i is an element $\mu_i \in \Delta(G_i)$. We note that G_i is compact if and only if $\Delta(G_i)$ is compact. For an overview of the notions which deal with the probability measures on metric spaces, the reader is referred to [16].

Notation. For each $i \in I$ and for each s_{-i} fixed, let us denote $V_i(., s_{-i}) : \Delta(G_i) \to \mathbb{R}$, $V_i(\mu_i, s_{-i}) = \int_{G_i} u_i(s_i, s_{-i})d\mu_i(s_i)$ for each $\mu_i \in \Delta(G_i)$.

Remark 5.1. The upper semicontinuity of $V_i(\cdot, s_{-i}) : \Delta(G_i) \to \mathbb{R}$, with s_{-i} fixed, is a consequence of Lemma 5.1.

Lemma 5.1. (see [19]) *Consider $u : K \to \mathbb{R}$ an upper semicontinuous function, where K is a compact metric space. It follows that $\int_K u d\mu$ is upper semicontinuous in μ : $\limsup_n \int_K u d\mu_n \leq \int_K u d\mu$ if $(\mu_n)_n, \mu \in \Delta(K)$, the set of probability measures on Borel sets of K and $\mu_n \overset{w}{\to} \mu$.*

5.1. Types of dominance relations and reductions

The aim of this subsection is to define new types of dominance relations and game reductions, by considering mixed strategies.

In the setting mentioned above, let G be a game. For each $i \in I$, we define the following extension of $\overset{H}{\succ}_i$, when a strategy $x \in G_i$ is dominated by a mixed strategy $\mu_i \in \Delta(G_i)$.

Let H be a paring of G.

Definition 5.1. *Let $i \in I$, $x_i \in G_i$ and $\mu_i \in \Delta(G_i)$. We say that μ_i strictly dominates x_i with respect to H and we denote $\mu_i \overset{H}{\succ}_i x_i$, if $H_{-i} \neq \emptyset$ and $V_i(\mu_i, s_{-i}) > u_i(x_i, s_{-i})$ for each $s_{-i} \in H_{-i}$.*

We note that if $H = G$, we obtain that $\mu_i \overset{G}{\succ}_i x_i$ if $G_{-i} \neq \emptyset$ and $V_i(\mu_i, s_{-i}) > u_i(x_i, s_{-i})$ for each $s_{-i} \in G_{-i}$. The last dominance relation will be used to define a game reduction.

The next result improves the Dufwenberg-Stegeman Lemma. It states the existence, in the initial game G, of an element $z_i^* \in H_i$, which is undominated by mixed strategies from $\Delta(G_i)$ for $i \in I$.

Lemma 5.2. *Let $G \to^* H$ for some compact and own-upper semicontinuous game G. Let $i \in I$ and $x_i, y_i \in G_i$ be such that $y_i \overset{H}{\succ}_i x_i$. Then, there exists $z_i^* \in H_i$ such that $\mu_i \overset{H}{\not\succ}_i z_i^* \overset{H}{\succ}_i x_i$ for each $\mu_i \in \Delta(G_i)$.*

Proof. The assumptions of Dufwenberg-Stegeman Lemma are fulfilled. According to this lemma, there exists $z_i^* \in H_i$ such that $z_i \overset{H}{\not\succ}_i z_i^* \overset{H}{\succ}_i x_i$ for each $z_i \in G_i$. It follows that $u_i(z_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for each $s_i \in G_i$ and $s_{-i} \in H_{-i}$, and therefore,

$$u_i(z_i^*, s_{-i}) \geq \int_{G_i} u_i(s_i, s_{-i}) d\mu_i(s_i) \text{ for each } s_{-i} \in H_{-i}. \tag{5.1}$$

We will prove that, in addition, $\mu_i \overset{H}{\not\succ}_i z_i^*$ for each $\mu_i \in \Delta(G_i)$.

Let us suppose, by way of contradiction, that $\mu_i \overset{H}{\succ}_i z_i^*$ for some $\mu_i \in \Delta(G_i)$. Then, $\int_{G_i} u_i(s_i, s_{-i})d\mu_i(s_i) > u_i(z_i^*, s_{-i})$ for each $s_{-i} \in H_{-i}$, relation which contradicts (5.1). Hence, $\mu_i \overset{H}{\not\succ}_i z_i^*$ for each $\mu_i \in \Delta(G_i)$. The proof is complete. $\square$

Further, we will work with borelian parings of the game G.

Definition 5.2. *We say that the paring H of the game G is borelian, if for each $i \in I$, H_i is a borelian subset of G_i.*

Let us consider the parings K and H of the game G such that K is borelian and for each $i \in I$, $H_i \subseteq K_i$.

In addition to the game reduction used by Dufwenberg and Stegeman in [10], we present the following one.

Definition 5.3. (i) *$K \mapsto H$ if, for each $i \in I$ and $x_i \in K_i \backslash H_i$, there exists $\mu_i \in \Delta(K_i)$ such that $\mu_i \overset{K}{\succ}_i x_i$.*

(ii) *The reduction $K \mapsto H$ is called fast, if for each $i \in I$, and for each $x_i \in K_i$, which is dominated by an element $\mu_i \in \Delta(K_i)$ (that is, $\mu_i \overset{K}{\succ}_i x_i$), we have $x_i \notin H_i$.*

Remark 5.2. Suppose that K fulfills the condition $(\mathcal{C})$:

$(\mathcal{C})$ K is borelian and for each $i \in I$, the set $C_i := \{x_i \in K_i : V_i(\mu_i, s_{-i}) \leq u_i(x_i, s_{-i})$ for each $\mu_i \in \Delta(K_i)\}$ is borelian.

