Annals of the University of Bucharest (mathematical series)
5 (LXIII) (2014), 383–402

Optimization of the blood flow in venous insufficiency

Ruxandra Stavre

Dedicated to Professor Nicolaie D. Cristescu on his 85th birthday

Abstract - We consider a simplified problem describing the interaction between a viscous fluid (blood) and an elastic structure (vein walls). The aim of this article is to propose a mathematical model that provides an exterior compression which prevents the blood recirculation through an inelasic vein. By means of a boundary control problem we determine some exterior forces necessary for compensating the rigidity of the vein walls, in such a way that medical complications such as leg edema and venous ulcers are attenuated as much as possible. We prove the existence of an optimal control and we establish the necessary conditions of optimality.

Key words and phrases : Fluid-structure interaction problem, blood recirculation, boundary control problem.

Mathematics Subject Classification (2010) **:** 74F10, 49J20, 76D55.

1. Introduction

The interaction of a fluid with a deformable structure has important applications in medicine. The purpose of the present work is to optimize the blood flow through a vein by means of a fluid-structure interaction model. The blood motion in a leg vein has an anti-gravity sense. When the vein loses its elasticity, from different reasons, the blood has no normal flow through it; consequently, stagnation and recirculation may appear. These phenomena lead to medical complications such as leg edema and venous ulcers. A medical solution for attenuating these effects is the use of elastic stockings. The compression determined by them compensates in a certain way the lack of elasticity of the vein walls, realising an almost normal blood flow. We propose in this article a mathematical model that allows us to determine an optimal compression that prevents the blood recirculation through the inelastic vein.

Due to its various applications, the fluid-structure interaction problem has been studied extensively in the last years. For instance, some results concerning the existence of weak or strong solutions can be found in [3], [8]. An asymptotic analysis of the fluid-structure interaction was developed in [1], [2], [10], [11], [12], [13], [14] and this list is non-exhaustive.

In all the previously cited papers the thickness of the elastic structure was neglected. The present work represents a first part of a more extensive approach. We consider the simplified model introduced in [10] slightly modified, described in the next section. Section 3 deals with the variational formulation of the problem which provides existence, uniqueness and regularity results. The main results of this article are contained in the next section. We introduce the boundary control problem with its physical motivation. After proving the existence of an optimal control, we establish the optimality conditions.

2. The physical problem

Recently, we published some results concerning the flow through the bloodstream. In [5], [6], [7] we considered a more complicated model for the fluid motion, but the flow domain was taken with rigid boundaries. The present article deals with a fluid-structure interaction problem, but with simplified geometry of the flow domain. Consider a cylindrical domain with lateral elastic boundary (vein) filled with an incompressible viscous fluid (blood). We suppose that in each axial section of the right circular cylinder the interaction problem is the same; so we study it in the 2-dimensional domain

$$\Omega_f = \left\{(x,y) \in \mathbb{R}^2 / x \in (-a/2, a/2)\,,\, y \in (0,b)\right\}, \tag{2.1}$$

with the elastic boundaries

$$\Gamma^{\pm} = \left\{(\pm\, a/2, y)\,/ y \in (0,b)\right\}, \tag{2.2}$$

where a and b are positive given constants. The anti-gravity blood flow in the vein being slow, we describe it be means of the non-steady Stokes' equations. For modeling the transversal deformation of the vein walls we use the Koiter's equations. This is in agreement with the reduced elasticity of the vein wall that appears in venous insufficiency. The longitudinal deformation is neglected. Supposing that on $\partial\Omega_f \backslash (\Gamma^+ \cup \Gamma^-)$ the velocity is given and considering the clamped ends condition for the elastic walls we are leaded to the following coupled system

$$\begin{cases} \rho_f \dfrac{\partial \mathbf{u}}{\partial t} - \mu\ \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega_f \times (0,T), \\ \text{div } \mathbf{u} = 0 \text{ in } \Omega_f \times (0,T), \\ \mathbf{u} = \dfrac{\partial d_\pm}{\partial t}\mathbf{i} \text{ on } \Gamma^\pm \times (0,T), \\ \rho h \dfrac{\partial^2 d_\pm}{\partial t^2} + \dfrac{h^3 E}{12(1-\sigma^2)}\ \dfrac{\partial^4 d_\pm}{\partial y^4} - \dfrac{\sigma}{6(1-\sigma^2)}\dfrac{h^3 E}{a^2}\dfrac{\partial^2 d_\pm}{\partial y^2} \\ \ +\dfrac{hE}{a^2(1-\sigma^2)}\left(1+\dfrac{h^2}{12a^2}\right) d_\pm = \pm p_{/_{x=\pm a/2}} + g_\pm \text{ on } \Gamma^\pm \times (0,T), \\ \mathbf{u} = \mathbf{u}_0 \text{ on } \ (\partial\Omega_f \backslash (\Gamma^+ \cup \Gamma^-)) \times (0,T), \\ d_\pm = \dfrac{\partial d_\pm}{\partial y} = 0 \text{ in } \{0,b\} \times (0,T). \\ \mathbf{u}(0) = \mathbf{0} \text{ in } \Omega_f, d_\pm(0) = \frac{\partial d_\pm}{\partial t}(0) = 0 \text{ in } (0,b), \end{cases} \tag{2.3}$$

with T a positive given constant defining the time interval and $\mathbf{i}, \mathbf{j}$ the axes versors. The data of the previous system are: ρ_f, ρ, μ, σ, h, E representing positive given constants in connection with the properties of the materials and the functions: $\mathbf{f}$, the forces that act on the fluid, $\mathbf{u}_0$ a given velocity, $g_\pm$ the forces that act from the exterior on the elastic boundaries. In our particular problem, $g_\pm$ represent the compression determined by the elastic stockings. Concerning the given velocity $\mathbf{u}_0$, it represents the trace of a function denoted also by $\mathbf{u}_0$ that has the following properties:

$$\begin{cases} \mathbf{u}_0 \in H^2(0,T;(H^2(\Omega_f))^2), \\ \mathbf{u}_0 = \mathbf{0} \text{ on } \Gamma^\pm \times (0,T), \\ \text{div } \mathbf{u}_0 = 0 \text{ in } \Omega_f \times (0,T), \\ \mathbf{u}_0(0) = \mathbf{0} \text{ in } \Omega_f. \end{cases} \tag{2.4}$$

There may exist several functions $\mathbf{u}_0$ satisfying (2.4). We choose one of these functions that will be fixed throughout the paper.

The unknowns of the previous system are: the velocity and the pressure of the fluid, $\mathbf{u}$ and p, respectively and the displacement of the elastic boundaries $\Gamma^\pm$, $d_\pm$.

As a consequence of the properties $(2.3)_{2,3,5,8}$ and $(2.4)_{2,3}$ we obtain the following compatibility condition

$$\int_0^b (d_+(y,t) - d_-(y,t))\,\mathrm{d}y = 0. \tag{2.5}$$

Denoting

$$\mathbf{F} = \mathbf{f} - \rho_f \frac{\partial \mathbf{u}_0}{\partial t} + \mu\ \Delta \mathbf{u}_0,$$

we replace the non homogeneous boundary value problem (2.3) by the homogeneous one

$$\begin{cases} \rho_f \dfrac{\partial \mathbf{v}}{\partial t} - \mu \, \Delta \mathbf{v} + \nabla p = \mathbf{F} \text{ in } \Omega_f \times (0,T), \\ \text{div } \mathbf{v} = 0 \text{ in } \Omega_f \times (0,T), \\ \mathbf{v} = \dfrac{\partial d_\pm}{\partial t} \mathbf{i} \text{ on } \Gamma^\pm \times (0,T), \\ \rho h \dfrac{\partial^2 d_\pm}{\partial t^2} + \dfrac{h^3 E}{12(1-\sigma^2)} \, \dfrac{\partial^4 d_\pm}{\partial y^4} - \dfrac{\sigma}{6(1-\sigma^2)} \dfrac{h^3 E}{a^2} \dfrac{\partial^2 d_\pm}{\partial y^2} \\ + \dfrac{hE}{a^2(1-\sigma^2)} \left(1 + \dfrac{h^2}{12a^2}\right) d_\pm = \pm p_{/_{x=\pm a/2}} + g_\pm \text{ on } \Gamma^\pm \times (0,T), \\ \mathbf{v} = \mathbf{0} \text{ on } \left(\partial\Omega_f \backslash (\Gamma^+ \cup \Gamma^-)\right) \times (0,T), \\ d_\pm = \dfrac{\partial d_\pm}{\partial y} = 0 \text{ in } \{0,b\} \times (0,T). \\ \mathbf{v}(0) = \mathbf{0} \text{ in } \Omega_f, \\ d_\pm(0) = \dfrac{\partial d_\pm}{\partial t}(0) = 0 \text{ in } (0,b), \end{cases} \tag{2.6}$$

where $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$.

