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Comparative growth measures of differential monomials and differential polynomials depending on their relative orders

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Abstract - In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative order (relative lower order) and differential monomials, differential polynomials generated by one of the factors.

Key words and phrases : Entire function, meromorphic function, order (lower order), relative order (relative lower order), Property (A), growth, differential monomial, differential polynomial.

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1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [22] and [28].

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, ... n_{kj} (k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. We call $M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} ... \left(f^{(k)}\right)^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{Mj} = \sum\limits_{i=0}^{k} n_{ij}$ and $\Gamma_{Mj} = \sum\limits_{i=0}^{k} (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ (see [21], [26]). The expression $P[f] = \sum\limits_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max\limits_{1 \leq j \leq s} \gamma_{Mj}$ and $\Gamma_P = \max\limits_{1 \leq j \leq s} \Gamma_{Mj}$ are called respectively the degree and weight of $P[f]$ (see [21], [26]). Also we call the numbers $\underline{\gamma}_P = \min\limits_{1 \leq j \leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\gamma_p = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e., for which $n_{0j} = 0$ for $j = 1, 2, \ldots s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

In the sequel the following definitions are well known:

Definition 1.1. *Let 'a' be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of 'a' with respect to a meromorphic function f are defined as*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N_f(r,a)}{T_f(r)} = \liminf_{r\to\infty} \frac{m_f(r,a)}{T_f(r)}$$

and

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N_f(r,a)}{T_f(r)} = \limsup_{r\to\infty} \frac{m_f(r,a)}{T_f(r)}.$$

Definition 1.2. *The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows*

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_f(r,a)}{T_f(r)}.$$

Definition 1.3. (see [30]) *For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n_{f|=1}(r,a)$, the number of simple zeros of $f-a$ in $|z| \le r$. $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way. We put*

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N_{f|=1}(r,a)}{T_f(r)},$$

the deficiency of 'a' corresponding to the simple a-points of f i.e., simple zeros of $f-a$.

Yang proved in [29] that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which one has $\delta_1(a;f) > 0$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \le 4$.

Definition 1.4. (see [23]) *For $a \in \mathbb{C}\cup\{\infty\}$, let $n_p(r,a;f)$ denote the number of zeros of $f-a$ in $|z| \le r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly* *p times and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define*

$$\delta_p(a;f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T_f(r)}.$$

Definition 1.5. (see [4]) *$P[f]$ is said to be admissible if*

(i) *$P[f]$ is homogeneous, or*

(ii) *$P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.*

During the past decades, several authors (see [5]−[16], [25]) made closed investigations on comparative study of the growth properties of composite entire or meromorphic functions in different directions using order (lower order) and differential polynomials and differential monomials generated by one of the factors. The growth indicator such as order (lower order) of entire or meromorphic function which is generally used in computational purpose is defined in terms of the growth of that function with respect to the exponential function is shown in the following definition:

Definition 1.6. *The order ρ_f (the lower order λ_f) of an entire function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)}\right).$$

When f is a meromorphic, one may easily prove that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

Both entire and meromorphic function have regular growth if their order coincides with thier lower order.

For a non-constant entire function f, $M_f(r)$ and $T_f(r)$ are both strictly increasing and continuous functions of r and their inverses $M_f^{-1}(r) : (|f(0)|, \infty) \to (0,\infty)$ and $T_f^{-1} : (T_f(0), \infty) \to (0,\infty)$ respectively exist where $\lim_{s\to\infty} M_g^{-1}(s) = \infty$ and $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. In this connection we just recall the following definition which is relevant to our study:

Definition 1.7. (see [3]) *A non-constant entire function f is said have the property* (A) *if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \le M_f(r^\sigma)$ holds. For examples of functions with or without the Property* (A)*, one may see* [3].

Bernal (see [2], [3]) initiated the idea of relative order of an entire function f with respect to another entire function g, symbolized by $\rho_g(f)$ to keep away from comparing growth just with $\exp z$ which is as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

The definition agrees with the classical one [27] if $g(z) = \exp z$.

Similarly, one may define the relative lower order of an entire function f with respect to another entire function g symbolized by $\lambda_g(f)$ in the following way:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Extending this idea, Lahiri and Banerjee (see [24]) established the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 1.8. (see [24]) *Let f be any meromorphic function and g be any entire function. The relative order of f with respect to g is defined as*

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \textit{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

Likewise, one may define the relative lower order of a meromorphic function f with respect to an entire function g in the following way:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

It is known (cf. [24]) that if $g(z) = \exp z$ then Definition 1.8 coincides with the classical definition of the order of a meromorphic function f .

