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A numerical approach for stress distribution during creep of the rock around horizontal tunnel

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Dedicated to Professor Nicolaie D. Cristescu in honour of his 85th birthday

Abstract - Study of stress distribution during creep of the rock surrounding a circular horizontal tunnel is a very important problem, mainly for mining engineering. At big depths, an opening excavated in rock can close completely after time intervals which are of the order of several tens of years. A non-associated elasto/viscoplastic constitutive equation is used to describe both compressibility and/or dilatancy during transient and steady-state creep, as well as evolutive damage possibly leading to failure. The variation of stress during creep convergence of a deep borehole excavated in rock salt is examined. An in-house FEM numerical method is used for this purpose.

Key words and phrases : elasto/viscoplastic constitutive equation, rock mechanics, compressibility, dilatancy, finite element method.

Mathematics Subject Classification (2010) **:** 74C10, 74S05.

1. Introduction

In this paper we determine a complete numerical solution about behaviour of elasto-viscoplastic rock around circular horizontal tunnel. The obtained numerical approach is comparative with simplified solution (creep solution) and linear elastic solution (instantaneous response).

The problem was studied for the other author analysing different aspects. In some cases was used linear elasticity (Obert and Duvall [9], Jumikis [6], Massier ([7], [8]) had used a linear viscoelastic model with plane state of deformations. A study of this problem using an elasto-viscoplastic model proposed by Cristescu [1] was done by Cristescu and Hunche [4], Paraschiv and Cristescu [10], supposed plane state of deformations and determined a creep solution. In this paper we determine a numerical solution without using hypothesis that stress state remains constant in time. For numerical solution we used the scheme proposed by Paraschiv-Munteanu in [11], [12], [15] using finit element method for spatial integration and a complet implicit method for integration in time. In most cases we observed that in proximity of underground opening the stress becomes relaxed relatively to the moment of excavation. However, for short period of time the creep solution and the numerical solution are very close.

2. The mathematical constitutive equation

This is done in this paper using an elastic-viscoplastic non-associated constitutive equation (see Cristescu [1], Cristescu and Hunsche [4]). The reference configuration, with respect to which the strains must be estimated, is the state *in situ* before excavation, there where the future excavation is envisaged. Using standard notation we have for the rate of deformation tensor:

$$\dot{\boldsymbol{\varepsilon}} = \frac{\dot{\boldsymbol{\sigma}}^{\mathrm{R}}}{2\mathrm{G}} + \left(\frac{1}{3\mathrm{K}} - \frac{1}{2\mathrm{G}}\right) \dot{\sigma}^{\mathrm{R}} \mathbf{1} + k_{\mathrm{T}} \left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\boldsymbol{\sigma})} \right\rangle \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}) + k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}), \quad (2.1)$$

where G and K are elastic moduli which may depend of the invariants of the stress and strain and possibly on the damage of rock, $H(\boldsymbol{\sigma}(t)) = W^{\mathrm{I}}(t)$ is the equation of the stabilization boundary for the transient creep, $F(\boldsymbol{\sigma})$ and $S(\boldsymbol{\sigma})$ are the viscoplastic potentials for the transient and stationary creep respectively, k_{T} and k_{S} are the corresponding viscosity coefficients, $\mathbf{1}$ is the unit tensor and $\sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ is the mean stress. We will use the notation τ for the octahedral shear stress:

$$\tau = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right].$$

In (2.1) the work-hardening parameter (the internal state variable) is the irreversible stress work per unit volume:

$$W^{\mathrm{I}}(t) = W^{\mathrm{P}} + \int_0^t \sigma(s)\dot{\varepsilon}_v^{\mathrm{I}}(s)\ \mathrm{d}s + \int_0^t \boldsymbol{\sigma}'(s) \cdot \left(\dot{\boldsymbol{\varepsilon}}^{\mathrm{I}}\right)'(s)\ \mathrm{d}s, \quad (2.2)$$