3. The variational problem

In order to obtain the weak formulation of the coupled system (2.6) we choose for the data the regularity

$$\mathbf{f} \in H^1(0,T;(L^2(\Omega_f))^2), \ g_+, g_- \in H^1(0,T;L^2(0,b)). \tag{3.1}$$

The properties $(2.4)_1$ and (3.1) give for $\mathbf{F}$ the same regularity as for $\mathbf{f}$, i.e.

$$\mathbf{F} \in H^1(0,T;(L^2(\Omega_f))^2). \tag{3.2}$$

We consider the spaces

$$\begin{aligned} &V = \{\mathbf{v} \in (H^1(\Omega_f))^2 / \text{div}\mathbf{v} = 0 \text{ in } \Omega_f, v_y = 0 \text{ on } \Gamma^\pm, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_f \backslash (\Gamma^+ \cup \Gamma^-)\}, \\ &B = \{\boldsymbol{\beta} = (\beta_+, \beta_-)/\beta_+, \beta_- \in H_0^2(0,b), \int_0^b (\beta_+(y) - \beta_-(y)) \, \mathrm{d}y = 0\}. \end{aligned} \tag{3.3}$$

Here and in what follows we denote by v_x, v_y the two components of a vector $\mathbf{v}$. For simplifying the writing we introduce the notations $A_1 = \dfrac{h^3 E}{12(1-\sigma^2)}$, $A_2 = \dfrac{\sigma h^3 E}{6a^2(1-\sigma^2)}$, $A_3 = \dfrac{hE}{a^2(1-\sigma^2)} \left(1 + \dfrac{h^2}{12a^2}\right)$. Standard techniques lead to the following variational formulation of the coupled problem (2.6)

$$\begin{cases} \text{Find } (\mathbf{v},\mathbf{d}) \in L^2(0,T;V)\times H^1(0,T;B) \text{ with} \\ \left(\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t},\dfrac{\mathrm{d}^2\mathbf{d}}{\mathrm{d}t^2}\right) \in L^2(0,T;V')\times L^2(0,T;B'); \mathbf{d}=(d_+,d_-) \text{ s. t.} \\ \rho_f \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{\Omega_f}\mathbf{v}(t)\cdot\mathbf{w} + \mu\int_{\Omega_f}\nabla\mathbf{v}(t):\nabla\mathbf{w} + \rho h \dfrac{\mathrm{d}}{\mathrm{d}t}\int_0^b \dfrac{\partial\mathbf{d}(t)}{\partial t}\cdot\boldsymbol{\beta} \\ +A_1\displaystyle\int_0^b \dfrac{\partial^2\mathbf{d}(t)}{\partial y^2}\cdot\boldsymbol{\beta}'' + A_2\int_0^b \dfrac{\partial\mathbf{d}(t)}{\partial y}\cdot\boldsymbol{\beta}' + A_3\int_0^b \mathbf{d}(t)\cdot\boldsymbol{\beta} = \int_{\Omega_f}\mathbf{F}(t)\cdot\mathbf{w} \\ +\displaystyle\int_0^b \mathbf{g}(t)\cdot\boldsymbol{\beta} \text{ a.e. in } (0,T),\ \forall\,\mathbf{w}\in V,\ \forall\,\boldsymbol{\beta}\in B,\ w_x=\beta_\pm \text{ on } \Gamma^\pm, \\ \mathbf{v} = \dfrac{\partial d_\pm}{\partial t}\mathbf{i} \text{ a.e. on } \Gamma^\pm\times(0,T), \\ \mathbf{v}(0)=\mathbf{0} \text{ in } \Omega_f,\ \mathbf{d}(0)=\dfrac{\partial\mathbf{d}}{\partial t}(0)=\mathbf{0} \text{ in } (0,b). \end{cases} \tag{3.4}$$

Remark 3.1. In what follows we shall obtain further regularity for the unknowns $\mathbf{v}$ and $\mathbf{d}$; the regularity stated in (3.4) for the functions and for their derivatives is the lowest that is necessary in order to give sense to the expressions appearing in (3.4).

We prove next the main result of this section.

Theorem 3.1. *The variational problem* (3.4) *has a unique solution* $(\mathbf{v},\mathbf{d})$, *with* $\mathbf{v} \in W^{1,\infty}(0,T;(L^2(\Omega_f))^2) \cap H^1(0,T;V) \cap L^2(0,T;(H^2(\Omega_f))^2)$, $\mathbf{d} \in W^{2,\infty}(0,T;(L^2(0,b))^2) \cap W^{1,\infty}(0,T;B)$. *Moreover, there exists a unique function* $p \in L^2(0,T;H^1(\Omega_f))$ *such that* $(\mathbf{v},p,\mathbf{d})$ *is a classical solution for* (2.6).

Proof. The uniqueness of the pair $(\mathbf{v},\mathbf{d})$ is obtained in a classical way; so we skip this proof. The existence and regularity are established by means of the Galerkin's method. The main ideas are similar with those from [10]. However, here the regularity results for the unknown functions rely on the regularity of the data, unlike in [10] where an additional "viscous" term was added in the equation for the elastic structure in order to ensure the desired regularity for the velocity and displacement. Let $\{\boldsymbol{\beta}_j\}_{j\in\mathbb{N}^*}$ and $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}^*}$ be some bases for the spaces B and $V_0 = V\cap(H_0^1(\Omega_f))^2$, respectively.

For any $\boldsymbol{\beta}_j$ consider the problem:

$$\begin{cases} -\mu\,\Delta\boldsymbol{\varphi}_j + \nabla p_j = 0 \text{ in } \Omega_f, \\ \operatorname{div}\boldsymbol{\varphi}_j = 0 \text{ in } \Omega_f, \\ \boldsymbol{\varphi}_j = \mathbf{0} \text{ on } \partial\Omega_f\backslash(\Gamma^+\cup\Gamma^-), \\ \boldsymbol{\varphi}_j = (\beta_j)_\pm\mathbf{i} \text{ on } \Gamma^\pm. \end{cases} \tag{3.5}$$

The problem (3.5) is a classical non homogeneous Stokes problem with unique solution.

By means of the functions $\{\boldsymbol{\beta}_j\}_{j\in\mathbb{N}^*}$, $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}^*}$ and $\{\boldsymbol{\varphi}_j\}_{j\in\mathbb{N}^*}$ we define the approximate functions

$$\begin{cases} \mathbf{d}_n(y,t) = \displaystyle\sum_{j=1}^{n} b_j(t)\ \boldsymbol{\beta}_j(y), \\ \mathbf{v}_n^m(x,y,t) = \displaystyle\sum_{k=1}^{m} a_k(t)\ \boldsymbol{\psi}_k(x,y) + \sum_{j=1}^{n} \dot{b}_j(t)\ \boldsymbol{\varphi}_j(x,y), \end{cases} \tag{3.6}$$

for each $n,\ m \in \mathbb{N}^*$. To determine the approximate functions means to determine the functions of t, $a_k,\ b_j : [0,T] \mapsto \mathbb{R}$, for all $k, j \in \mathbb{N}^*$. This can be done by solving the problem written below

$$\begin{cases} \rho_f \displaystyle\int_{\Omega_f} \frac{\partial \mathbf{v}_n^m}{\partial t}(t)\cdot\boldsymbol{\psi}_k + \mu\int_{\Omega_f}\nabla\mathbf{v}_n^m(t):\nabla\boldsymbol{\psi}_k = \int_{\Omega_f}\mathbf{F}(t)\cdot\boldsymbol{\psi}_k,\ k=1,...,m, \\ \rho_f \displaystyle\int_{\Omega_f} \frac{\partial \mathbf{v}_n^m}{\partial t}(t)\cdot\boldsymbol{\varphi}_j + \mu\int_{\Omega_f}\nabla\mathbf{v}_n^m(t):\nabla\boldsymbol{\varphi}_j + \rho h\int_0^b \frac{\partial^2\mathbf{d}_n}{\partial t^2}(t)\cdot\boldsymbol{\beta}_j \\ \quad + A_1\displaystyle\int_0^b \frac{\partial^2\mathbf{d}_n}{\partial y^2}(t)\cdot\boldsymbol{\beta}_j'' + A_2\int_0^b\frac{\partial\mathbf{d}_n}{\partial y}(t)\cdot\boldsymbol{\beta}_j' + A_3\int_0^b \mathbf{d}_n(t)\cdot\boldsymbol{\beta}_j \\ \quad = \displaystyle\int_{\Omega_f}\mathbf{F}(t)\cdot\boldsymbol{\varphi}_j + \int_0^b \mathbf{g}(t)\cdot\boldsymbol{\beta}_j,\ j=1,...,n,\ \text{a.e. in } (0,T), \\ \mathbf{v}_n^m(0) = \mathbf{0} \text{ in } \Omega_f;\ \mathbf{d}_n(0) = \dfrac{\partial\mathbf{d}_n}{\partial t}(0) = \mathbf{0} \text{ in } (0,b). \end{cases} \tag{3.7}$$

We notice that the condition $\mathbf{v}_n^m = \dfrac{\partial(d_n)_\pm}{\partial t}\mathbf{i}$ on $\Gamma^\pm\times(0,T)$ is fulfilled due to the construction of the functions $\boldsymbol{\varphi}_j$.