If $K \mapsto H$ fast, then, $K_i \backslash H_i$ and H_i are borelian.

Definition 5.4. *The reduction $K \mapsto^* H$ is defined by the existence of (finite or countable infinite) sequence of borelian parings A^t of K, $t = 0, 1, 2...$, such that for each $t \geq 0$, $A^0 = K$, $A^t \mapsto A^{t+1}$, and for each $i \in I$, $H_i = \cap_t A_i^t$.*

Now, we are defining the maximal reduction of the game.

Definition 5.5. *H is said to be a maximal $(\mapsto^*)$-reduction of the game G, if $G \mapsto^* H$ and $H \mapsto H'$, only for $H' = H$.*

Remark 5.3. If the game G satisfies the condition $(\mathcal{C})$, then H, the maximal $(\mapsto^*)$-reduction of G, is borelian, since for each $i \in I$, H_i is the intersection of borelian subsets of G_i.

Now, let H be a borelian paring of G. In this context, we make the following remarks.

Remark 5.4. Let us consider $i \in I$. The mixed strategy $m_i \in \Delta(G_i)$ is strictly dominated by the mixed strategy $\mu_i \in \Delta(G_i)$ (and it is denoted $\mu_i \overset{\Delta(H)}{\succ_i} m_i$) if $U_i(\mu_i, \mu_{-i}) > U_i(m_i, \mu_{-i})$ for each $\mu_{-i} \in \prod_{j \neq i} \Delta(H_j)$, where $U_i : \Delta(G_i) \times \prod_{j \neq i} \Delta(H_j) \to \mathbb{R}$ is defined by

$$U_i(\mu_i, \mu_{-i}) = \int_{G_i \times H_{-i}} u_i(s_i, s_{-i}) d\mu_1(s_1) \times ... \times d\mu_n(s_n)$$

for each $(\mu_i)_{i \in I} \in \Delta(G_i) \times \prod_{j \neq i} \Delta(H_j)$.

In the particular case when $\int_{G_i} u_i(s_i, s_{-i}) d\mu_i(s_i) > \int_{G_i} u_i(s_i, s_{-i}) dm_i(s_i)$ for each $s_{-i} \in H_{-i}$, we will denote $\mu_i \overset{H}{\succ_i} m_i$. Obviously, $\mu_i \overset{\Delta(H)}{\succ_i} m_i \Rightarrow \mu_i \overset{H}{\succ_i} m_i$.

Let us consider the borelian parings K and H of G such that for each $i \in I$, $H_i \subseteq K_i$. The associated reduction of game is defined as follows.

Definition 5.6. $\Delta(K) \hookrightarrow \Delta(H)$ *if, for each $i \in I$ and $m_i \in \Delta(K_i) \backslash \Delta(H_i)$, there exists $\mu_i \in \Delta(K_i)$ such that $\mu_i \overset{K}{\succ_i} m_i$.*

The next lemma provides the condition under which the reduction $\Delta(K) \hookrightarrow \Delta(H)$ is obtained.

Lemma 5.3. *Let us consider the game G and the borelian parings K and H of G, such that $H_i \subseteq K_i$ for each $i \in I$. Let us assume that $K \mapsto H$. If for each $i \in I$ there exists $s_i^* \in K_i \backslash H_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for each $s_i \in K_i \backslash H_i$ and $s_{-i} \in K_{-i}$, then $\Delta(K) \hookrightarrow \Delta(H)$.*

Proof. Let $i \in I$ and let s_i^* be such that $s_i^* \in K_i \backslash H_i$ and $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for each $s_i \in K_i \backslash H_i$ and $s_{-i} \in K_{-i}$. Then,

$$\int_{K_i \backslash H_i} u_i(s_i, s_{-i}) dm_i(s_i) \leq u_i(s_i^*, s_{-i}) \text{ for each } m_i \in \Delta(K_i \backslash H_i). \tag{5.2}$$

Since $s_i^* \in K_i \backslash H_i$ and $K \mapsto H$, it follows that there exists $\mu_i \in \Delta(K_i)$ such that $\mu_i \overset{K}{\succ_i} s_i^*$, that is

$$\int_{K_i} u_i(s_i, s_{-i}) d\mu_i(s_i) > u_i(s_i^*, s_{-i}) \text{ for each } s_{-i} \in K_{-i}. \tag{5.3}$$

The relations (5.2) and (5.3) guarantee that for each $i \in I$ and $m_i \in \Delta(K_i) \backslash \Delta(H_i)$, there exists $\mu_i \in \Delta(K_i)$ such that $\int_{K_i} u_i(s_i, s_{-i}) d\mu_i(s_i) > \int_{K_i \backslash H_i} u_i(s_i, s_{-i}) dm_i(s_i)$ for each $s_{-i} \in K_{-i}$, that is $\mu_i \overset{K}{\succ_i} m_i$. Therefore, $\Delta(K) \hookrightarrow \Delta(H)$. □

5.2. Existence of undominated strategies

We obtain the next result concerning the game reduction $G \mapsto^* H$. It is an extension of the Dufwenberg and Stegeman Lemma in the mixed strategies case.

