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Some diffusion effects in Hele-Shaw immiscible displacements

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Dedicated to Professor Nicolaie D. Cristescu in honour of his 85th birthday

Abstract - Saffman and Taylor (1958) proved the linear instability of the displacement of two immiscible fluids in a Hele-Shaw cell, when the displacing fluid is less viscous. We minimize this instability by using a Middle-Layer, containing a polymer solute with a variable viscosity, between the two initial fluids. A diffusion process is considered here, governing the variable viscosity. The linear stability of the displacement is governed by a Sturm-Liouville system. In some previous papers was proved the improvement of stability, for large enough diffusion coefficient. In this paper we give simpler estimates of the growth constant (in time) of the linear perturbations, by using an analysis of the maximum points of the eigenfunctions of the governing Sturm-Liouville system.

Key words and phrases : Linear stability, Hele-Shaw immiscible displacements, porous media, diffusion.

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1. Introduction

Consider the displacement of two immiscible fluids (water and oil) in a horizontal Hele-Shaw cell. This model can be used for the study of the Secondary Oil Recovery from a porous medium. The instability of the flow, when the displacing fluid (water) is less viscous, was first studied by Saffman and Taylor (see [10]). This instability can be minimized by using a Middle Layer region (denoted by $M.L.$) between the initial immiscible fluids, where a polymer-solute exists. We suppose that the viscosity here is an unknown invertible function μ in terms of the polymer concentration c.

Numerical and experimental results were given by Slobod and Lestz (see [11]), Mungan (see [6]), Uzoigve *et al.* (see [13]), Shah and Schecter (see [12]), Gorell and Homsy (see [5]), Wang and Dong ([14]), related with some exponential viscosity profiles in $M.L.$. Theoretical results were obtained by Carasso and Paşa (see [1]), Paşa (see [7]), Daripa and Paşa (see [2]), Paşa and Titaud (see [9]). In these models, the polymer concentration of the fluid particles is constant.

In the present paper we consider a diffusion process in $M.L.$ The polymer "concentration" of each fluid particle is not constant, but verifies a diffusion equation with the diffusion coefficient η. We study the linear stability of the basic flow given by a basic linear viscosity $\mu(x) = ax + b$ in $M.L.$. This time, two eigenfunctions appear in the governing Sturm-Liouville system, related with the "amplitude" of the perturbations of horizontal velocity and viscosity. The eigenfunctions of the governing Sturm-Liouville system are the growth constant (in time) of the perturbations.

Daripa and Paşa (see [3]) proved the stabilizing effect of the diffusion process for large enough η, but an exact sufficient stability condition in terms of η was not given. Paşa (see [8]) obtained estimates of σ by using a generalization of the Gerschgorin's localization theorem and proved that the diffusion coefficient η can not improve the displacement stability for small wave numbers of perturbations.

In the present paper we improve these results. We use a numerical procedure only for the first equation (2.3) of the stability system. The exact form of the second equation (2.4) is used to get a direct estimate of σ, only by using the rectangle rule for approximating the integrals on a finite interval and a Poincaré type inequality for the eigenfunction h. The same conclusion is obtained: the diffusion coefficient η alone can not improve the displacement stability for small wavenumbers α of perturbations. On the contrary, a strong improvement of the stability is given by the diffusion process, for large α.

2. The stability analysis

A three-layer Hele-Shaw cell with horizontal plates is considered in the fixed x_1Oy plane. We neglect the gravity effects. The Hele-Shaw cell is filled by water (with constant viscosity μ_1), a polymer-solute (with variable viscosity μ) and oil (with constant viscosity μ_2). We suppose $\mu_1 \leq \mu < \mu_2$. The middle-layer between water and oil is the segment $Ut - l < x_1 < Ut$. The polymer-solute with the variable viscosity μ is contained in this region. The water velocity U far upstream is giving the displacement in the positive Ox_1 direction.