Introducing (3.6) into (3.7) we obtain for determining the unknown functions $a_k,\ b_j$ the following second order system of $m+n$ linear differential equations

$$\begin{cases}\rho_f\sum_{l=1}^{m}\delta_{kl}\dot{a}_l(t)+\rho_f\sum_{i=1}^{n}\left(\int_{\Omega_f}\boldsymbol{\varphi}_i\cdot\boldsymbol{\psi}_k\right)\ddot{b}_i(t)+\mu\sum_{l=1}^{m}\left(\int_{\Omega_f}\nabla\boldsymbol{\psi}_l:\nabla\boldsymbol{\psi}_k\right)a_l(t)\\ \quad=\int_{\Omega_f}\mathbf{F}(t)\cdot\boldsymbol{\psi}_k,\quad k=1,...,m,\\ \rho_f\sum_{l=1}^{m}\left(\int_{\Omega_f}\boldsymbol{\psi}_l\cdot\boldsymbol{\varphi}_j\right)\dot{a}_l(t)+\sum_{i=1}^{n}\left(\rho_f\left(\int_{\Omega_f}\boldsymbol{\varphi}_i\cdot\boldsymbol{\varphi}_j\right)+\rho h\delta_{ij}\right)\ddot{b}_i(t)\\ +\mu\sum_{i=1}^{n}\left(\int_{\Omega_f}\nabla\boldsymbol{\varphi}_i:\nabla\boldsymbol{\varphi}_j\right)\dot{b}_i(t)+\sum_{i=1}^{n}\left(A_1\int_0^b\boldsymbol{\beta}_i''\cdot\boldsymbol{\beta}_j''+A_2\int_0^b\boldsymbol{\beta}_i'\cdot\boldsymbol{\beta}_j'+A_3\delta_{ij}\right)b_i(t)\\ \quad=\int_{\Omega_f}\mathbf{F}(t)\cdot\boldsymbol{\varphi}_j+\int_0^b\mathbf{g}(t)\cdot\boldsymbol{\beta}_j,\ j=1,...,n,\\ a_l(0)=b_i(0)=\dot{b}_i(0)=0,\ l=1,...,m,\ i=1,...,n.\end{cases}\tag{3.8}$$

The functions a_l, b_i are uniquely determined from (3.8) if and only if the matrix of order $m+n$

$$\mathcal{M}=\begin{pmatrix}\rho_f\left(\delta_{kl}\right)_{1\le k,l\le m} & \rho_f\left(\int_{\Omega_f}\boldsymbol{\varphi}_i\cdot\boldsymbol{\psi}_k\right)_{1\le k\le m,1\le i\le n}\\ \rho_f\left(\int_{\Omega_f}\boldsymbol{\varphi}_j\cdot\boldsymbol{\psi}_l\right)_{1\le j\le n,1\le l\le m} & \left(\rho_f\int_{\Omega_f}\boldsymbol{\varphi}_i\cdot\boldsymbol{\varphi}_j+\rho h\delta_{ij}\right)_{1\le i,j\le n}\end{pmatrix}\tag{3.9}$$

is non singular. We obtain this result by showing that

$$\begin{cases}\mathcal{M}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\ge 0\quad\forall\boldsymbol{\xi}\in\mathbb{R}^m,\ \boldsymbol{\eta}\in\mathbb{R}^n,\\ \mathcal{M}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}=0\Rightarrow\boldsymbol{\xi}=\mathbf{0},\ \boldsymbol{\eta}=\mathbf{0}.\end{cases}\tag{3.10}$$

Obvious computations lead to

$$\mathcal{M}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}=\rho_f\int_{\Omega_f}\left(\sum_{k=1}^{m}\xi_k\boldsymbol{\psi}_k+\sum_{j=1}^{n}\eta_j\boldsymbol{\varphi}_j\right)^2+\rho h\boldsymbol{\eta}^2\ \forall\boldsymbol{\xi}\in\mathbb{R}^m,\ \boldsymbol{\eta}\in\mathbb{R}^n,\tag{3.11}$$

which yields $(3.10))_1$. Supposing next that $\mathcal{M}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{pmatrix}=0$ we obtain $(3.10)_2$.

Computing $\sum_{k=1}^{m} a_k(t)\cdot(3.7)_1 + \sum_{j=1}^{n} \dot{b}_j(t)\cdot(3.7)_2$ and using standard techniques we obtain the first estimates for $\mathbf{v}_n^m$ and $\mathbf{d}_n$

$$\begin{aligned}&\max\Big\{\rho_f^{1/2}\,\|\mathbf{v}_n^m\|_{L^\infty(0,T;(L^2(\Omega_f))^2)}\,;(2\mu)^{1/2}\,\|\mathbf{v}_n^m\|_{L^2(0,T;(H_0^1(\Omega_f))^2)}\,;\\ &(\rho h)^{1/2}\left\|\frac{\partial \mathbf{d}_n}{\partial t}\right\|_{L^\infty(0,T;(L^2(0,b))^2)};A_1^{1/2}\|\mathbf{d}_n\|_{L^\infty(0,T;(H_0^2(0,b))^2)}\Big\}\le c(\mathbf{F},\mathbf{g}),\end{aligned} \tag{3.12}$$

where

$$c^2(\mathbf{F},\mathbf{g}) = e^T\left(\frac{1}{\rho_f}\int_0^T\int_{\Omega_f}\mathbf{F}^2 + \frac{1}{\rho h}\int_0^T\int_0^b \mathbf{g}^2\right).$$

Due to the regularity of the data with respect to t given by (2.4) and (3.1) we obtain further estimates, by computing, as above $\sum_{k=1}^{m}\dot{a}_k(t)\cdot(3.7)_1$ $+\sum_{j=1}^{n}\ddot{b}_j(t)\cdot(3.7)_2$:

$$\begin{aligned}&\max\Big\{\rho_f^{1/2}\left\|\frac{\partial \mathbf{v}_n^m}{\partial t}\right\|_{L^\infty(0,T;(L^2(\Omega_f))^2)};(2\mu)^{1/2}\left\|\frac{\partial \mathbf{v}_n^m}{\partial t}\right\|_{L^2(0,T;(H_0^1(\Omega_f))^2)};\\ &(\rho h)^{1/2}\left\|\frac{\partial^2 \mathbf{d}_n}{\partial t^2}\right\|_{L^\infty(0,T;(L^2(0,b))^2)};A_1^{1/2}\left\|\frac{\partial \mathbf{d}_n}{\partial t}\right\|_{L^\infty(0,T;(H_0^2(0,b))^2)}\Big\}\le \tilde{c}(\mathbf{F},\mathbf{g}),\end{aligned} \tag{3.13}$$

with

$$\tilde{c}^2(\mathbf{F},\mathbf{g}) = c^2\left(\frac{\partial \mathbf{F}}{\partial t},\frac{\partial \mathbf{g}}{\partial t}\right) + e^T\left(\frac{1}{\rho_f}\int_0^T\int_{\Omega_f}\mathbf{F}^2(0) + \frac{1}{\rho h}\int_0^T\int_0^b \mathbf{g}^2(0)\right).$$

These estimates give the following regularity of the approximations $\mathbf{v}_n^m$ and $\mathbf{d}_n$

$$\begin{cases}\mathbf{v}_n^m \in W^{1,\infty}(0,T;(L^2(\Omega_f))^2)\cap H^1(0,T;V),\\ \mathbf{d}_n \in W^{2,\infty}(0,T;(L^2(0,b))^2)\cap W^{1,\infty}(0,T;B)\end{cases} \tag{3.14}$$

and the following convergences, on subsequences

$$\begin{cases}\mathbf{v}_{n_l}^{m_q} \rightharpoonup \mathbf{v} \text{ weakly-}\star \text{ in } W^{1,\infty}(0,T;(L^2(\Omega_f))^2),\\ \mathbf{v}_{n_l}^{m_q} \rightharpoonup \mathbf{v} \text{ weakly in } H^1(0,T;V),\\ \mathbf{d}_{n_l} \rightharpoonup \mathbf{d} \text{ weakly-}\star \text{ in } W^{2,\infty}(0,T;(L^2(0,b))^2),\\ \mathbf{d}_{n_l} \rightharpoonup \mathbf{d} \text{ weakly-}\star \text{ in } W^{1,\infty}(0,T;B),\end{cases} \tag{3.15}$$

when $l,q\to\infty$. We consider next an arbitrary function $\tau\in L^2(0,T)$ and we compute $\int_0^T (3.7)_1\cdot\tau(t)\mathrm{d}t$. We pass to the limit on subsequences and using the fact that $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}^*}$ is a basis of V_0 and De Rham's theorem we establish the existence of a distribution $\tilde{p}$ such that

$$\rho_f \frac{\partial \mathbf{v}}{\partial t} - \mu\Delta\mathbf{v} - \mathbf{F} = -\nabla\tilde{p} \text{ in } \mathcal{D}'(0,T;(\mathcal{D}'(\Omega_f))^2). \tag{3.16}$$