Lemma 5.4. *Let $G = (I, (G_i)_{i\in I}, (u_i)_{i\in I})$, where I is a finite set, and for each $i \in I$, G_i is a compact subset of a metric space X considered with its borelian sets and $u_i : \prod_{k\in I} G_k \to \mathbb{R}_+$ is upper semicontinuous in each argument. Let $G \mapsto^* H$ and suppose that $\Delta(G) \hookrightarrow^* \Delta(H)$. Let $i \in I$, $x_i \in G_i$ and $\mu'_i \in \Delta(G_i)$ such that $\mu'_i \overset{H}{\succ}_i x_i$. Then, there exists $\mu^*_i \in \Delta(H_i)$ with the property that $\mu_i \overset{H}{\not\succ}_i \mu^*_i \overset{H}{\succ}_i x_i$ for each $\mu_i \in \Delta(G_i)$.*

Proof. We note that if the game G is compact, then $\Delta(G)$ is also compact. According to Remark 5.1, if for $i \in I$, $u_i(., s_{-i})$ is upper semicontinuous for each $s_{-i} \in G_{-i}$, then $V_i(., s_{-i})$ is also upper semicontinuous on $\Delta(G_i)$ for each $s_{-i} \in G_{-i}$, where $V_i(\mu_i, s_{-i}) = \int_{G_i} u_i(s_i, s_{-i}) d\mu_i(s_i)$.

Since $G \mapsto^* H$, there exists a sequence of borelian parings A^t, $t = 0, 1, 2...$ such that $A^0 = G$, $A^t \mapsto A^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t A^t_i$, for each $i \in I$. It follows that for each $i \in I$, H_i is a borelian subset of G_i.

Let $i \in I$, $x_i \in G_i$ and $\mu'_i \in \Delta(G_i)$, such that $\mu'_i \overset{H}{\succ}_i x_i$. Hence, according to the definition of $\overset{H}{\succ}_i$, H_{-i} is nonempty and the set Z_i is well defined, where $Z_i = \{\mu_i \in \Delta(G_i) : V_i(\mu_i, s_{-i}) \geq V_i(\mu'_i, s_{-i})$ for each $s_{-i} \in H_{-i}\}$.

Notice that Z_i is nonempty, since $\mu'_i \in Z_i$. The set Z_i is also closed (as intersection of the closed sets $G_i(s_{-i}) = \{\mu_i \in \Delta(G_i) : V_i(\mu_i, s_{-i}) \geq V_i(\mu'_i, s_{-i}),\ s_{-i} \in H_{-i}\}$) and therefore, compact. Now, for $s^*_{-i} \in H_{-i}$ fixed, let us define $f_i : Z_i \to \mathbb{R}$, $f_i(\mu_i) = V_i(\mu_i, s^*_{-i})$ for each $\mu_i \in Z_i$. The function f_i is upper semicontinuous and it reaches its maximum on the compact set Z_i. Let us denote $\mu^*_i = \arg\max_{\mu_i \in Z_i} f_i(\mu_i)$.

By using a similar argument to the one in the proof of the Dufwenberg and Stegeman Lemma, we obtain that there exists $\mu^*_i \in \Delta(G_i)$, such that $\mu_i \overset{H}{\not\succ}_i \mu^*_i \overset{H}{\succ}_i x_i$ for each $\mu_i \in \Delta(G_i)$. Therefore, $\mu_i \overset{H}{\not\succ}_i \mu^*_i$ for each $\mu_i \in \Delta(G_i)$ and then $\mu_i \overset{A^t}{\not\succ}_i \mu^*_i$ for each $\mu_i \in \Delta(G_i)$ and $t \geq 0$. Since $\Delta(G) \hookrightarrow^* \Delta(H)$, we conclude that $\mu^*_i \in \Delta(A^t)$ for each $t \geq 0$, and this proves that one has $\mu^*_i \in \Delta(H_i)$. □

For $H = G$, we obtain the following corollary.

Corollary 5.1. *Let $G = (I, (G_i)_{i\in I}, (u_i)_{i\in I})$, where I is a finite set, and for each $i \in I$, G_i is a compact subset of a metric space X considered with its borelian sets and $u_i : \prod_{k\in I} G_k \to \mathbb{R}_+$ is upper semicontinuous in each* *argument. Let $i \in I$, $x_i \in G_i$ and $\mu_i' \in \Delta(G_i)$ such that $\mu_i' \overset{G}{\succ}_i x_i$. Then, there exists $\mu_i^* \in \Delta(G_i)$ with the property that $\mu_i \overset{G}{\not\succ}_i \mu_i^* \overset{G}{\succ}_i x_i$ for each $\mu_i \in \Delta(G_i)$.*

The following result is a consequence of Lemma 5.3.