In the paper we prove some comparative growth properties of composite entire or meromorphic functions in almost a new direction in the light of their relative orders and relative lower orders and differential monomials, differential polynomials generated by one of the factor.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. (see [1]) *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 2.2. (see [7]) *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_f(\exp(r^{\mu})).$$

Lemma 2.3. (see [7]) *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_g(\exp(r^{\mu})).$$

Lemma 2.4. (see [18]) *Let f be an entire function which satisfy the Property* (A)*, $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

Lemma 2.5. (see [19]) *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty}\delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a\neq\infty}\delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite order and $\Theta(\infty; g) = \sum\limits_{a\neq\infty}\delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum\limits_{a\neq\infty}\delta(a; g) = 1$. Then the relative order and relative lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g where $P_0[f]$ and $P_0[g]$ are homogeneous. i.e.,*

$$\rho_{P_0[g]}(P_0[f]) = \rho_g(f) \ \text{ and } \lambda_{P_0[g]}(P_0[f]) = \lambda_g(f).$$

Lemma 2.6. (see [17]) *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; f) = 4$. Also let g be a transcendental entire function of regular growth having non zero finite order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; g) = 4$. Then the relative order and relative* *lower order of $M[f]$ with respect to $M[g]$ are same as those of f with respect to g. i.e.,*

$$\rho_{M[g]}\left(M\left[f\right]\right)=\rho_{g}\left(f\right) \text{ and } \lambda_{M[g]}\left(M\left[f\right]\right)=\lambda_{g}\left(f\right).$$

3. Theorems

In this section we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogenity of $P_0\left[f\right]$ for meromorphic f will be needed as per the requirements of the theorems.

Theorem 3.1. *Let g be an entire function and f be a meromorphic function either of finite order and non-zero lower order with $\Theta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;f\right)=1$ or $\delta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta\left(a;f\right)=1$. Let also h be an entire function of regular growth having non zero finite order with $\Theta\left(\infty;h\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;h\right)=1$ or $\delta\left(\infty;h\right)=\sum\limits_{a\neq\infty}\delta\left(a;h\right)=1$ and $0<\lambda_h\left(f\right)\leq\rho_h\left(f\right)<\infty$. Then for every positive constant μ and each $\alpha\in\left(-\infty,\infty\right)$,*

$$\liminf_{r\to\infty}\frac{\left\{\log T_h^{-1}T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{P_0(h)}^{-1}T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right)}=0 \text{ if } \mu>\lambda_g.$$

Proof. If $1+\alpha\leq 0$, then the theorem is obvious. We consider $1+\alpha>0$. Since $T_h^{-1}\left(r\right)$ is an increasing function of r, it follows from Lemma 2.2 for a sequence of values of r tending to infinity that

$$\begin{aligned}\log T_h^{-1}T_{f\circ g}(r) &< \log T_h^{-1}T_f\left(\exp\left(r^{\mu}\right)\right)\\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &< \left(\rho_h\left(f\right)+\varepsilon\right)r^{\mu}.\end{aligned}\tag{3.1}$$

Again for all sufficiently large values of r, we get in view of Lemma 2.5 that

$$\begin{aligned}\log T_{P_0(h)}^{-1}T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right) &\geq \left(\lambda_{P_0(h)}\left(P_0(f)\right)-\varepsilon\right)r^{\mu}\\ i.e.,\ \log T_{P_0(h)}^{-1}T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right) &\geq \left(\lambda_h\left(f\right)-\varepsilon\right)r^{\mu}.\end{aligned}\tag{3.2}$$

Therefore for a sequence of values of r tending to infinity, we obtain from (3.1) and (3.2) that

$$\frac{\left\{\log T_h^{-1}T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{P_0(h)}^{-1}T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right)}\leq\frac{\left(\rho_h\left(f\right)+\varepsilon\right)^{1+\alpha}r^{\mu(1+\alpha)}}{\left(\lambda_h\left(f\right)-\varepsilon\right)r^{\mu}}.\tag{3.3}$$

So from (3.3) we obtain that

$$\liminf_{r\to\infty}\frac{\left\{\log T_h^{-1}T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{P_0(h)}^{-1}T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right)}=0.$$

This proves the theorem. $\square$

In the line of Theorem 3.1 and with the help of Lemma 2.6, we may state the following theorem without its proof.