With the technique of [17], Ch. 3 we improve the regularity of the functions $\mathbf{v}, \tilde{p}$ as follows

$$\begin{cases} \mathbf{v} \in L^2(0,T;(H^2(\Omega_f))^2), \\ \tilde{p} \in L^2(0,T;H^1(\Omega_f)) \end{cases} \tag{3.17}$$

and now the relation (3.16) makes sense in $L^2(0,T;(L^2(\Omega_f))^2)$. In the same way as before we pass to the limit in $(3.7)_2$; this yields

$$\begin{aligned} &\rho_f \int_{\Omega_f} \frac{\partial \mathbf{v}}{\partial t}(t)\cdot\boldsymbol{\varphi}_j + \mu\int_{\Omega_f} \nabla\mathbf{v}(t) : \nabla\boldsymbol{\varphi}_j + \rho h\int_0^b \frac{\partial^2\mathbf{d}}{\partial t^2}(t)\cdot\boldsymbol{\beta}_j \\ &+ A_1\int_0^b \frac{\partial^2\mathbf{d}}{\partial y^2}(t)\cdot\boldsymbol{\beta}_j'' + A_2\int_0^b \frac{\partial\mathbf{d}}{\partial y}(t)\cdot\boldsymbol{\beta}_j' + A_3\int_0^b \mathbf{d}(t)\cdot\boldsymbol{\beta}_j \\ &= \int_{\Omega_f} \mathbf{F}(t)\cdot\boldsymbol{\varphi}_j + \int_0^b \mathbf{g}(t)\cdot\boldsymbol{\beta}_j,\ \forall j\in\mathbb{N}^*,\ \text{a.e. in } (0,T). \end{aligned} \tag{3.18}$$

We also compute $\displaystyle\int_{\Omega_f}(3.16)$ (written in $L^2(0,T;(L^2(\Omega_f))^2)$) $\cdot\boldsymbol{\varphi}_j$ which gives

$$\begin{aligned} &\rho_f \int_{\Omega_f} \frac{\partial \mathbf{v}}{\partial t}(t)\cdot\boldsymbol{\varphi}_j + \mu\int_{\Omega_f} \nabla\mathbf{v}(t) : \nabla\boldsymbol{\varphi}_j + \int_{\Gamma^+} \tilde{p}(t)(\beta_j)_+ \\ &- \int_{\Gamma^-} \tilde{p}(t)(\beta_j)_- = \int_{\Omega_f} \mathbf{F}(t)\cdot\boldsymbol{\varphi}_j\ \forall j\in\mathbb{N}^*,\ \text{a.e. in } (0,T). \end{aligned} \tag{3.19}$$

Computing (3.18)-(3.19) and taking into account that $\{\boldsymbol{\beta}_j\}_{j\in\mathbb{N}^*}$ is a basis of the space B we get

$$\begin{aligned} &\rho h\int_0^b \frac{\partial^2\mathbf{d}}{\partial t^2}(t)\cdot\boldsymbol{\beta} + A_1\int_0^b \frac{\partial^2\mathbf{d}}{\partial y^2}(t)\cdot\boldsymbol{\beta}'' + A_2\int_0^b \frac{\partial\mathbf{d}}{\partial y}(t)\cdot\boldsymbol{\beta}' + A_3\int_0^b \mathbf{d}(t)\cdot\boldsymbol{\beta} \\ &= \int_0^b \mathbf{g}(t)\cdot\boldsymbol{\beta} + \int_{\Gamma^+} \tilde{p}(t)\beta_+ - \int_{\Gamma^-} \tilde{p}(t)\beta_-,\ \forall\boldsymbol{\beta}\in B,\ \text{a.e. in } (0,T). \end{aligned} \tag{3.20}$$

Consider next $\mathbf{w}\in V$ with $w_x = \beta_\pm$ on $\Gamma^\pm$ and compute, as before, $\displaystyle\int_{\Omega_f}(3.16)$ (written in $L^2(0,T;(L^2(\Omega_f))^2)$) $\cdot\mathbf{w}$. We obtain, as before

$$\begin{aligned} &\rho_f \int_{\Omega_f} \frac{\partial \mathbf{v}}{\partial t}(t)\cdot\mathbf{w} + \mu\int_{\Omega_f} \nabla\mathbf{v}(t) : \nabla\mathbf{w} + \int_{\Gamma^+} \tilde{p}(t)\beta_+ - \int_{\Gamma^-} \tilde{p}(t)\beta_- \\ &= \int_{\Omega_f} \mathbf{F}(t)\cdot\mathbf{w}\ \forall\mathbf{w}\in V,\ w_x = \beta_\pm \text{ on } \Gamma^\pm,\ \text{a.e. in } (0,T). \end{aligned} \tag{3.21}$$

We are now in a position to obtain some of the assertions of the theorem: the existence of a pair $(\mathbf{v},\mathbf{d})$ satisfying the variational equality of (3.4) follows from (3.20) and (3.21); the regularity of the functions $\mathbf{v}$ and $\mathbf{d}$ is a consequence of (3.14), (3.15) and $(3.17)_1$. This supplementary regularity allows us to write in $(3.4)_2$ the time derivatives under the integrals.

To achieve the proof, it remains to obtain the boundary and initial conditions satisfied by $\mathbf{v}$ and $\mathbf{d}$ and the properties of the pressure.

From the estimates (3.13) we also obtain that $\{\mathbf{v}_n^m\}_{n,m\in\mathbb{N}^*}$ is bounded at least in $L^2(0,T;(H^{1/2}(\Gamma^\pm))^2)$; hence the convergence $\mathbf{v}_{n_l}^{m_q}\to\mathbf{v}$ strongly in $L^2(0,T;(L^2(\Gamma^\pm))^2)$ holds, for $l,q\to\infty$. Multiplying the boundary condition for the approximate functions $\mathbf{v}_n^m/_{\Gamma^\pm}=\dfrac{\partial(d_n)_\pm}{\partial t}\mathbf{i}$ with test functions $\boldsymbol{\tau}\in L^2(0,T;(L^2(\Gamma^\pm))^2)$ and integrating on $\Gamma^\pm\times(0,T)$ we obtain $(3.4)_3$ by passing to the limit on subsequences, with $l,q\to\infty$.

For establishing the initial conditions $(3.4)_4$ we proceed as follows. We obtain for the approximate functions $\mathbf{v}_n^m$ and $\mathbf{d}_n$ the relation that corresponds to $(3.4)_2$, i.e.

$$\begin{aligned}&\rho_f\int_{\Omega_f}\frac{\partial\mathbf{v}_n^m}{\partial t}(t)\cdot\mathbf{w}+\mu\int_{\Omega_f}\nabla\mathbf{v}_n^m(t):\nabla\mathbf{w}+\rho h\int_0^b\frac{\partial^2\mathbf{d}_n}{\partial t^2}(t)\cdot\boldsymbol{\beta}\\&+A_1\int_0^b\frac{\partial^2\mathbf{d}_n(t)}{\partial y^2}\cdot\boldsymbol{\beta}''+A_2\int_0^b\frac{\partial\mathbf{d}_n(t)}{\partial y}\cdot\boldsymbol{\beta}'+A_3\int_0^b\mathbf{d}_n(t)\cdot\boldsymbol{\beta}=\int_{\Omega_f}\mathbf{F}(t)\cdot\mathbf{w}\\&+\int_0^b\mathbf{g}(t)\cdot\boldsymbol{\beta}\ \ \forall\,\mathbf{w}\in V,\ \forall\,\boldsymbol{\beta}\in B,\ w_x=\beta_\pm\text{ on }\Gamma^\pm;\text{ a.e. in }(0,T).\end{aligned}\tag{3.22}$$

We apply next a standard technique, that can be found, e.g., in [15]. We present in what follows only the main ideas of this technique. We consider first a function $\tau\in C^1([0,T])$ with $\tau(T)=0$. We compute $\int_0^T(3.4)_2\tau$ and $\int_0^T(3.22)\tau$; in the second calculus we pass to the limit on subsequences after integrating by parts. In this way we obtain the first and the third initial condition of (3.4). Some details are necessary for the third initial condition. With the technique previously described we get

$$\int_0^b\frac{\partial\mathbf{d}}{\partial t}(0)\cdot\boldsymbol{\beta}=0\ \ \forall\,\boldsymbol{\beta}\in B,\tag{3.23}$$

which gives only $\dfrac{\partial d_\pm}{\partial t}(0)=c_\pm$, with c_+,c_- constants. Due to the regularity of $\mathbf{d}$ we also have $\displaystyle\int_0^b\left(\frac{\partial d_+}{\partial t}(0)-\frac{\partial d_-}{\partial t}(0)\right)=0$, which gives, combined with the previous property, $c_+=c_-=:c$. Introducing this into (3.23) we obtain $c\displaystyle\int_0^b(\beta_++\beta_-)=0\ \forall\,\boldsymbol{\beta}\in B$, which yields $c=0$.