Corollary 5.2. *Lemma 5.4 is true, if, instead of having the assumption $\Delta(G) \hookrightarrow^* \Delta(H)$, we have the following one: for each $i \in I$, there exists $s_i^* \in G_i \backslash H_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for each $s_i \in G_i \backslash H_i$ and $s_{-i} \in G_{-i}$.*

Proof. The proof of the corollary is based on Lemma 5.3. □

5.3. Maximal reductions; existence and uniqueness

The main result of Section 5 is Theorem 5.1. It states the existence of a unique maximal ($\mapsto^*$) reduction of an own-upper semicountinuous game.

Theorem 5.1. *Let $G = (I, (G_i)_{i \in I}, (u_i)_{i \in I})$ be a strategic game, such that I is a finite set and for each $i \in I$, G_i is a nonempty compact subset of a metric space, $u_i : \prod_{k \in I} G_k \to \mathbb{R}$ is upper semicontinuous in each argument and the set $C_i := \{x_i \in G_i : u_i(y_i, s_{-i}) \leq u_i(x_i, s_{-i})$ for each $y_i \in G_i\}$ is borelian. Assume that for each $G \mapsto H$, $\Delta(G) \hookrightarrow \Delta(H)$ (or, for each $i \in I$, there exists $s_i^* \in G_i \backslash H_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for each $s_i \in G_i \backslash H_i$ and $s_{-i} \in G_{-i}$.) Then, any nonempty maximal ($\mapsto^*$) reduction M of G is the unique maximal ($\mapsto^*$) reduction of G. In addition, M is compact and own-upper semicontinuous.*

Proof. We begin by making the following remarks. The game $(I, (\Delta(G_i))_{i \in I}, (V_i)_{i \in I})$ is also compact and own-upper semicontinuous. According to Lemma 5.4, we have that: if $\mu_i' \overset{H}{\succ}_i x_i$ for some $x_i \in G_i$ and $\mu_i' \in \Delta(G_i)$, $i \in I$, and if H_i is borelian, then there exists $\mu_i^* \in \Delta(H_i)$, such that $\mu_i \overset{H}{\not\succ}_i \mu_i^* \overset{H}{\succ}_i x_i$ for each $\mu_i \in \Delta(G_i)$. The set $\Delta(H_i)$ is nonempty, if H_i is nonempty.

The proof of the uniqueness of a nonempty maximal ($\mapsto^*$) reduction M of G follows the same line as the proof of Theorem 1a) of Dufwenberg-Stegeman.

Recall that the nonempty maximal ($\mapsto^*$) reduction M is the result of an iterative process of reductions. In order to prove that M is compact, we will firstly show that if G is compact and own-upper semicontinuous, then, for each nonempty paring H of G with the property that $G \mapsto H$ fast, it is true that H is compact.

Let $G \mapsto H$ fast and choose $i \in I$, such that $\emptyset \neq H_i \neq G_i$. According to Remark 5.1, we know that H_i is borelian. It remains to show that H_i is compact. Choose $\mu_i \in \Delta(H_i)$ and let

$$Z_i(\mu_i) = \{(s_i, s_{-i}) \in \prod_{k\in I} G_k : V_i(\mu_i, s_{-i}) \leq u_i(s_i, s_{-i})\},\ P_i(\mu_i) = \mathrm{pr}_i Z_i(\mu_i)$$

and $P_{-i}(\mu_i) = \mathrm{pr}_{-i} Z_i(\mu_i)$.

The set $P_i(\mu_i)$ is nonempty. In order to prove this fact, we will assume the opposite: $P_i(\mu_i) = \emptyset$. In this case, $V_i(\mu_i, s_{-i}) > u_i(s_i, s_{-i})$ for each $s_i \in G_i$ and for each $s_{-i} \in G_{-i}$. We can conclude that $\mu_i \overset{G}{\succ_i} s_i$ for each $s_i \in G_i$, and, since $G \mapsto H$ fast, we have that each $s_i \in G_i$ implies $s_i \notin H_i$. Then, H_i is an empty set, and we reached a contradiction.

Now, let us define $F_i(\mu_i, s_{-i}) = \{s_i \in G_i : V_i(\mu_i, s_{-i}) \leq u_i(s_i, s_{-i})\}$ for each $\mu_i \in \Delta(H_i)$ and $s_{-i} \in G_{-i}$. Then, for each $\mu_i \in \Delta(H_i)$, $P_i(\mu_i) = \cap_{s_{-i}\in Z_{-i}(\mu_i)} F_i(x_i, s_{-i})$. Since $u_i(\cdot, s_{-i})$ is upper semicontinuous for each $s_{-i} \in G_{-i}$, we have that, for each $\mu_i \in \Delta(H_i)$, $P_i(\mu_i)$ is closed as being an intersection of closed sets.