In the "left" end of $M.L.$, $x = -l$, we consider a continuous viscosity and zero surface tension. In the "right" end, $x = 0$, a viscosity jump $[\mu_2 - \mu_0]$ and a surface tension T exist. The left limit of the viscosity μ in the "right" end $x = 0$ is denoted by $\mu_0 = \mu(0) = \mu^-(0)$.

The fluid velocity and pressure are denoted by (u, v) and p. The polymer concentration c verifies a diffusion equation in $M.L.$. As the viscosity in $M.L.$ is supposed invertible in terms of c, in $M.L.$ the viscosity μ verifies the equation

$$\mu_t + u\nabla\mu = \eta\Delta\mu. \tag{2.1}$$

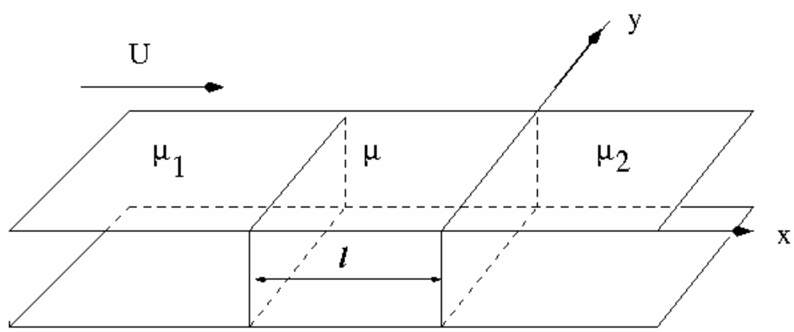

Figure 1: Basic flow in the three-layer Hele-Shaw cell: water, polymer-solute, oil.

We consider a basic viscosity μ in $M.L.$ which is linear in terms of the moving spatial coordinate $x = x_1 - Ut$:

$$\mu(x,t) = ax + b = x\frac{\mu_0 - \mu_1}{l} + \mu_0, \quad x \in (-l, 0). \tag{2.2}$$

The basic velocity components of the basic velocity are $(U, 0)$. The basic pressure is given by $P_x = -\mu U,\ P_y = 0$. The basic initial straight interfaces are $\Gamma_1 : x = -l;\ \Gamma_2 : x = 0$. The basic solution is described in Figure 1.

The flow in all porous medium is governed by the Darcy's law and the continuity equation. We consider the Laplace's law on the interfaces: the pressure jump is balanced by the surface tension multiplied with the curvature and the normal velocity is continuous.

u', v', p', μ' are the perturbations of the basic velocity, pressure and viscosity. Far upstream and downstream we have $u', v' = 0$ and $\mu' = 0 \ \ on \ \ \Gamma_i$.

As the problem is linear, the horizontal perturbation is decomposed in Fourier modes:

$$u'(x,y,t) = f(x) \ \exp\left(i\alpha y + \sigma t\right)$$

where σ, α are the growth constant (in time) and the wave numbers of the perturbations. The free divergence conditions gives us v' in terms of u'. The Darcy's law gives us p' in terms of σ and α - see eqs. (24), (25), (26) from [8]. The viscosity perturbation is

$$\mu'(x,y,t) = h(x) \ \exp\left(i\alpha y + \sigma t\right)$$

The perturbed interfaces and the limit values of the pressure on Γ_i are given in [8] - see eqs. (28), (29), (30). The cross derivation of the pressure derivatives and the diffusion equation for the basic viscosity are giving the following system which governs the stability of the flow in our three-layer Hele-Shaw cell:

$$- (\mu f_x)_x + \mu\alpha^2 f = -\alpha^2 U h, \quad x \in (-l, 0); \tag{2.3}$$

$$\eta h_{xx} - (\sigma + \eta\alpha^2)h = af, \quad x \in (-l, 0); \tag{2.4}$$

$$f_x^+(-l) = \alpha f(-l), \quad f_x^-(0) = f(0)\{e/\sigma + q\}; \tag{2.5}$$

$$h(0) = h(-l) = 0; \tag{2.6}$$

$$e = \frac{U\alpha^2(\mu_2 - \mu(0)) - \alpha^4 T}{\mu(0)}, \quad q = -\frac{\mu_2\alpha}{\mu(0)}. \tag{2.7}$$

In [3] were obtained estimates of the growth constant σ in term of η, but depending on the eigenfunctions f, h. A variational formulation of the problem described by the above system (2.3)-(2.7) was used. In [8] we obtained an estimate of σ in terms of the problem data, not depending on f, h, by using a generalization of the Gerschgorin's localization theorem for a system with two eigenfunctions.