Theorem 3.2. *Let g be an entire function and f be a transcendental meromorphic function either of finite order and of non-zero lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Let also h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and $0<\lambda_h(f)\le\rho_h(f)<\infty$. Then for every positive constant μ and each $\alpha\in(-\infty,\infty)$,*

$$\liminf_{r\to\infty}\frac{\left\{\log T_h^{-1}T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{M(h)}^{-1}T_{M(f)}\left(\exp\left(r^{\mu}\right)\right)}=0 \text{ if } \mu>\lambda_g.$$

Theorem 3.3. *Let f be a meromorphic function with non zero finite order and lower order, g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$ and h be an entire function of regular growth having non zero finite order and $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$. Also let $\rho_h(f)<\infty$ and $\lambda_h(g)>0$. Then for every positive constant μ and each $\alpha\in(-\infty,\infty)$,*

$$\liminf_{r\to\infty}\frac{\left\{\log T_h^{-1}T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{P_0(h)}^{-1}T_{P_0(g)}\left(\exp\left(r^{\mu}\right)\right)}=0 \text{ if } \mu>\lambda_g.$$

The proof is omitted as it can be carried out in the line of Theorem 3.1.

Theorem 3.4. *Let f be a meromorphic function with non zero finite order and lower order, g be a transcendental entire function either of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ and h be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. Also let $\rho_h(f)<\infty$ and $\lambda_h(g)>0$. Then for every positive constant μ and each $\alpha\in(-\infty,\infty)$,*

$$\liminf_{r\to\infty}\frac{\left\{\log T_h^{-1}T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{M(h)}^{-1}T_{M(g)}\left(\exp\left(r^{\mu}\right)\right)}=0 \text{ if } \mu>\lambda_g.$$

The proof of the above theorem is omitted as it can be carried out in the line of Theorem 3.3 and with the help of Lemma 2.6.

Theorem 3.5. *Let f be a meromorphic function of finite order with $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be an entire function with non zero finite lower order and h be an entire function of regular growth having non zero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let $\lambda_h(f) > 0$ and $\rho_h(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{P_0(h)}^{-1} T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right)} = 0 \text{ if } \mu > \lambda_g.$$

Theorem 3.6. *Let f be a meromorphic function with finite order, g be an entire function non zero finite lower order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$ and h be an entire function of regular growth having non zero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let $0 < \lambda_h(g) \le \rho_h(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{P_0(h)}^{-1} T_{P_0(g)}\left(\exp\left(r^{\mu}\right)\right)} = 0 \text{ if } \mu > \lambda_g.$$

We omit the proofs of Theorem 3.5 and Theorem 3.6 as those can be carried out in the line of Theorem 3.1 and Theorem 3.3 respectively and with the help of Lemma 2.3.

In the line of Theorem 3.5 and Theorem 3.6 and with the help of Lemma 2.6 we may state the following two theorems without their proofs :

Theorem 3.7. *Let f be a transcendental meromorphic function of finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$, g be an entire function with non zero finite lower order and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; h) = 4$. Also let $\lambda_h(f) > 0$ and $\rho_h(g) < \infty$. Then for every positive constant μ and each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log T_{M(h)}^{-1} T_{M(f)}\left(\exp\left(r^{\mu}\right)\right)} = 0 \text{ if } \mu > \lambda_g.$$

Theorem 3.8. *Let f be a meromorphic function with finite order, g be a transcendental entire function non zero finite lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g)$*

$= 4$ *and* h *be a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. *Also let* $0 < \lambda_h(g) \leq \rho_h(g) < \infty$. *Then for every positive constant* μ *and each* $\alpha \in (-\infty, \infty)$,

$$\liminf_{r \to \infty} \frac{\left\{\log T_h^{-1} T_{f \circ g}(r)\right\}^{1+\alpha}}{\log T_{M(h)}^{-1} T_{M(g)}\left(\exp\left(r^\mu\right)\right)} = 0 \text{ if } \mu > \lambda_g.$$