We claim that $H_i = \cap_{\mu_i\in\Delta(H_i)} P_i(\mu_i)$. To see this, let us consider $x_i \in G_i$. For any $\mu_i \in \Delta(H_i)$, if $x_i \notin P_i(\mu_i)$, we have that $V_i(\mu_i, s_{-i}) > u_i(x_i, s_{-i})$ for each $s_{-i} \in G_{-i}$ and, therefore, $\mu_i \overset{G}{\succ_i} x_i$. Thus, $x_i \notin H_i$ and $H_i \subseteq \cap_{\mu_i\in\Delta(H_i)} P_i(\mu_i)$. To obtain the reverse inclusion, notice that, if $x_i \notin H_i$, then $\mu_i \overset{G}{\succ_i} x_i$ for some $\mu_i \in \Delta(G_i)$ and Lemma 5.4 implies that there exists $\mu_i^* \in \Delta(H_i)$ with the property that $\mu_i^* \overset{H}{\succ_i} x_i$ and then, $x_i \notin P_i(\mu_i^*)$. We can conclude that $x_i \notin \cap_{\mu_i\in\Delta(H_i)} P_i(\mu_i)$. Hence, $H_i \supseteq \cap_{\mu_i\in\Delta(H_i)} P_i(\mu_i)$. The equality $H_i = \cap_{\mu_i\in\Delta(H_i)} P_i(\mu_i)$ holds and, since $P_i(\mu_i)$ is closed for all $\mu_i \in \Delta(H_i)$, H_i is also closed and therefore, compact.

Consequently, we proved that H is compact.

Let $C(t)$, $t = 0, 1, ...$ denote the unique sequence of subgames of G such that $C(0) = G$ and $C(t) \mapsto C(t+1)$ is fast for each $t \geq 0$. By induction, we have that $C(t)$ is compact for each $t \geq 0$.

Let us also consider the case when $C(t)$ is nonempty for each $t \geq 0$. It follows that $M_i = \cap_t C(t)_i$ is compact and nonempty for each $i \in I$.

We will show that M is a maximal $(\mapsto^*)$ reduction of G. Let us consider $i \in I$, $x_i \in M_i$ and $\mu_i \in \Delta(M_i)$. Let $X_{-i}(t) = \{s_{-i} \in C(t)_{-i} : V_i(\mu_i, s_{-i}) \leq u_i(x_i, s_{-i})\}$. If $X_{-i}(t) = \emptyset$ for each t such that $C(t) \neq M$, then $\mu_i \overset{C(t)}{\succ_i} x_i$, which is a contradiction. Therefore, $X_{-i}(t) \neq \emptyset$. The set $C(t)_{-i}$ is compact for each t such that $C(t) \neq M$. Then, $M_{-i} \neq \emptyset$ and it follows that the set $X_{-i} = \{s_{-i} \in M_{-i} : V_i(\mu_i, s_{-i}) \leq u_i(x_i, s_{-i})\}$ is nonempty. We conclude that $\mu_i \overset{M}{\not\succ_i} x_i$, and this proves that M is maximal. $\square$

5.4. Other types of game reductions and open problems

The aim of this subsection is to open a direction of study which deals with the problem of order independence. Thus, we provide new types of game reduction in the context of mixed strategies. The object of the future research is to prove new results of the type of those obtained in this paper.

Let us consider the parings K and H of the game G, such that $H_i \subseteq K_i$ for each $i \in I$.

We introduce the next definition, concerning the removal of the strategies which are strictly dominated by strategies from the paring (H, respectively ΔH), and not from the initial game (K, respectively ΔK) .

Definition 5.7. (see [11]) $K \Rightarrow H$, *if, for each* $i \in I$ *and* $x_i \in K_i \backslash H_i$, *there exists* $y_i \in H_i$ *such that* $y_i \overset{K}{\succ_i} x_i$.

If, in addition, H and K are borelian, we introduce the following definition.

Definition 5.8. (i) $K \rightrightarrows H$, *if, for each* $i \in I$ *and* $x_i \in K_i \backslash H_i$, *there exists* $\mu_i \in \Delta(H_i)$ *such that* $\mu_i \overset{K}{\succ_i} x_i$.

(ii) $\Delta(K) \Rrightarrow \Delta(H)$, *if, for each* $i \in I$ *and* $m_i \in \Delta(K_i)\backslash\Delta(H_i)$, *there exists* $\mu_i \in \Delta(H_i)$ *such that* $\mu_i \overset{K}{\succ_i} m_i$.

We will need the following theorem, which considers the reductions defined above, and studies the way they are related.

Theorem 5.2. *There are the following relations amongst the former types of reductions.*

(i) $(K \Rightarrow H) \Longrightarrow (K \to H)$
$(K \rightrightarrows H) \Longrightarrow (K \mapsto H)$
$(\Delta(K) \Rrightarrow \Delta(H)) \Longrightarrow (\Delta(K) \hookrightarrow \Delta(H))$

(ii) $(\Delta(K) \Rightarrow \Delta(H)) \Longrightarrow (\Delta(K) \Rrightarrow \Delta(H)) \Longrightarrow (K \rightrightarrows H)$

(iii) $(\Delta(K) \to \Delta(H)) \Longrightarrow (\Delta(K) \hookrightarrow \Delta(H)) \Longrightarrow (K \mapsto H)$.

Proof.

(i) The proof is obvious.

(ii) Suppose $(\Delta(K) \Rightarrow \Delta(H))$. It follows that for each $i \in I$, $\Delta(H_i) \subset \Delta(K_i)$ and for each $m_i \in \Delta(K_i)\backslash\Delta(H_i)$, there exists $\mu_i \in \Delta(H_i)$, such that $\mu_i \overset{\Delta(K)}{\succ_i} m_i$. Then, we obtain that, for each $i \in I$, $H_i \subset K_i$ and for each $m_i \in \Delta(K_i)\backslash\Delta(H_i)$, there exists $\mu_i \in \Delta(H_i)$ such that $\mu_i \overset{K}{\succ_i} m_i$. Consequently, $\Delta(K) \Rrightarrow \Delta(H)$.