where $t = 0$ is the moment of excavation and the W^{P} is the primary (initial) value of W^{I} (Cristescu [1]). In (2.2) $\varepsilon_v^{\mathrm{I}}$ is the irreversible part of the volumetric deformation and $\left(\dot{\boldsymbol{\varepsilon}}^{\mathrm{I}}\right)'$ is the deviator of the irreversible part of the rate of deformation tensor. Finally, the bracket from (2.1) has the meaning of the positive part the mentioned function, i.e.: $\langle A \rangle = A^+ = \frac{1}{2}(A + |A|)$.

All constitutive functions and parameters involved in (2.1) are determined from experimental data. The constitutive equation (2.1) can describe the following mechanical properties exhibited by most rocks: transitory and stationary creep, work-hardening during transient creep, volumetric compressibility and/or dilatancy, as well as short-term failure. All these properties are incorporated into the constitutive equation via the procedure used to determine the constitutive functions (Cristescu [1], [2], Cristescu and Hunche [4]).

3. Mechanical problem formulation

We assume that the problem to solve is formulated in cylindrical coordinate (r, θ, z). Let a the initial radius of the cavity and $m \in \mathbf{N}$, $m \geq m_0 \geq 5$, number of radius which defined the limits of the domain, $\Omega = [a, ma] \times [0, 2\pi)$ (that represent the circular crown $D = \left\{ (x, y) \mid a^2 \leq x^2 + y^2 \leq (ma)^2 \right\}$ in cylindrical coordinate).

We assume that in all horizontal directions the primary stresses are the same, σ_h, and the depth is sufficient great to consider that σ_v, the vertical initial (primary) stress, is not variable in the domain Ω (σ_v corresponds for the axis of the tunnel). The conditions for $r \to \infty$ (in case of infinitely domain) has been considered on the external boundary of the domain Ω.

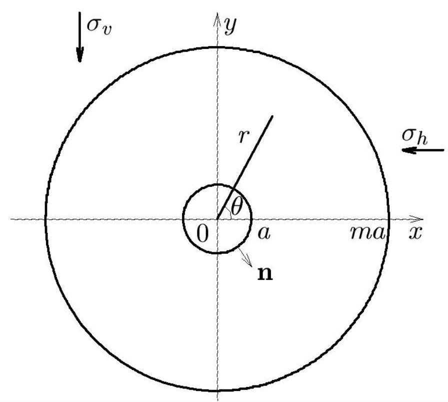

Figure 1: Cross vertical section of the domain.

Because it is assumed the plane state of deformation, the domain for the studying problem is a vertical cross section at depth h and is represented in figure 1. Depth h corresponds to the tunnel axis Oz. Also we take consider small deformations. So, we obtain

$$\begin{gathered} u_z = 0\,, \quad \frac{\partial}{\partial z} = 0\,, \\ \varepsilon_{rr} = \frac{\partial u_r}{\partial r}\,,\ \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),\ \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\,, \end{gathered} \tag{3.1}$$

The general formulation of the problem of determination the stress distribution around a circular horizontal tunnel in elasto-viscoplastic rock, like a cvasistatic problem, is:

to determine the displacement function $(u_r, u_\theta) : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}^2$, *the stress function* $\boldsymbol{\sigma} : \mathbf{R}_+ \times \Omega \longrightarrow \mathcal{S}_3$ *and the irreversible stress work function* $W^{\mathrm{I}} : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}$ *such that:*

$$Div\, \boldsymbol{\sigma}^{\mathrm{R}}(t, r, \theta) = 0 \quad in\ \ \mathbf{R}_+ \times \Omega\,, \tag{3.2}$$