Theorem 3.9. *Suppose* f *be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum\limits_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum\limits_{a \neq \infty} \delta(a; f) = 1$. *Also let* h *be an entire function of regular growth having non zero finite order with* $\Theta(\infty; h) = \sum\limits_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum\limits_{a \neq \infty} \delta(a; h) = 1$ *and* g *be any entire function such that* $\rho_h(f) < \infty$ *and* $\lambda_h(f \circ g) = \infty$. *Then*

$$\lim_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity,

$$\log T_h^{-1} T_{f \circ g}(r) \leq \beta \log T_{P_0[h]}^{-1} T_{P_0[f]}(r). \tag{3.4}$$

Again from the definition of $\rho_{P_0[h]}(P_0[f])$, it follows for all sufficiently large values of r and in view of Lemma 2.5 that

$$\begin{aligned} \log T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\leq \left(\rho_{P_0[h]}(P_0[f]) + \varepsilon\right) \log r \\ \text{i.e., } \log T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\leq (\rho_h(f) + \varepsilon) \log r. \end{aligned} \tag{3.5}$$

Thus from (3.4) and (3.5), we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &\leq \beta(\rho_h(f) + \varepsilon) \log r \\ \text{i.e., } \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log r} &\leq \frac{\beta(\rho_h(f) + \varepsilon) \log r}{\log r} \\ \text{i.e., } \liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log r} &= \lambda_h(f \circ g) < \infty. \end{aligned}$$

This is a contradiction. Hence the theorem follows. □

Remark 3.1. Theorem 3.9 is also valid with "limit superior" instead of "limit" if $\lambda_h(f\circ g)=\infty$ is replaced by $\rho_h(f\circ g)=\infty$ and the other conditions remain the same.

Corollary 3.1. *Under the assumptions of Theorem* 3.9 *and Remark* 3.1*,*

$$\lim_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}=\infty \text{ and } \limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}=\infty$$

respectively hold.

The proof is omitted.

Analogously one may also state the following theorem and corollaries without their proofs as those may be carried out in the line of Remark 3.1, Theorem 3.9 and Corollary 3.1 respectively.

Theorem 3.10. *Let g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. Also let h be an entire function of regular growth having non zero finite order with $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$ and f be any meromorphic function such that $\rho_h(g)<\infty$ and $\rho_h(f\circ g)=\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[g]}(r)}=\infty.$$

Remark 3.2. Theorem 3.10 is also valid with "limit" instead of "limit superior" if $\rho_h(f\circ g)=\infty$ is replaced by $\lambda_h(f\circ g)=\infty$ and the other conditions remain the same.

Corollary 3.2. *Under the assumptions of Theorem* 3.10 *and Remark* 3.2*,*

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)}=\infty \text{ and } \lim_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)}=\infty$$

respectively hold.

In the line of Theorem 3.9 and Theorem 3.10 and with the help of Lemma 2.6, we may state the following two theorems without their proofs.

Theorem 3.11. *Suppose f be a transcendental meromorphic function either of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$.*

Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and g be any entire function such that* $\rho_h(f)<\infty$ *and* $\lambda_h(f\circ g)=\infty$. *Then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[f]}(r)}=\infty.$$

Remark 3.3. Theorem 3.11 is also valid with "limit superior" instead of "limit" if $\lambda_h(f\circ g)=\infty$ is replaced by $\rho_h(f\circ g)=\infty$ and the other conditions remain the same.

Corollary 3.3. *Under the assumptions of Theorem* 3.11 *and Remark* 3.3*,*

$$\lim_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}=\infty \text{ and } \limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r)}=\infty$$

respectively hold.

The proof is omitted.

Theorem 3.12. *Let g be a transcendental entire function either of finite order or of non-zero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Also let h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and f be any meromorphic function such that* $\rho_h(g)<\infty$ *and* $\rho_h(f\circ g)=\infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[g]}(r)}=\infty.$$

Remark 3.4. Theorem 3.12 is also valid with "limit" instead of "limit superior" if $\rho_h(f\circ g)=\infty$ is replaced by $\lambda_h(f\circ g)=\infty$ and the other conditions remain the same.

Corollary 3.4. *Under the assumptions of Theorem* 3.12 *and Remark* 3.4*,*

$$\limsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}=\infty \text{ and } \lim_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{M[h]}^{-1}T_{M[g]}(r)}=\infty$$

respectively hold.