Now, let us consider that, for each $i \in I$, $m_i = \delta_{x_i}$, where $x_i \in K_i\backslash H_i$ and δ_{x_i} is the Dirac measure with unit mass at x_i, that is, $\delta_{x_i}(E_i) = \begin{cases} 1, & \text{if } x_i \in E_i; \\ 0, & \text{if } x_i \notin E_i \end{cases}$ for each Borel subset E_i of $K_i\backslash H_i$. We have that $H_i \subset K_i$

and for each $x_i \in K_i \backslash H_i$, there exists $\mu_i \in \Delta(H_i)$ such that $\mu_i \overset{K}{\succ}_i x_i$, which is equivalent with $K \rightrightarrows H$.

(iii) The implication is true from (i) and (ii). □

In the case of a finite game, we obtain Theorem 5.3, which states the equivalence between two types of game reduction. For the proof, we firstly recall that: if F is a finite set with a discrete $\sigma-$algebra $\mathcal{F}$, then, every probability μ on this measurable space can be uniquely represented in the form $\mu = \sum_{x \in F} c_x \delta_x$, where $c_x \in [0,1]$ for each $x \in X$, $\sum_{x \in F} c_x = 1$ and δ_x is the Dirac measure with unit mass at x. Thus, $\mu(E) = \sum_{x \in E} c_x$ for all $E \subset F$.

Theorem 5.3. *Let us consider G a finite game and the parings K and H of G, such that $H_i \subseteq K_i$ for each $i \in I$. Then, $(\Delta(K) \Rrightarrow \Delta(H)) \iff (K \rightrightarrows H)$.*

Proof. The direct implication comes from Theorem 5.2 (ii).

In order to prove the converse implication, let us consider $i \in I$ and let $x_i \in K_i \backslash H_i$. Since $K \rightrightarrows H$, there exists $\mu_{x_i} \in \Delta(H_i)$ such that $\mu_{x_i} \overset{K}{\succ}_i x_i$. Obviuously, $\mu_{x_i} \overset{K}{\succ}_i \delta_{x_i}$.

Let $m_i \in \Delta(K_i) \backslash \Delta(H_i)$. Hence, m_i can be uniquely represented as a convex combination of Dirac measures δ_{x_i}, $x_i \in K_i \backslash H_i$. It is clear that there exists unique $c_{x_i} \in [0,1]$, $\sum_{x_i \in K_i \backslash H_i} c_{x_i} = 1$ such that $m_i = \sum_{x_i \in K_i \backslash H_i} c_{x_i} \delta_{x_i}$. But, as we noted above, for each δ_{x_i} with $x_i \in K_i \backslash H_i$, there exists $\mu_{x_i} \in \Delta(H_i)$ such that $\mu_{x_i} \overset{K}{\succ}_i \delta_{x_i}$. Therefore, for each $i \in I$, $\mu_i = \sum_{x_i \in K_i \backslash H_i} c_{x_i} \mu_{x_i}$ is a probability measure on $K_i \backslash H_i$ and $\mu_{x_i} \overset{K}{\succ}_i m_i$. This proves that $\Delta(K) \Rrightarrow \Delta(H)$. □

The equivalence stated above does not maintain outside the class of finite games. The following theorem provides a condition under which the reverse of Theorem 5.2 (ii) is true.

Theorem 5.4. *Let us consider the game G and the borelian parings K and H of G, such that $H_i \subseteq K_i$ for each $i \in I$. Let $K \rightrightarrows H$. If for each $i \in I$, there exists $s_i^* \in K_i \backslash H_i$, such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for each $s_i \in K_i \backslash H_i$ and $s_{-i} \in K_{-i}$, then $\Delta(K) \Rrightarrow \Delta(H)$.*

Proof. The proof follows the same line as the one of Lemma 5.3. □

6. Concluding remarks

We underline that the motivation of our work is the fact that many economic models have discontinuous payoff functions. The economists have been searching for weaker conditions that can guarantee the existence of an equilibrium. Important results concerning the equilibrium existence for games with discontinuous payoff functions are due to Dasgupta and Maskin (see [9]), Simon (see [20]), Simon and Zame (see [21]), Baye, Tian, and Zhou (see [5]), Reny (see [18]), Nessah and Tian (see [14]) or Barelli and Soza (see [4]). The conditions of transfer upper continuity and transfer weakly upper continuity, introduced by Tian and Zhou in [23], are satisfied in many economic games and are often quite simple to be checked.

In this paper, we identified a class of discontinuous games for which the iterated elimination of strictly dominated strategies produce a unique maximal reduction that is nonempty. We also provided conditions under which order independence remains valid for the case that the pure strategies are dominated by mixed strategies. Our results expel Dufwenberg and Stegeman's idea in [10] that 'the proper definition and the role of iterated strict dominance is unclear for games that are not compact and continuous'. Tian and Zhou's notion of transfer upper continuity proved to be a suitable assumption for the payoff functions of a game in order to obtain our results. Their Weierstrass-like theorem for transfer weakly upper continuous functions defined on a compact set was the key of the proofs of Lemma 3.1 and Lemma 3.2. We can conclude and emphasize that, even outside the continuous class of games, the iterated elimination of strictly dominated strategies remains an interesting procedure. A further attention is needed for rigorous formalization of the concepts concerning rationality in different classes of games, and for reopening the discussion of the problems in a unified framework which implies economic settings.