Taking next a more regular test function, $\tau \in C^2([0,T])$, with $\tau(T) = \tau'(T) = 0$ and repeating the previous technique we also obtain $\mathbf{d}(0) = \mathbf{0}$.

We establish next the part of the theorem concerning the pressure. We have already introduced in (3.16) a function $\tilde{p}$, unique up to an additive function of t, with the regularity given by $(3.17)_2$. These properties give an expression for $\tilde{p}$ of the form

$$\tilde{p}(x,y,t) = q(x,y,t) + \pi(t) \quad \text{a.e. in } \Omega_f \times (0,T), \tag{3.24}$$

with $q \in L^2(0,T;H^1(\Omega_f))$ unique and $\pi \in L^2(0,T)$ an arbitrary function.

Define the set

$$A_q = \left\{\tilde{p} \in L^2(0,T;H^1(\Omega_f))/\tilde{p} = q + \pi,\ \pi \in L^2(0,T)\right\} \tag{3.25}$$

and notice that $(2.6)_1$ is verified by $(\mathbf{v},\tilde{p})$ a.e. in $\Omega_f \times (0,T)$ for any $\tilde{p} \in A_q$.

Taking now $\boldsymbol{\beta} = (\beta_+, 0)$ and then $\boldsymbol{\beta} = (0, \beta_-)$ in (3.20), integrating by parts and using the property $\displaystyle\int_0^b \beta_\pm = 0$ we get two equations involving the displacements $d_\pm$ and the pressure of the form

$$\rho h \frac{\partial^2 d_\pm}{\partial t^2} + A_1 \frac{\partial^4 d_\pm}{\partial y^4} - A_2 \frac{\partial^2 d_\pm}{\partial y^2} + A_3 d_\pm = g_\pm \pm \tilde{p}/_{x=\pm a/2} + \alpha_\pm \tag{3.26}$$

in $L^2((0,b)\times(0,T))$, with $\alpha_\pm$ arbitrary functions in $L^2(0,T)$. Since $\tilde{p}$ is an element of A_q we can write instead (3.26)

$$\rho h \frac{\partial^2 d_\pm}{\partial t^2} + A_1 \frac{\partial^4 d_\pm}{\partial y^4} - A_2 \frac{\partial^2 d_\pm}{\partial y^2} + A_3 d_\pm = g_\pm \pm q/_{x=\pm a/2} \pm \pi + \alpha_\pm. \tag{3.27}$$

The two relations of (3.27) represent $(2.6)_4$ for $p = q + \pi + \alpha$, if $\alpha_+ = -\alpha_- =: \alpha$. Let us introduce the notation

$$E_+(t) = \rho h \frac{\partial^2 d_+}{\partial t^2} + A_1 \frac{\partial^4 d_+}{\partial y^4} - A_2 \frac{\partial^2 d_+}{\partial y^2} + A_3 d_+ - g_+ - q/_{x=a/2}.$$

We notice that E_+ is uniquely determined due to the uniqueness of the functions d_+ and q. The relation (3.27) corresponding to the subscript $+$ gives $\pi + \alpha = E_+$ and so, the unique function p that satisfies, together with $\mathbf{v}, \mathbf{d}$ the problem (2.6) is $p = q + E_+$, that completes the proof. □

Corollary 3.1. *The following estimate yields*

$$\|\mathbf{v}\|_{H^1(0,T;(H_0^1(\Omega_f))^2)} \leq \tilde{\tilde{c}}(\mathbf{F},\mathbf{g}), \tag{3.28}$$

with $\tilde{\tilde{c}}^2(\mathbf{F},\mathbf{g}) = (2\mu)^{-1}\left(c^2(\mathbf{F},\mathbf{g}) + \tilde{c}^2(\mathbf{F},\mathbf{g})\right)$.

Proof. In the same way as (3.12) and (3.13) we obtain the corresponding estimates for $\mathbf{v}, \mathbf{d}$ instead of $\mathbf{v}_n^m, \mathbf{d}_n$, which lead to (3.28). □

4. The boundary control problem

There exist various optimal control problems of practical interest in connection with the blood flow through the bloodstream. For instance, in a previous paper, [16], we studied the possibility to obtain an optimal configuration for the blood pressure. The present article deals with a boundary control problem inspired by the physical situation described below. The blood flow in a leg vein in an anti-gravity sense is expressed in our model by the condition

$$\mathbf{u}\cdot\mathbf{j} \geq 0. \tag{4.1}$$

When the vein walls lose their elasticity, recirculation phenomena may appear, which means that there exists $\omega \subset \Omega_f$ with $\mathrm{meas}(\omega) > 0$ and there exists a time interval $(\tau_1, \tau_2) \subset (0,T)$ such that $u_y < 0$ a.e. in $\omega \times (\tau_1, \tau_2)$. This situation leads to leg edema and venous ulcers. The inelasticity of the vein walls is compensated by the compression exerted by the elastic stockings. In our problem, this compression is represented by the functions g_+ and g_- that act from the exterior on the vein walls, $\Gamma^{\pm}$. Taking into account the phenomenon previously described, we want to determine some forces $g_\pm$ which realise a blood flow without recirculation. For this purpose we consider the following boundary control problem: *Find some forces* $\mathbf{g}^* = (g_+^*, g_-^*)$ *such that* $u_y^* \geq 0$ *a.e. in* $\Omega_f \times (0,T)$, *where* $(\mathbf{u}^*, p^*, d_\pm^*)$ *represents the unique solution of* (2.3) *corresponding to* $\mathbf{g}^*$. In order to study the control problem presented above, we consider the cost functional $J : H^1(0,T;(L^2(0,b))^2) \mapsto \mathbb{R}$ with the expression

$$J(\mathbf{g}) = \frac{1}{2}\int_0^T \int_{\Omega_f} (\min(u_y, 0))^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}t, \tag{4.2}$$

with $(\mathbf{u}, p, d_\pm)$ the unique solution of (2.3) corresponding to $\mathbf{g}$. We also consider a bounded subset of the space for $\mathbf{g}$ chosen in (3.1), i.e.

$$B_r = \left\{\mathbf{g} \in H^1(0,T;(L^2(0,b))^2) / \|\mathbf{g}\|_{H^1(0,T;(L^2(0,b))^2)} \leq r\right\}, \tag{4.3}$$

where r is a positive given constant that may be taken arbitrarily large.

In what follows we study the following boundary control problem

$$\text{Find } \mathbf{g}^* \in B_r \text{ such that } J(\mathbf{g}^*) \leq J(\mathbf{g}) \;\; \forall\, \mathbf{g} \in B_r. \tag{CP}$$

Limiting the search of the exterior forces $\mathbf{g}$ to the bounded set B_r does not represent a generality restriction of the practical problem for which we propose this mathematical model, since the compression of the elastic stockings varies, from the medical point of view, between some admissible values.

Remark 4.1. a) If $J(\mathbf{g}^*) = 0$, it means that we found an exterior compression that prevents the blood recirculation in the vein.

b) If, for any admissible choice of the bounded set B_r $J(\mathbf{g}^*) > 0$, this means that there is no compression that ensures a blood flow without recirculation, but we founded those exterior forces which give a blood flow with minimum recirculation.

The existence of an optimal control is given in the next theorem.

Theorem 4.1. *The control problem (CP) has at least one solution.*

Proof. For obtaining this result we use a Weierstrass theorem (see e.g. [4]). We apply this result for the space $H^1(0,T;(L^2(0,b))^2)$, the set B_r and the functional J. Since the necessary properties for $H^1(0,T;(L^2(0,b))^2)$ and B_r are obvious, it remains to prove that J is weakly lower semicontinuous. In fact, we shall prove in what follows a stronger result, that J is weakly continuous. Let us consider a sequence $\{\mathbf{g}_n\}_{n\in\mathbb{N}^*} \subset B_r$ with $\mathbf{g}_n \rightharpoonup \mathbf{g}$ weakly in $H^1(0,T;(L^2(0,b))^2)$. From the definition of $c(\mathbf{F},\mathbf{g}), \tilde{c}(\mathbf{F},\mathbf{g}), \tilde{\tilde{c}}(\mathbf{F},\mathbf{g})$, and from (3.28) it follows that $\{\mathbf{v}_n\}_{n\in\mathbb{N}^*}$ is bounded in $H^1(0,T;(H_0^1(\Omega_f))^2)$ by a constant depending on r, where $(\mathbf{v}_n,\mathbf{d}_n)$ is the unique solution of (3.4) corresponding to $\mathbf{g}_n$. Hence, there exists an element $\mathbf{v} \in H^1(0,T;(H_0^1(\Omega_f))^2)$ such that on a subsequence we have

$$\begin{cases} \mathbf{v}_{n_k} \rightharpoonup \mathbf{v} \text{ weakly in } H^1(0,T;(H_0^1(\Omega_f))^2), \\ \mathbf{v}_{n_k} \rightharpoonup \mathbf{v} \text{ strongly in } L^2(0,T;(L^2(\Omega_f))^2), \end{cases} \tag{4.4}$$

when $k \to \infty$.