Theorem 3.13. *Let f be a meromorphic function either of finite order or of non-zero lower order with* $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=$

$\sum_{a\neq\infty}\delta(a;f) = 1$, *g be an entire function and h be an entire function of regular growth having non zero finite order with* $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ *and satisfy the Property* (A). *Also let* $\lambda_g < \lambda_h(f) \leq \rho_h(f) < \infty$. *Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} = 0.$$

Proof. Let $\beta > 2$ and $\delta > 1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1 and Lemma 2.4, for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\}T_f(M_g(r))\right] \\ i.e.,\ T_h^{-1}T_{f\circ g}(r) &\leqslant \beta\left[T_h^{-1}T_f(M_g(r))\right]^{\delta} \\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\leqslant \delta\log T_h^{-1}T_f(M_g(r)) + O(1). \end{aligned}$$

Therefore from above, we get for a sequence of values of r tending to infinity that

$$\log T_h^{-1}T_{f\circ g}(r) \leq \delta(\rho_h(f)+\varepsilon)\log M_g(r) + O(1) \tag{3.6}$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}(r) \leq \delta(\rho_h(f)+\varepsilon)r^{\lambda_g+\varepsilon} + O(1). \tag{3.7}$$

Again from the definition of relative order, we obtain in view of Lemma 2.5 for all sufficiently large values of r that

$$\begin{aligned} T_{P_0[h]}^{-1}T_{P_0[f]}(r) &\geqslant r^{\left(\lambda_{P_0(h)}(P_0(f))-\varepsilon\right)} \\ i.e.,\ T_{P_0[h]}^{-1}T_{P_0[f]}(r) &\geqslant r^{(\lambda_h(f)-\varepsilon)}. \end{aligned} \tag{3.8}$$

Thus in view of (3.7) and (3.8), we get for a sequence of values of r tending to infinity,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} < \frac{\delta(\rho_h(f)+\varepsilon)r^{\lambda_g+\varepsilon}+O(1)}{r^{(\lambda_h(f)-\varepsilon)}}. \tag{3.9}$$

Now as $\lambda_g < \lambda_h(f)$, we can choose $\varepsilon(>0)$ in such a way that $\lambda_g+\varepsilon < \lambda_h(f)-\varepsilon$ and the theorem follows from (3.9). $\square$

Remark 3.5. If we take $\rho_g < \lambda_h(f) \leq \rho_h(f) < \infty$ instead of $\lambda_g < \lambda_h(f) \leq \rho_h(f) < \infty$ and the other conditions remain the same, the conclusion of Theorem 3.13 remains valid with "limit inferior " replaced by " limit ".

Theorem 3.14. *Let f be a transcendental meromorphic function either of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, *$g$ be an entire function and h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$ *and satisfy the Property* (A). *Also let $\lambda_g < \lambda_h(f) \le \rho_h(f) < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} = 0.$$

The proof of the above theorem is omitted as it can be carried out in the line of Theorem 3.13 and with the help of Lemma 2.6.

Remark 3.6. If we consider $\rho_g < \lambda_h(f) \le \rho_h(f) < \infty$ instead of $\lambda_g < \lambda_h(f) \le \rho_h(f) < \infty$ and the other conditions remain the same, the conclusion of Theorem 3.14 remains valid with “limit inferior”replaced by “limit”.

Theorem 3.15. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with* $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$ *and h be an entire function of regular growth having non zero finite order with* $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$. *Also let $\rho_h(f\circ g) < \infty$ and $\lambda_h(g) > 0$. Then*

$$\lim_{r\to\infty} \frac{\left[\log T_h^{-1} T_{f\circ g}(r)\right]^2}{T_{P_0(h)}^{-1} T_{P_0(g)}(\exp r)\cdot \log T_{P_0(h)}^{-1} T_{P_0(g)}(r)} = 0.$$

Proof. For any arbitrary positive ε, we have in view of Lemma 2.5 for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \le (\rho_h(f\circ g) + \varepsilon)\log r \tag{3.10}$$

and

$$\begin{aligned} \log T_{P_0(h)}^{-1} T_{P_0(g)}(r) &\ge \left(\lambda_{P_0(h)}(P_0(g)) - \varepsilon\right)\log r \\ \textit{i.e.},\ \log T_{P_0(h)}^{-1} T_{P_0(g)}(r) &\ge (\lambda_h(g) - \varepsilon)\log r. \end{aligned} \tag{3.11}$$