In the present paper we use a simpler method for estimate the growth constant, based on the analysis of the points where the maximum value of $|f(x)|$ is attained on the interval $[-l, 0]$. We have three possibilities, as follows.

i) Max $|f(x)| = |f(0)|$. In this case we use an approximation of the second boundary condition (2.5):

$$f_x(0) \approx [f(0) - f(-d)]/d = (e/\sigma + q)f(0), \tag{2.8}$$

where d is a small positive number and e, q are given by (2.7). The precision order of the above approximation is $O(d)$. The diffusion coefficient η is not appearing in the above relation. We estimate the real part σ_R of the growth constant as follows:

$$f(0)(1 - dq) - f(-d) = \frac{ed}{\sigma} f(0), \; (1 - dq) = \frac{f(-d)}{f(0)} + \frac{ed}{\sigma} \leq 1 + \frac{ed}{|\sigma|} \Rightarrow$$

$$\sigma_R \leq |\sigma| \leq \frac{e}{-q} = \frac{\alpha U(\mu_2 - \mu(0)) - \alpha^3 T}{\mu_2} \tag{2.9}$$

$\square$

ii) Max $|f(x)| = |f_M| := |f(x_M)|$, $-l < x_M < 0$. Let * denote the complex conjugate. We introduce the notation

$$F = -\int_{-l}^{0} f(x)h^*(x)dx = F_1 + iF_2. \tag{2.10}$$

We multiply (2.4) with h^*, we integrate on $(-l, 0)$ and obtain

$$\sigma_R + \alpha^2\eta = \frac{aF_1 - \eta\int_{-l}^0 |h_x|^2}{\int_{-l}^0 |h|^2}. \tag{2.11}$$

The Cauchy-Schwartz inequality is giving

$$|F_1|^2 \leq \int_{-l}^{0} |f|^2 \int_{-l}^{0} |h|^2,$$

therefore from the last two relations it follows

$$\sigma_R + \alpha^2\eta \leq a\frac{\sqrt{|f|^2}}{\sqrt{|h|^2}} - \eta\frac{\int_{-l}^{0} |h_x|^2}{\int_{-l}^{0} |h|^2}. \tag{2.12}$$

A mean formula is used for the two integrals in the first term of the right hand-side of the above inequality. There exists the points $r, s \in (-l, 0)$ such that

$$\int_{-l}^{0} |f|^2 = l|f(r)|^2; \quad \int_{-l}^{0} |h|^2 = l|h(s)|^2. \tag{2.13}$$

We have $|f(r)| \leq |f_M|$, then

$$\sigma_R + \alpha^2\eta =\leq a\frac{|f(r)|}{|h(s)|} - \eta\frac{\int_{-l}^{0} |h_x|^2}{\int_{-l}^{0} |h|^2} \leq a\frac{|f_M|}{|h(s)|} - \eta\frac{\int_{-l}^{0} |h_x|^2}{\int_{-l}^{0} |h|^2}. \tag{2.14}$$

In [3] was not possible to estimate the ratio $|f(r)|/|h(s)|$. To overcome this difficulty, we use the above inequality (2.14).