Moreover, from (3.12) and (3.13) written for the solution $(\mathbf{v}_n,\mathbf{d}_n)$ of (3.4) corresponding to $\mathbf{g}_n$ we also have

$$\max\left\{ \left\| \frac{\partial^2 \mathbf{d}_n}{\partial t^2} \right\|_{L^\infty(0,T;(L^2(0,b))^2)} ; \|\mathbf{d}_n\|_{W^{1,\infty}(0,T;(H_0^2(0,b))^2)} \right\} \le c(r), \tag{4.5}$$

that gives

$$\begin{cases} \dfrac{\partial^2 \mathbf{d}_{n_k}}{\partial t^2} \rightharpoonup \dfrac{\partial^2 \mathbf{d}}{\partial t^2} \text{ weakly in } L^2(0,T;(L^2(0,b))^2), \\ \mathbf{d}_{n_k} \rightharpoonup \mathbf{d} \text{ weakly in } H^1(0,T;(H_0^2(0,b))^2) \end{cases} \tag{4.6}$$

for $k \to \infty$. We notice that, in fact, the convergences (4.4), (4.6) hold on the whole sequences, due to the uniqueness of the solution of (3.4). Consequently

$$\mathbf{u}_n \to \mathbf{u} \text{ strongly in } L^2(0,T;(L^2(\Omega_f))^2). \tag{4.7}$$

Using the inequality

$$\int_{\Omega_f} (\min(f,0) - \min(h,0))^2 \le \|f-h\|_{L^2(\Omega_f)} \quad \forall\, f,h \in L^2(\Omega_f),$$

we obtain with standard computations

$$|J(\mathbf{g}_n) - J(\mathbf{g})| \leq C \|\mathbf{u}_n - \mathbf{u}\|_{L^2(0,T;(L^2(\Omega_f))^2)}, \tag{4.8}$$

with the constant C depending on $\mathbf{u}$, which means the weak continuity of J, and achieves the proof. □

5. The optimality conditions

The first result of this section concerns the G-differentiability of the functional J. Because the expression of J, this property is more difficult to obtain than usually.

Proposition 5.1. *The functional J is G-differentiable on $H^1(0,T;(L^2(0,b))^2)$. Let $\mathbf{g}, \mathbf{g}^*$ be two arbitrary elements of the space $H^1(0,T;(L^2(0,b))^2)$. Then*

$$\langle DJ(\mathbf{g}^*), \mathbf{g} - \mathbf{g}^* \rangle = \int_0^T \int_{\Omega_f} \hat{u}_y \min(u_y^*, 0) \mathrm{d}x\mathrm{d}y\mathrm{d}t, \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^1(0,T;(L^2(0,b))^2)$ and its dual, $(\mathbf{u}^*, p^*, \mathbf{d}^*)$ is the unique solution of (2.3) corresponding to $\mathbf{g}^*$ and $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{d}})$ is the unique solution of the auxiliary problem

$$\begin{cases} \rho_f \dfrac{\partial \hat{\mathbf{u}}}{\partial t} - \mu \ \Delta \hat{\mathbf{u}} + \nabla \hat{p} = \mathbf{0} \text{ in } \Omega_f \times (0,T), \\ \text{div } \hat{\mathbf{u}} = 0 \text{ in } \Omega_f \times (0,T), \\ \hat{\mathbf{u}} = \dfrac{\partial \hat{d}_\pm}{\partial t} \mathbf{i} \ \text{ on } \ \Gamma^\pm \times (0,T), \\ \rho h \dfrac{\partial^2 \hat{d}_\pm}{\partial t^2} + A_1 \ \dfrac{\partial^4 \hat{d}_\pm}{\partial y^4} - A_2 \dfrac{\partial^2 \hat{d}_\pm}{\partial y^2} + A_3 \hat{d}_\pm \\ \qquad = \pm \hat{p}_{/_{x=\pm a/2}} + (g - g^*)_\pm \text{ on } \Gamma^\pm \times (0,T), \\ \hat{\mathbf{u}} = \mathbf{0} \text{ on } \ (\partial \Omega_f \backslash (\Gamma^+ \cup \Gamma^-)) \times (0,T), \\ \hat{d}_\pm = \dfrac{\partial \hat{d}_\pm}{\partial y} = 0 \text{ in } \{0,b\} \times (0,T). \\ \hat{\mathbf{u}}(0) = \mathbf{0} \text{ in } \Omega_f, \\ \hat{d}_\pm(0) = \dfrac{\partial \hat{d}_\pm}{\partial t}(0) = 0 \text{ in } (0,b). \end{cases} \tag{5.2}$$

Proof. We first notice that the existence of a unique solution for (5.2) is given by Theorem 3.1, since the problem (5.2) is the same as (2.6) with a different right hand side.

Let us consider $\alpha \in (0,1)$, let us denote $\mathbf{g}_\alpha = \mathbf{g}^* + \alpha(\mathbf{g} - \mathbf{g}^*)$ and let $(\mathbf{u}_{\alpha g}, p_{\alpha g}, \mathbf{d}_{\alpha g})$ be the unique solution of the coupled problem (2.3) corresponding to $\mathbf{g}_\alpha$. Since the problem (2.3) is linear, we obtain with obvious computations that $\left(\dfrac{\mathbf{u}_{\alpha g}-\mathbf{u}^*}{\alpha}, \dfrac{p_{\alpha g}-p^*}{\alpha}, \dfrac{\mathbf{d}_{\alpha g}-\mathbf{d}^*}{\alpha}\right)$ is the unique solution of (5.2) for any $\alpha \in (0,1)$. Introduce next the notation

$$\begin{cases} J_1^\alpha = \dfrac{1}{2\alpha}\displaystyle\int_{\{\hat{u}_y<0\}\cap\{u_y^*<0\}} \left((u^*+\alpha\hat{u})_y^2-(u^*)_y^2\right), \\ J_2^\alpha = \dfrac{1}{2\alpha}\displaystyle\int_{\{\hat{u}_y<0\}\cap\{u_y^*\geq 0\}} \left(\min\left((u^*+\alpha\hat{u})_y,0\right)\right)^2, \\ J_3^\alpha = \dfrac{1}{2\alpha}\displaystyle\int_{\{\hat{u}_y\geq 0\}\cap\{u_y^*<0\}} \left(\left(\min\left((u^*+\alpha\hat{u})_y,0\right)\right)^2-(u^*)_y^2\right) \end{cases} \tag{5.3}$$

and notice that

$$\lim_{\alpha\searrow 0}\frac{J(\mathbf{g}^*+\alpha(\mathbf{g}-\mathbf{g}^*))-J(\mathbf{g}^*)}{\alpha} = \lim_{\alpha\searrow 0}\left(J_1^\alpha+J_2^\alpha+J_3^\alpha\right). \tag{5.4}$$

We compute in what follows the three limits of the right hand side of (5.4).

$$\bullet \lim_{\alpha\searrow 0} J_1^\alpha = \int_{\{\hat{u}_y<0\}\cap\{u_y^*<0\}} \hat{u}_y u_y^* = \int_{\{\hat{u}_y<0\}} \hat{u}_y \min(u_y^*,0).$$

• The second limit is more difficult to compute. We have $J_2^\alpha \leq a_\alpha b_\alpha$, where

$$\begin{cases} a_\alpha^2 = \displaystyle\int_{\{\hat{u}_y<0\}\cap\{u_y^*\geq 0\}} \left(\min\left(\left(\frac{u^*}{\alpha}+\hat{u}\right)_y,0\right)\right)^2, \\ b_\alpha^2 = \dfrac{1}{4}\displaystyle\int_{\{\hat{u}_y<0\}\cap\{u_y^*\geq 0\}} \left(\min\left((u^*+\alpha\,\hat{u})_y,0\right)\right)^2. \end{cases}$$

We analyze next the sequences $\{a_\alpha\}_\alpha$, $\{b_\alpha\}_\alpha$. We prove first that $\{a_\alpha\}_\alpha$ is a bounded sequence. Let $\alpha_1<\alpha_2$ be two arbitrary elements of $(0,1)$. On $\{\hat{u}_y<0\}\cap\{u_y^*\geq 0\}$ we obviously have $\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y \geq \dfrac{1}{\alpha_2}u_y^*+\hat{u}_y$ for any $0<\alpha_1<\alpha_2<1$. We compare a_{α_1} and a_{α_2} taking into account the following possibilities:

i) If $\dfrac{1}{\alpha_2}u_y^*+\hat{u}_y\geq 0$, then $\min\left(\frac{1}{\alpha_1}u_y^*+\hat{u}_y,0\right)=\min\left(\frac{1}{\alpha_2}u_y^*+\hat{u}_y,0\right)=0$;

ii) If $\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y\geq 0>\dfrac{1}{\alpha_2}u_y^*+\hat{u}_y$, then $0=\left(\min\left(\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y,0\right)\right)^2\leq \left(\min\left(\dfrac{1}{\alpha_2}u_y^*+\hat{u}_y,0\right)\right)^2$;

iii) If $0>\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y$, then $\min\left(\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y,0\right)=\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y\geq\dfrac{1}{\alpha_2}u_y^*+\hat{u}_y=\min\left(\dfrac{1}{\alpha_2}u_y^*+\hat{u}_y,0\right)$.