Similarly, for all sufficiently large values of r we have

$$\begin{aligned} \log T_{P_0(h)}^{-1} T_{P_0(g)}(\exp r) &\ge \left(\lambda_{P_0[h]}(P_0[g]) - \varepsilon\right) r \\ \textit{i.e.},\ T_{P_0(h)}^{-1} T_{P_0(g)}(\exp r) &\ge \exp\left[(\lambda_h(g) - \varepsilon)\, r\right]. \end{aligned} \tag{3.12}$$

From (3.10) and (3.11), we have for all sufficiently large values of r that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(g)}(r)} \leq \frac{(\rho_h(f\circ g)+\varepsilon)\log r}{(\lambda_h(g)-\varepsilon)\log r}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(g)}(r)} \leq \frac{\rho_h(f\circ g)}{\lambda_h(g)}. \tag{3.13}$$

Again from (3.10) and (3.12), we get for all sufficiently large values of r that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0(h)}^{-1} T_{P_0(g)}(\exp r)} \leq \frac{(\rho_h(f\circ g)+\varepsilon)\log r}{\exp\left[(\lambda_h(g)-\varepsilon)r\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0(h)}^{-1} T_{P_0(g)}(\exp r)} &= 0 \\ i.e.,\ \lim_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0(h)}^{-1} T_{P_0(g)}(\exp r)} &= 0. \end{aligned} \tag{3.14}$$

Thus the theorem follows from (3.13) and (3.14). □

In view of Theorem 3.15, the following two theorems can be carried out. Hence their proofs are omitted.

Theorem 3.16. *Let f a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ or $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$, g be any entire function and h be an entire function of regular growth having non zero finite order with $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$. Also let $\rho_h(f\circ g) < \infty$ and $\lambda_h(f) > 0$. Then*

$$\lim_{r\to\infty} \frac{\left[\log T_h^{-1} T_{f\circ g}(r)\right]^2}{T_{P_0(h)}^{-1} T_{P_0(f)}(\exp(r)) \cdot \log T_{P_0(h)}^{-1} T_{P_0(f)}(r)} = 0.$$

Theorem 3.17. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ or $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$, g be an entire function either of finite order or of non-zero lower order such that $\sum_{a\neq\infty} \Theta(a;g) = 2$ and h be an entire function of* *regular growth having non zero finite order and* $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ *or* $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$. *Also let* $\rho_h(f\circ g)<\infty$, $\lambda_h(g)>0$ *and* $\lambda_h(f)>0$. *Then*

$$\text{(i)}\ \lim_{r\to\infty}\frac{\left[\log T_h^{-1}T_{f\circ g}(r)\right]^2}{T_{P_0(h)}^{-1}T_{P_0(g)}(\exp r)\cdot\log T_{P_0(h)}^{-1}T_{P_0(f)}(r)}=0\ \textit{and}$$
$$\text{(ii)}\ \lim_{r\to\infty}\frac{\left[\log T_h^{-1}T_{f\circ g}(r)\right]^2}{T_{P_0(h)}^{-1}T_{P_0(f)}(\exp r)\cdot\log T_{P_0(h)}^{-1}T_{P_0(g)}(r)}=0.$$

In the line of Theorem 3.15, Theorem 3.16 and Theorem 3.17 and with the help of Lemma 2.6 we may state the following three theorems without their proofs.

Theorem 3.18. *Let* f *be a meromorphic function,* g *be a transcendental entire function either of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and* h *be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. *Also let* $\rho_h(f\circ g)<\infty$ *and* $\lambda_h(g)>0$. *Then*

$$\lim_{r\to\infty}\frac{\left[\log T_h^{-1}T_{f\circ g}(r)\right]^2}{T_{M(h)}^{-1}T_{M(g)}(\exp r)\cdot\log T_{M(h)}^{-1}T_{M(g)}(r)}=0.$$

Theorem 3.19. *Let* f *a transcendental meromorphic function either of finite order or of non-zero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g *be any entire function and* h *be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. *Also let* $\rho_h(f\circ g)<\infty$ *and* $\lambda_h(f)>0$. *Then*

$$\lim_{r\to\infty}\frac{\left[\log T_h^{-1}T_{f\circ g}(r)\right]^2}{T_{M(h)}^{-1}T_{M(f)}(\exp r)\cdot\log T_{M(h)}^{-1}T_{M(f)}(r)}=0.$$

Theorem 3.20. *Let* f *be a transcendental meromorphic function either of finite order or of non-zero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, g *be a transcendental entire function either of finite order or of non-zero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and* h *be a transcendental entire function of regular growth having non zero finite order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$.