Acknowledgements

We thank to Professor Krzysztof Apt for the precious idea under which this paper has been developed and for the hospitality he proved during our visit at the Institute for Logic, Language and Computation at the University of Amsterdam.

7. Appendix

We add a list with the main notations used in this paper, in order to make the reading easier.

List of notations

Correspondence (set valued map): $T : X \to 2^Y$.

Strategic game: $\Gamma = (G_i, u_i)_{i \in I}$, where $u_i : \prod_{k \in I} G_k \to \mathbb{R}$.

$G_{-i} := \prod_{j \in I \setminus \{i\}} G_j$.

$s_{-i}=(s_1,...,s_{i-1},s_{i+1},...)\in G_{-i}$, if $s\in\prod_{i\in I}G_i$.

Paring of $G: H=(H_i,u_{i|\prod_{k\in I}H_k})_{i\in I}$, where $H_i\subseteq G_i$.

$y\overset{H}{\succ_i}x$: $x,y\in G_i$, $H_{-i}\neq\emptyset$ and $u_i(y_i,s_{-i})>u_i(x_i,s_{-i})$, for each $s_{-i}\in H_{-i}$.

$K\to H$: for each $i\in I$ and $x\in K_i\backslash H_i$, there exists $y\in K_i$ such that $y\overset{K}{\succ_i}x$.

$K\to H$ is fast: for each $i\in I$, $K_{-i}\neq\emptyset$ and $y\overset{H}{\succ_i}x$ for some $x\in K_i$ implies $x\notin H_i$.

$K\to^* H$: there exists a sequence of restrictions R^t of H, $t=0,1,2...$, such that $R^0=K$, $R^t\to R^{t+1}$ fast for each $t\geq 0$ and $H_i=\cap_t R_i^t$ for each $i\in I$.

$K\to^* H$ is maximal: $K\to^* H$ and $H\to H'$ only for $H=H'$.

$\Delta(G_i)-$ the set of probability measures on the family of Borel sets of G_i.

$V_i(.,s_{-i}):\Delta(G_i)\to\mathbb{R}$, $V_i(\mu_i,s_{-i})=\int_{G_i}u_i(s_i,s_{-i})d\mu_i(s_i)$ for each $\mu_i\in\Delta(G_i)$ and for each fixed $s_{-i}\in H_{-i}$.

$\mu_i\overset{H}{\succ_i}x_i$: $\mu_i\in\Delta(G_i)$, $x_i\in G_i$, $H_{-i}\neq\emptyset$ and $V_i(\mu_i,s_{-i})>u_i(x_i,s_{-i})$ for each $s_{-i}\in H_{-i}$.

$K\mapsto H$: for each $i\in I$ and $x_i\in K_i\backslash H_i$, there exists $\mu_i\in\Delta(K_i)$ such that $\mu_i\overset{K}{\succ_i}x_i$.

$U_i:\Delta(G_i)\times\prod_{j\neq i}\Delta(H_j)\to\mathbb{R}, U_i(\mu_i,\mu_{-i})=\int_{G_i\times H_{-i}}u_i(s_i,s_{-i})d\mu_1(s_1)\times...\times d\mu_n(s_n)$ for each $(\mu_i)_{i\in I}\in\Delta(G_i)\times\prod_{j\neq i}\Delta(H_j)$.

$\mu_i\overset{\Delta(H)}{\succ_i}m_i$: $m_i,\mu_i\in\Delta(G_i)$, $U_i(\mu_i,\mu_{-i})>U_i(m_i,\mu_{-i})$ for each $\mu_{-i}\in\prod_{j\neq i}\Delta(H_j)$.

$\mu\ _i\overset{H}{\succ_i}m_i:\int_{G_i}u_i(s_i,s_{-i})d\mu_i(s_i)>\int_{G_i}u_i(s_i,s_{-i})dm_i(s_i)$ for each $s_{-i}\in H_{-i}$.

$\Delta(K)\hookrightarrow\Delta(H)$: for each $i\in I$ and $m_i\in\Delta(K_i)\backslash\Delta(H_i)$, there exists $\mu_i\in\Delta(K_i)$ such that $\mu_i\overset{K}{\succ_i}m_i$.

$K\Rightarrow H$: for each $i\in I$ and $x_i\in K_i\backslash H_i$, there exists $y_i\in H_i$ such that $y_i\overset{K}{\succ_i}x_i$.

$K\rightrightarrows H$: for each $i\in I$ and $x_i\in K_i\backslash H_i$, there exists $\mu_i\in\Delta(H_i)$ such that $mu_i\overset{K}{\succ_i}x_i$.

$\Delta(K) \Rrightarrow \Delta(H)$: for each $i \in I$ and $m_i \in \Delta(K_i) \backslash \Delta(H_i)$, there exists $\mu_i \in \Delta(H_i)$ such that $\mu_i \overset{K}{\succ_i} m_i$.