A first new element of this paper, compared with the previous results, is a Poincaré type inequality for the second (negative) term in the right hand side of the relation (2.14). As $h(-l) = 0$, we have $h(x) = \int_{-l}^{x} h_x(y)dy$, then

$$\int_{-l}^{0} |h|^2 dy \leq l\int_{-l}^{0} |h_x|^2 dy. \tag{2.15}$$

Recall a is the slope of the viscosity, then from (2.2)

$$a = (\mu_0 - \mu_1)/l. \tag{2.16}$$

From the last three relations we get

$$\sigma_R + \alpha^2\eta \leq \frac{1}{l}\{(\mu_0 - \mu_1)\frac{|f_M|}{|h(s)|} - \eta\}. \tag{2.17}$$

The ratio $|f_M|/|h(s)|$ is considered in the equivalent form

$$\frac{|f_M|}{|h(s)|} = \frac{|f_M|}{|h_M|}\frac{|h_M|}{|h(s)|}, \quad h_M := h(x_M). \tag{2.18}$$

We must prove $h_M \neq 0$. For this, we use the first stability equation (2.3) in the point x_M. We approximate the derivative of f in a point $y \in (-l, 0)$ by the formula

$$f_x(y) \approx \frac{f(y + d/2) - h(y - d/2)}{d},$$

whith the precision order $O(d^2)$. The *discretized* form of the equation (2.3) in the point $y = x_M$ is

$$- f(y+d)\frac{\mu(y+d/2)}{d^2} + f(y)[\frac{\mu(y+d/2)+\mu(y-d/2)}{d^2} + \alpha^2\mu(y)] - \tag{2.19}$$

$$-f(y-d)\frac{\mu(y-d/2)}{d^2} = -\alpha^2 U h(y).$$

Suppose now $h(y) = h(x_M) = 0$, then from the last equality we obtain

$$|\mu(y+d/2) + \mu(y-d/2) + d^2\alpha^2\mu(y)| \leq |\mu(y+d/2)| + |\mu(y-d/2)|$$

because $|f(y)|$ is the maximum value of $|f(x)|$. This last last inequality is giving $d^2\alpha^2 \leq 0$, which is false, then indeed we have $h_M \neq 0$.

A second new element of this paper is a direct estimate of the ratio $|f_M|/|h_M|$ in terms of the problem data, as follows. We use again the discretized form (2.19) of the equation (2.3) in $y = x_M$, we divide with $|f_M|$ and get

$$|A_{MM}| \leq \alpha^2 U\frac{|h_M|}{|f_M|} + |A_{M.M-1}| + |A_{M,M+1}|, \tag{2.20}$$

where the elements of the matrix A are given by the following formulas:

$$A_{MM} = \frac{\mu(y+d/2)+\mu(y-d/2)}{d^2} + \alpha^2\mu(y), \quad A_{M,M-1} = -\frac{\mu(y-d/2)}{d^2}$$

$$A_{M,M+1} = -\frac{\mu(y+d/2)}{d^2}.$$

It follows that A is diagonal dominant: $|A_{MM}| > |A_{M,M-1}| + |A_{M,M+1}|$. From (2.20) we get

$$\frac{|f_M|}{|h_M|} \leq \alpha^2 U\frac{1}{|A_{MM}| - |A_{M,M-1}| - |A_{M,M+1}|} = \frac{\alpha^2 U}{\alpha^2\mu(x_M)} \leq \frac{U}{\mu_1}. \tag{2.21}$$

The relations (2.17), (2.18) and (2.21) give us

$$\sigma_R + \alpha^2\eta \leq \frac{1}{l}\{\frac{U(\mu_0-\mu_1)}{\mu_1}\cdot\frac{|h_M|}{|h(s)|} - \eta\}. \tag{2.22}$$

As we don't know the (finite) value of $|h_M|/|h(s)|$, the final general result in the case **ii)** is:

There exists a large enough η such that

$$\sigma_R \leq \frac{U(\mu_0-\mu_1)}{l\mu_1} - \alpha^2\eta. \tag{2.23}$$

□

Remark 1. We not used any discretization of the stability equation (2.4) but only a mean formula for the integrals of the eigenfunctions f, h on $[-l, 0]$. However, a numerical approximation of h_{xx} only in a particular point can give us the same estimate (2.23). For this, we consider the maximum value $|h(x_P)|$ of $|h(x)|$, attained in the interior point x_P, and we approximate (2.4) in the point x_P as follows:

$$\frac{\eta}{d^2}\{h(x_P-d)-2h(x_P)+h(x_P+d)\}-(\sigma+\alpha^2\eta)h(x_P)=af(x_P), \quad (2.24)$$

then we get

$$(\sigma+\alpha^2\eta)+2\frac{\eta}{d^2}=-a\frac{f(x_P)}{h(x_P)}+\frac{\eta}{d^2}\cdot\frac{h(x_P-d)+h(x_P+d)}{h(x_P)},$$

$$|\sigma+\alpha^2\eta+2\frac{\eta}{d^2}|\leq a\frac{|f(x_P)|}{|h(x_P)|}+\frac{\eta}{d^2}\cdot 2\Rightarrow\sigma_R+\alpha^2\eta\leq a\frac{|f(x_P)|}{|h(x_P)|},\quad\forall\eta>0.$$

As $|f(x_M)|$ is the maximum value of $|f(x)|$, we have the inequalities

$$\frac{|f(x_P)|}{|h(x_P)|}\leq\frac{|f(x_M)|}{|h(x_M)|}\cdot\frac{|h(x_M)|}{|h(x_P)|},\quad\frac{|h(x_M)|}{|h(x_P)|}\leq 1$$

and we obtain

$$\sigma_R+\alpha^2\eta\leq a\frac{|f(x_M)|}{|h(x_M)|}.$$

This last relation and the inequality (2.21) are giving the same previous estimate (2.23). □

iii) Max $|f(x)|=|f(-l)|$. In this case we approximate the first boundary condition (2.5) in the point $-l$ by the formula

$$f_x(-l)\approx[f(-l+d)-f(-l)]/d=\alpha f(-l). \quad (2.25)$$

Then $f(-l+d)-f(-l)\approx\alpha df(-l)$, the maximum value of f is attained in the interior point $(-l+d)$ and we use the analysis of the previous case **ii)** . □

From the above formulas (2.23) and (2.9) we get the following estimate of the real part of the growth constant:

$$\sigma_R\leq MAX\{\frac{U(\mu_0-\mu_1)}{l\mu_1}-\alpha^2\eta;\quad\frac{\alpha U(\mu_2-\mu_0)-\alpha^3T}{\mu_2}\} \quad (2.26)$$

We can see that the possible upper bound of σ_R obtained in the case **i)** can not be improved by the diffusion coefficient η. On the contrary, in the case **ii)**, the diffusion process can improve the displacement stability, but only for large values of the wave number α. For small α, an improved stability is obtained for small $U, (\mu_2 - \mu_0)$ and large values of l, T.

Remark 2. The estimate (2.21) in the point **ii)** above is a particular case of a more general result. Let A be a square diagonal dominant matrix and f a vector with the maximum modulus value $|f_p|$. Then the following inequality holds:

$$\frac{|f_p|}{|A_{pk} f_k|} \leq \frac{1}{|A_{pp}| - \sum_{k \neq p} |A_{pk}|}. \tag{2.27}$$

Indeed, we have

$$A_{pk} f_k = A_{pp} f_p + \sum_{k \neq p} A_{pk} f_k, \quad A_{pp} = \frac{A_{pk} f_k}{f_p} - \sum_{k \neq p} A_{pk} \frac{f_k}{f_p},$$

$$|A_{pp}| \leq |\frac{A_{pk} f_k}{f_p}| + \sum_{k \neq p} |A_{pk}|, \quad |\frac{A_{pk} f_k}{f_p}| \geq |A_{pp}| - \sum_{k \neq p} |A_{pk}|$$

because $|f_k / f_p| \leq 1$. □

Remark 3. The largest upper bound of σ_R is obtained from the maximum value in terms of α of the right hand side of (2.26):

$$\sigma_R \leq MAX\{\frac{U(\mu_0 - \mu_1)}{l\mu_1}, \quad \frac{[U(\mu_2 - \mu_0)]^{3/2}}{6\mu_2\sqrt{3T}}\} \tag{2.28}$$

□

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