Hence $\left(\min\left(\dfrac{1}{\alpha_1}u_y^*+\hat{u}_y,0\right)\right)^2\leq\left(\min\left(\dfrac{1}{\alpha_2}u_y^*+\hat{u}_y,0\right)\right)^2$.

From i), ii), iii) it follows that $a_{\alpha_1} \le a_{\alpha_2}\ \forall\, \alpha_1 < \alpha_2,\ \alpha_1, \alpha_2 \in (0,1)$ which yields $0 \le a_\alpha \le a_1\ \forall\, \alpha \in (0,1)$.

For the second sequence we have $\lim\limits_{\alpha\searrow 0} b_\alpha^2 = \int_{\{\hat{u}_y<0\}\cap\{u_y^*=0\}} (u_y^*)^2 = 0$. Consequently, $\lim\limits_{\alpha\searrow 0} J_2^\alpha = 0$.

• For determining the third limit we write

$$J_3^\alpha = \frac{1}{2\alpha}\int_{\{\hat{u}_y\ge 0\}\cap\{u_y^*<0\}\cap\{(u^*+\alpha\hat{u})_y<0\}} 2\alpha u_y^*\hat{u}_y + \alpha^2(\hat{u}_y)^2 - \frac{1}{2\alpha}\int_{\{\hat{u}_y\ge 0\}\cap\{u_y^*<0\}\cap\{(u^*+\alpha\hat{u})_y\ge 0\}} (u_y^*)^2.$$

The second term satisfies

$$\frac{1}{2\alpha}\int_{\{\hat{u}_y\ge 0\}\cap\{u_y^*<0\}\cap\{(u^*+\alpha\hat{u})_y\ge 0\}} (u_y^*)^2 \le \frac{\alpha}{2}\int_{\{\hat{u}_y\ge 0\}\cap\{u_y^*<0\}\cap\{(u^*+\alpha\hat{u})_y\ge 0\}} (\hat{u}_y)^2.$$

Hence, $\lim\limits_{\alpha\searrow 0} J_3^\alpha = \int_{\{\hat{u}_y\ge 0\}\cap\{u_y^*<0\}} \hat{u}_y u_y^* = \int_{\{\hat{u}_y\ge 0\}} \hat{u}_y \min(u_y^*, 0)$.

Combining the previous computations we get

$$\langle DJ(\mathbf{g}^*), \mathbf{g}-\mathbf{g}^*\rangle = \lim_{\alpha\searrow 0} (J_1^\alpha + J_2^\alpha + J_3^\alpha) = \int_{\{\hat{u}_y<0\}} \hat{u}_y \min(u_y^*, 0) + \int_{\{\hat{u}_y\ge 0\}} \hat{u}_y \min(u_y^*, 0),$$

which represents (5.1). □

Corollary 5.1. *Let* $\mathbf{g}^*$ *be an optimal control and* $\mathbf{g} \in B_r$ *an arbitrary element. Then*

$$\int_0^T \int_{\Omega_f} \hat{u}_y \min(u_y^*, 0) \ge 0. \tag{5.5}$$

In order to establish the necessary conditions of optimality we consider the variational problem

$$\begin{cases} \text{Find } (\boldsymbol{\omega}, \boldsymbol{\delta}) \in L^2(0,T;V) \times H^1(0,T;B) \text{ with} \\ \left(\dfrac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}, \dfrac{\mathrm{d}^2\boldsymbol{\delta}}{\mathrm{d}t^2}\right) \in L^2(0,T;V') \times L^2(0,T;B'); \boldsymbol{\delta} = (\delta_+, \delta_-) \text{ s. t.} \\ -\rho_f \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{\Omega_f} \boldsymbol{\omega}(t)\cdot\mathbf{w} + \mu\int_{\Omega_f} \nabla\boldsymbol{\omega}(t) : \nabla\mathbf{w} + \rho h \dfrac{\mathrm{d}}{\mathrm{d}t}\int_0^b \dfrac{\partial\boldsymbol{\delta}(t)}{\partial t}\cdot\boldsymbol{\beta} \\ \quad + A_1\displaystyle\int_0^b \dfrac{\partial^2\boldsymbol{\delta}(t)}{\partial y^2}\cdot\boldsymbol{\beta}'' + A_2\int_0^b \dfrac{\partial\boldsymbol{\delta}(t)}{\partial y}\cdot\boldsymbol{\beta}' + A_3\int_0^b \boldsymbol{\delta}(t)\cdot\boldsymbol{\beta} \\ \quad = \displaystyle\int_{\Omega_f} w_y \min(u_y^*(t), 0) \\ \qquad\qquad \text{a.e. in } (0,T), \forall\, \mathbf{w}\in V,\ \forall\, \boldsymbol{\beta}\in B,\ w_x = \beta_\pm \text{ on } \Gamma^\pm, \\ \boldsymbol{\omega} = -\dfrac{\partial\delta_\pm}{\partial t}\mathbf{i} \text{ on } \Gamma^\pm \times (0,T), \\ \boldsymbol{\omega}(T) = \mathbf{0} \text{ in } \Omega_f,\ \boldsymbol{\delta}(T) = \dfrac{\partial\boldsymbol{\delta}}{\partial t}(T) = \mathbf{0} \text{ in } (0,b). \end{cases} \tag{5.6}$$

For obtaining existence, uniqueness and regularity results for problem (5.6) we introduce the new functions

$$\left\{\begin{array}{l}\tilde{\boldsymbol{\omega}}(t)=\boldsymbol{\omega}(T-t),\\ \tilde{\boldsymbol{\delta}}(t)=\boldsymbol{\delta}(T-t)\end{array}\right. \qquad \forall\ t\in[0,T] \tag{5.7}$$

and we prove

Proposition 5.2. $(\boldsymbol{\omega},\boldsymbol{\delta})$ *is solution for* (5.6) *if and only if* $(\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\delta}})$ *verifies:*

$$\left\{\begin{array}{l}\text{Find } (\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\delta}})\in L^2(0,T;V)\times H^1(0,T;B) \text{ with}\\ \left(\dfrac{\mathrm{d}\tilde{\boldsymbol{\omega}}}{\mathrm{d}t},\dfrac{\mathrm{d}^2\tilde{\boldsymbol{\delta}}}{\mathrm{d}t^2}\right)\in L^2(0,T;V')\times L^2(0,T;B');\tilde{\boldsymbol{\delta}}=(\tilde{\delta}_+,\tilde{\delta}_-)\text{ s. t.}\\ \rho_f\dfrac{\mathrm{d}}{\mathrm{d}\tau}\displaystyle\int_{\Omega_f}\tilde{\boldsymbol{\omega}}(\tau)\cdot\mathbf{w}+\mu\int_{\Omega_f}\nabla\tilde{\boldsymbol{\omega}}(\tau):\nabla\mathbf{w}+\rho h\dfrac{\mathrm{d}}{\mathrm{d}\tau}\int_0^b\dfrac{\partial\tilde{\boldsymbol{\delta}}(\tau)}{\partial\tau}\cdot\boldsymbol{\beta}\\ \quad +A_1\displaystyle\int_0^b\dfrac{\partial^2\tilde{\boldsymbol{\delta}}(\tau)}{\partial y^2}\cdot\boldsymbol{\beta}''+A_2\int_0^b\dfrac{\partial\tilde{\boldsymbol{\delta}}(\tau)}{\partial y}\cdot\boldsymbol{\beta}'+A_3\int_0^b\tilde{\boldsymbol{\delta}}(\tau)\cdot\boldsymbol{\beta}\\ \quad =\displaystyle\int_{\Omega_f}w_y\min(u_y^*(T-\tau),0)\\ \qquad\qquad\qquad \text{a.e. in } (0,T),\forall\,\mathbf{w}\in V,\ \forall\,\boldsymbol{\beta}\in B,\ w_x=\beta_\pm \text{ on } \Gamma^\pm,\\ \tilde{\boldsymbol{\omega}}=\dfrac{\partial\tilde{\delta}_\pm}{\partial\tau}\mathbf{i}\text{ on } \Gamma^\pm\times(0,T),\\ \tilde{\boldsymbol{\omega}}(0)=\mathbf{0}\text{ in }\Omega_f,\ \tilde{\boldsymbol{\delta}}(0)=\dfrac{\partial\tilde{\boldsymbol{\delta}}}{\partial\tau}(0)=\mathbf{0}\text{ in }(0,b).\end{array}\right. \tag{5.8}$$