Relations amongst the former types of reductions:

i) $(K \Rightarrow H) \Longrightarrow (K \rightarrow H)$

$(K \rightrightarrows H) \Longrightarrow (K \mapsto H)$

$(\Delta(K) \Rrightarrow \Delta(H)) \Longrightarrow (\Delta(K) \hookrightarrow \Delta(H))$

ii) $(\Delta(K) \Rightarrow \Delta(H)) \Longrightarrow (\Delta(K) \Rrightarrow \Delta(H)) \Longrightarrow (K \rightrightarrows H)$

iii) $(\Delta(K) \rightarrow \Delta(H)) \Longrightarrow (\Delta(K) \hookrightarrow \Delta(H)) \Longrightarrow (K \mapsto H)$.

References

[1] K. Apt, Order Independence and Rationalizability, *Proceedings of the 10th Conference on Theoretical Aspects of Rationality and Knowledge (TARK X)*, pp. 22-38, 2005.

[2] K. Apt, The Many Faces of Rationalizability, *B. E. J. Theor. Econ.*, **7**, *1* (2007), http://dx.doi.org/10.2202/1935-1704.1339.

[3] C.W. Bach and J. Cabessa, *Limit knowledge of rationality, Theory and decision*, 2012.

[4] P. Barelli and I. Soza, *On the Existence of Nash Equilibria in Discontinuous and Qualitative Games*, mimeo, 2009.

[5] M.R. Baye, G. Tian and J. Zhou, Characterizations of the Existence of Equilibria in Games with Discontinuous and Non-Quasiconcave payoffs, *Rev. Econ. Stud.*, **60** (1993), 935-948, http://dx.doi.org/10.2307/2298107.

[6] B.D. Bernheim, Rationalizable strategic behavior, *Econometrica*, **52** (1984), 1007-1028, http://dx.doi.org/10.2307/1911196.

[7] P. Billingsley, *Convergence in Probability Measures*, Wiley, New York, 1968, http://dx.doi.org/10.1002/9780470316962.

[8] Y.C. Chen, N.V. Long and X. Luo, Iterated strict dominance in general games, *Games Econom. Behav.*, **61** (2007), 219-315, http://dx.doi.org/10.1016/j.geb.2007.02.002.

[9] P. Dasgupta and E. Maskin, The Existence of Equilibrium in Discontinuous Economic Games, I: Theory, *Rev. Econ. Stud.*, **53** (1986), 1-26, http://dx.doi.org/10.2307/2297588.

[10] M. Dufwenberg and M. Stegeman, Existence and uniqueness of maximal reductions under iterated strict dominance, *Econometrica*, **70** (2002), 2007-2023, http://dx.doi.org/10.1111/1468-0262.00360.

[11] I. Gilboa, E. Kalai and E. Zemel, On the order of eliminating dominated strategies, *Oper. Res. Lett.*, **9** (1990), 85-89, http://dx.doi.org/10.1016/0167-6377(90)90046-8.

[12] L. Marx and J. Swinkels, Order independence for iterated weak dominance, *Games Econom. Behav.*, **18** (1997), 219-245, http://dx.doi.org/10.1006/game.1997.0525.

[13] J. Morgan and V. Scalzo, Pseudocontinuous functions and existence of Nash equilibria, *J. Math. Econom.*, **43**, *2* (2007), 174-183, http://dx.doi.org/10.1016/j.jmateco.2006.10.004.

[14] R. Nessah and G. Tian, *The Existence of Equilibria in Discontinuous and Nonconvex Games*, working paper, 2008.

[15] M.J. OSBORNE and A. RUBINSTEIN, *A Course in Game Theory*, The MIT Press, Cambridge, Massachusetts, 1994.

[16] K.R. PARTHASARATHY, *Probability Measures on Metric Spaces*, AMS Chelsea Publishing, Providence, Rhode Island, 2005.

[17] D.G. PEARCE, Rationalizable strategic behavior and the problem of perfection, *Econometrica*, **52** (1984), 1029-1050, http://dx.doi.org/10.2307/1911197.

[18] J.P. RENY, On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games, *Econometrica*, **67** (1999), 1029-1056, http://dx.doi.org/10.1111/1468-0262.00069.

[19] A. ROBSON, *Existence of Nash Equilibrium in mixed strategies for games where payoffs need not be continuous in pure strategies*, Discussion paper, 1990.

[20] L. SIMON, Games with Discontinuous payoffs, *Rev. Econ. Stud.*, **54** (1987), 569-597, http://dx.doi.org/10.2307/2297483.

[21] L. SIMON and W. ZAME, Discontinuous Games and Endogenous Sharing Rules, *Econometrica*, **58** (1990), 861-872, http://dx.doi.org/10.2307/2938353.

[22] G. TIAN, *Existence of Equilibria in Games with Arbitrary Strategy Spaces and payoffs: A Full Characterization*, Working paper, 2009.

[23] G. TIAN and J. ZHOU, Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization, *J. Math. Econom.*, **24** (1995), 281-303, http://dx.doi.org/10.1016/0304-4068(94)00687-6.

Monica Patriche
University of Bucharest, Faculty of Mathematics and Computer Science
14 Academiei Street, 010014 Bucharest, Romania
E-mail: monica.patriche@yahoo.com