Also let $\rho_h(f \circ g) < \infty$, $\lambda_h(g) > 0$ *and* $\lambda_h(f) > 0$. *Then*

$$\text{(i)} \lim_{r\to\infty} \frac{\left[\log T_h^{-1} T_{f\circ g}(r)\right]^2}{T_{M(h)}^{-1} T_{M(g)}(\exp r) \cdot \log T_{M(h)}^{-1} T_{M(f)}(r)} = 0 \text{ and}$$

$$\text{(ii)} \lim_{r\to\infty} \frac{\left[\log T_h^{-1} T_{f\circ g}(r)\right]^2}{T_{M(h)}^{-1} T_{M(f)}(\exp r) \cdot \log T_{M(h)}^{-1} T_{M(g)}(r)} = 0.$$

Theorem 3.21. *Let* f *be a meromorphic function either of finite order or of non-zero lower order with* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g *be an entire function with finite order and* h *be an entire function of regular growth having non zero finite order with* $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ *and satisfying the Property* (A). *Also let* $0 < \lambda_h(f) \leq \rho_h(f) < \infty$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(f)}(r)} \leq \frac{\rho_g}{\lambda_h(f)}.$$

Proof. From (3.6) and in view of Lemma 2.5, it follows for all sufficiently large values of r that

$$\log^{[2]} T_h^{-1} T_{f\circ g}(r) \leq \log^{[2]} M_g(r) + O(1)$$

$$\text{i.e.,} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(f)}(r)} \leq \frac{\log^{[2]} M_g(r) + O(1)}{\log r} \cdot \frac{\log r}{\log T_{P_0(h)}^{-1} T_{P_0(f)}(r)}$$

$$\text{i.e.,} \limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(f)}(r)} \leq \limsup_{r\to\infty} \frac{\log^{[2]} M_g(r) + O(1)}{\log r} \cdot \limsup_{r\to\infty} \frac{\log r}{\log T_{P_0(h)}^{-1} T_{P_0(f)}(r)}$$

$$\text{i.e.,} \limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(f)}(r)} \leq \rho_g . \frac{1}{\lambda_{P_0(h)}(P_0(f))} = \frac{\rho_g}{\lambda_h(f)}.$$

This proves the theorem. □

Theorem 3.22. *Let* f *be a meromorphic function,* g *be an entire function of finite order with* $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ *or* $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$ *and* h *be an entire function of regular growth having non zero finite order with* $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ *and satisfying the Property* (A). *Also let* $\rho_h(f) < \infty$ *and* $\lambda_h(g) > 0$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1} T_{P_0(g)}(r)} \leq \frac{\rho_g}{\lambda_h(g)}.$$

The proof of Theorem 3.22 is omitted as it can be carried out in the line of Theorem 3.21.

Theorem 3.23. *Let f be a transcendental meromorphic function either of finite order or of non-zero lower order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$*, g be an entire function with finite order and h be a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and satisfy the Property* (A)*. Also let $0<\lambda_h(f)\leq\rho_h(f)<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[2]}T_h^{-1}T_{f\circ g}(r)}{\log T_{M(h)}^{-1}T_{M(f)}(r)}\leq\frac{\rho_g}{\lambda_h(f)}.$$

Theorem 3.24. *Let f be a meromorphic function, g be a transcendental entire function of finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and h be a transcendental entire function of regular growth having non zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and satisfy the Property* (A)*. Also let $\rho_h(f)<\infty$ and $\lambda_h(g)>0$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[2]}T_h^{-1}T_{f\circ g}(r)}{\log T_{M(h)}^{-1}T_{M(g)}(r)}\leq\frac{\rho_g}{\lambda_h(g)}.$$

The proof of the above two theorems are omitted as those can be carried out in the line of Theorem 3.21 and Theorem 3.22 respectively and with the help of Lemma 2.6.