Proof. Denote $\tau=T-t$. The result is obvious if we notice that $\dfrac{\partial}{\partial\tau}=-\dfrac{\partial}{\partial t}$ and $\dfrac{\partial^2}{\partial\tau^2}=\dfrac{\partial^2}{\partial t^2}$. ☐

We are now in a position to prove

Theorem 5.1. *Problem* (5.6) *has a unique solution* $(\boldsymbol{\omega}^*,\boldsymbol{\delta}^*)$ *with the regularity*

$$\left\{\begin{array}{l}\boldsymbol{\omega}^*\in W^{1,\infty}(0,T;(L^2(\Omega_f))^2)\cap H^1(0,T;V)\cap L^2(0,T;(H^2(\Omega_f))^2),\\ \boldsymbol{\delta}^*\in W^{2,\infty}(0,T;(L^2(0,b))^2)\cap W^{1,\infty}(0,T;B).\end{array}\right. \tag{5.9}$$

Moreover, there exists a unique function $\pi^*\in L^2(0,T;H^1(\Omega_f))$ *s. t.* $(\boldsymbol{\omega}^*,\pi^*,\boldsymbol{\delta}^*)$

satisfies the adjoint system

$$\begin{cases} -\rho_f \dfrac{\partial \boldsymbol{\omega}^*}{\partial t} - \mu\ \Delta \boldsymbol{\omega}^* + \nabla \pi^* = \min(u_y^*, 0)\mathbf{j} \text{ in } \Omega_f \times (0,T), \\ \text{div } \boldsymbol{\omega}^* = 0 \text{ in } \Omega_f \times (0,T), \\ \boldsymbol{\omega}^* = -\dfrac{\partial \delta_\pm^*}{\partial t}\mathbf{i} \text{ on } \Gamma^\pm \times (0,T), \\ \rho h \dfrac{\partial^2 \delta_\pm^*}{\partial t^2} + A_1 \dfrac{\partial^4 \delta_\pm^*}{\partial y^4} - A_2 \dfrac{\partial^2 \delta_\pm^*}{\partial y^2} + A_3 \delta_\pm^* = \pm \pi^*_{/_{x=\pm a/2}} \text{ on } \Gamma^\pm \times (0,T), \\ \boldsymbol{\omega}^* = \mathbf{0} \text{ on } (\partial\Omega_f \backslash (\Gamma^+ \cup \Gamma^-)) \times (0,T), \\ \delta_\pm^* = \dfrac{\partial \delta_\pm^*}{\partial y} = 0 \text{ in } \{0,b\} \times (0,T). \\ \boldsymbol{\omega}^*(T) = \mathbf{0} \text{ in } \Omega_f, \\ \delta_\pm^*(T) = \dfrac{\partial \delta_\pm^*}{\partial t}(T) = 0 \text{ in } (0,b). \end{cases} \tag{5.10}$$

Proof. From $\mathbf{u}^* = \mathbf{v}^* + \mathbf{u}_0$, the regularity of $\mathbf{v}^*$ given by Theorem 3.1 and $(2.4)_1$ we get $u_y^* \in H^1(0,T;(H^1(\Omega_f))$ and, applying e.g. Theorem A.1, [9], it follows that $\min(u_y^*, 0)\mathbf{j} \in H^1(0,T;(H^1(\Omega_f))^2)$. The proof is achieved if we notice that the variational problem (5.8) represents exactly (3.4) with the changes: $\mathbf{F} \mapsto \min(u_y^*, 0)\mathbf{j}$, $\mathbf{g} \mapsto \mathbf{0}$ and, consequently, we apply Theorem 3.1. ☐

The last result of this article is represented by the necessary conditions of optimality, obtained in the theorem below:

Theorem 5.2. *Let $\mathbf{g}^*$ be an optimal control. Then there exist the unique triplets $(\mathbf{u}^*, p^*, \mathbf{d}^*)$ and $(\boldsymbol{\omega}^*, \pi^*, \boldsymbol{\delta}^*)$ such that:*

i) $(\mathbf{u}^*, p^*, \mathbf{d}^*)$ is the unique solution of (2.3) corresponding to $\mathbf{g}^*$;

ii) $(\boldsymbol{\omega}^*, \pi^*, \boldsymbol{\delta}^*)$ is the unique solution of (5.10);

iii) $\displaystyle\int_0^T \int_0^b \frac{\partial \boldsymbol{\delta}^*}{\partial t} \cdot (\mathbf{g}^* - \mathbf{g}) \mathrm{d}y\mathrm{d}t \geq 0 \quad \forall\, \mathbf{g} \in B_r.$

Proof. The assertions i) and ii) follow from Theorem 3.1 and Theorem 5.1, respectively. Hence, it remains to prove iii). For this purpose, we take $\left(\boldsymbol{\omega}^*(t), -\dfrac{\partial \boldsymbol{\delta}^*}{\partial t}(t)\right)$ as test function in $(5.2)_{1,4}$, we integrate on $\Omega_f \times (0,T)$ and on $(0,b) \times (0,T)$, respectively and we use $(5.6)_4$. These give

$$\begin{aligned} &-\rho_f \int_0^T\!\!\int_{\Omega_f} \frac{\partial \boldsymbol{\omega}^*}{\partial t} \cdot \hat{\mathbf{u}} + \mu \int_0^T\!\!\int_{\Omega_f} \nabla \boldsymbol{\omega}^* : \nabla \hat{\mathbf{u}} + \rho h \int_0^T\!\!\int_0^b \frac{\partial^2 \boldsymbol{\delta}^*}{\partial t^2} \cdot \frac{\partial \hat{\mathbf{d}}}{\partial t} \\ &+ A_1 \int_0^T\!\!\int_0^b \frac{\partial^2 \boldsymbol{\delta}^*}{\partial y^2} \cdot \frac{\partial^3 \hat{\mathbf{d}}}{\partial y^2 \partial t} + A_2 \int_0^T\!\!\int_0^b \frac{\partial \boldsymbol{\delta}^*}{\partial y} \cdot \frac{\partial^2 \hat{\mathbf{d}}}{\partial y \partial t} + A_3 \int_0^T\!\!\int_0^b \boldsymbol{\delta}^* \cdot \frac{\partial \hat{\mathbf{d}}}{\partial t} \\ &= -\int_0^T \int_0^b \frac{\partial \boldsymbol{\delta}^*}{\partial t} \cdot (\mathbf{g} - \mathbf{g}^*). \end{aligned} \tag{5.11}$$

We take next $\left(\hat{\mathbf{u}}(t), \dfrac{\partial \hat{\mathbf{d}}}{\partial t}(t)\right)$ as test function in (5.6) and we integrate from 0 to T. This yields

$$\begin{aligned}
&-\rho_f \int_0^T \int_{\Omega_f} \frac{\partial \boldsymbol{\omega}^*}{\partial t} \cdot \hat{\mathbf{u}} + \mu \int_0^T \int_{\Omega_f} \nabla \boldsymbol{\omega}^* : \nabla \hat{\mathbf{u}} + \rho h \int_0^T \int_0^b \frac{\partial^2 \boldsymbol{\delta}^*}{\partial t^2} \cdot \frac{\partial \hat{\mathbf{d}}}{\partial t} \\
&+A_1 \int_0^T \int_0^b \frac{\partial^2 \boldsymbol{\delta}^*}{\partial y^2} \cdot \frac{\partial^3 \hat{\mathbf{d}}}{\partial y^2 \partial t} + A_2 \int_0^T \int_0^b \frac{\partial \boldsymbol{\delta}^*}{\partial y} \cdot \frac{\partial^2 \hat{\mathbf{d}}}{\partial y \partial t} + A_3 \int_0^T \int_0^b \boldsymbol{\delta}^* \cdot \frac{\partial \hat{\mathbf{d}}}{\partial t} \qquad (5.12) \\
&= \int_0^T \int_{\Omega_f} \hat{u}_y \min(u_y^*, 0).
\end{aligned}$$

Finaly, combining (5.11), (5.12) and (5.5) we get iii), which completes the proof. □

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Ruxandra Stavre
Institute of Mathematics "Simion Stoilow", Romanian Academy, Research Unit 6
P.O. Box 1-764, 014700 Bucharest, Romania
E-mail: `ruxandra.stavre@imar.ro`