Theorem 3.25. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$*, g be an entire function with finite order and h be an entire function of regular growth having non zero finite order with* $\Theta(\infty;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ *or* $\delta(\infty;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ *and satisfy the Property* (A)*. Also let $0<\lambda_h(f)\leq\rho_h(f)<\infty$. Then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P_0(h)}^{-1}T_{P_0(f)}(\exp r^{\mu})}=\infty,$$

where $\rho_g<\mu<\infty$.

Proof. Let us consider $\beta>2$ and $\delta>1$. As $T_h^{-1}(r)$ is an increasing function of r, in view of Lemma 2.1 we get from(3.6) for all sufficiently large values of r,

$$\log T_h^{-1}T_{f\circ g}(r)\leq\delta(\rho_h(f)+\varepsilon)r^{\rho_g+\varepsilon}+O(1). \tag{3.15}$$

Also from the definition of the relative lower order and in view of Lemma 2.5, we obtain for all sufficiently large values of r that

$$\log T_{P_0(h)}^{-1} T_{P_0(f)}\left(\exp\left(r^{\mu}\right)\right) \geq \left(\lambda_{P_0(h)}\left(P_0(f)\right) - \varepsilon\right) \log\left\{\exp\left(r^{\mu}\right)\right\}$$
$$i.e.,\ \ \log T_{P_0(h)}^{-1} T_{P_0(f)}\left(\exp r^{\mu}\right) \geq \left(\lambda_h\left(f\right) - \varepsilon\right) r^{\mu}. \tag{3.16}$$

Now from (3.15) and (3.16), it follows for all sufficiently large values of r that

$$\frac{\log T_h^{-1} T_{f\circ g}\left(r\right)}{\log T_{P_0(h)}^{-1} T_{P_0(f)}\left(\exp r^{\mu}\right)} \leq \frac{\delta\left(\rho_h\left(f\right) + \varepsilon\right) r^{\rho_g + \varepsilon} + O(1)}{\left(\lambda_h\left(f\right) - \varepsilon\right) r^{\mu}}. \tag{3.17}$$

As $\rho_g < \mu$, we can choose $\varepsilon\,(> 0)$ in such a way that

$$\rho_g + \varepsilon < \mu. \tag{3.18}$$

Thus from (3.17) and (3.18), we obtain that

$$\lim_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}\left(r\right)}{\log T_{P_0(h)}^{-1} T_{P_0(f)}\left(\exp r^{\mu}\right)} = 0.$$

Thus the theorem follows. □

In the line of Theorem 3.25, we may state the following theorem without its proof.

Theorem 3.26. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with $\Theta\left(\infty; g\right) = \sum\limits_{a\neq\infty} \delta_p\left(a; g\right) = 1$ or $\delta\left(\infty; g\right) = \sum\limits_{a\neq\infty} \delta\left(a; g\right) = 1$ and h be an entire function of regular growth having non zero finite order with $\Theta\left(\infty; h\right) = \sum\limits_{a\neq\infty} \delta_p\left(a; h\right) = 1$ or $\delta\left(\infty; h\right) = \sum\limits_{a\neq\infty} \delta\left(a; h\right) = 1$ and satisfy the Property* (A). *Also let $\lambda_h\left(g\right) > 0$ and $\rho_h\left(f\right) < \infty$. Then for every μ with $\rho_g < \mu < \infty$,*

$$\lim_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}\left(r\right)}{\log T_{P_0(h)}^{-1} T_{P_0(g)}\left(\exp r^{\mu}\right)} = 0.$$

In the line of Theorem 3.25 and Theorem 3.26 and with the help of Lemma 2.6, we may state the following two theorems without their proofs.

Theorem 3.27. *Let f be a transcendental meromorphic function either of finite order or of non-zero lower order such that $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$, g be an entire function with finite order and h be a transcendental entire function* *of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and satisfy the Property* (A)*. Also let* $0<\lambda_h(f)\le\rho_h(f)<\infty$*. Then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M(h)}^{-1}T_{M(f)}(\exp r^{\mu})}=\infty,$$

where $\rho_g<\mu<\infty$*.*

Theorem 3.28. *Let f be a meromorphic function, g be a transcendental entire function either of finite order or of non-zero lower order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *and h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and satisfy the Property* (A)*. Also let* $\lambda_h(g)>0$ *and* $\rho_h(f)<\infty$*. Then for every μ with* $\rho_g<\mu<\infty$*,*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M(h)}^{-1}T_{M(g)}(\exp r^{\mu})}=0.$$

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