$$\begin{aligned} \dot{\boldsymbol{\sigma}}^{\mathrm{R}} \; = \; & 2G\dot{\boldsymbol{\varepsilon}} + (3K - 2G)\,\dot{\varepsilon}\mathbf{1} + \\ & k_{\mathrm{T}} \left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\boldsymbol{\sigma})} \right\rangle \left[\frac{(2G - 3K)}{3}\frac{\partial F}{\partial \sigma}\mathbf{1} - 2G\frac{\partial F}{\partial \boldsymbol{\sigma}} \right] + \\ & k_{\mathrm{S}} \left[\frac{(2G - 3K)}{3}\frac{\partial S}{\partial \sigma}\mathbf{1} - 2G\frac{\partial S}{\partial \boldsymbol{\sigma}} \right] \quad in\ \ \mathbf{R}_+ \times \Omega\,, \end{aligned} \tag{3.3}$$

$$\dot{W}^{\mathrm{I}} = k_{\mathrm{T}} \left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\boldsymbol{\sigma})} \right\rangle \frac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma} \qquad \textit{in} \;\; \mathbf{R}_+ \times \Omega \,, \tag{3.4}$$

$$\begin{cases} \begin{cases} \sigma^{\mathrm{R}}_{rr}(t,a,\theta) = p - \sigma^{\mathrm{P}}_{rr}(\theta) \\ \sigma^{\mathrm{R}}_{r\theta}(t,a,\theta) = 0 \end{cases} , \quad (\forall)\, t > 0 \,,\, \theta \in [0, 2\pi) \,,\, \textit{(on } \Gamma_1\textit{)} \,, \\ u(t, ma, \theta) = 0 \,,\, (\forall)\, t > 0 \,,\, \theta \in [0, 2\pi) \,,\, \textit{(on } \Gamma_2\textit{)} \,, \end{cases} \tag{3.5}$$

$$\begin{cases} \boldsymbol{\sigma}^{\mathrm{S}}(0,r,\theta) = \boldsymbol{\sigma}^{\mathrm{P}}(\theta) + \tilde{\boldsymbol{\sigma}}(r,\theta) \\ (u_r, u_\theta)(0,r,\theta) = (\tilde{u}_r, \tilde{u}_\theta)(r,\theta) \\ W^{\mathrm{I}}(0,r,\theta) = H(\boldsymbol{\sigma}^{\mathrm{P}}(\theta)) \end{cases} , \qquad (\forall)\, (r,\theta) \in \Omega \,, \tag{3.6}$$

where $\tilde{\boldsymbol{\sigma}}$ and $(\tilde{u}_r, \tilde{u}_\theta)$ are the stress and, respectively, the displacement corresponding for the moment of excavation, $\boldsymbol{\sigma}^{\mathrm{S}} = \boldsymbol{\sigma}^{\mathrm{R}} + \boldsymbol{\sigma}^{\mathrm{P}}$, $u = u^{\mathrm{S}}$, p is the pressure on the inner wall of the cavity.

4. Instantaneous response

The stress distribution after excavation is obtained by exact elastic solution (the instantaneous response). The problem is:

to determine the displacement function $(\tilde{u}_r, \tilde{u}_\theta) : \Omega \longrightarrow \mathbf{R}^2$, *the stress function* $\tilde{\boldsymbol{\sigma}}^{\mathrm{R}} : \Omega \longrightarrow \mathcal{S}_3$ *such that*

$$\begin{cases} Div\, \tilde{\boldsymbol{\sigma}}^{\mathrm{R}}(r,\theta) = 0 \qquad in \;\; \Omega \,, \\ \tilde{\boldsymbol{\sigma}}^{\mathrm{R}} = 2G\tilde{\boldsymbol{\varepsilon}} + (3K - 2G)\, \tilde{\varepsilon} \mathbf{1} \qquad in \;\; \Omega \,, \\ \tilde{\boldsymbol{\sigma}}^{\mathrm{R}}(a,\theta)\mathbf{n} = p\, \mathbf{n} - \boldsymbol{\sigma}^{\mathrm{P}}\mathbf{n} \,,\; (\tilde{u}_r, \tilde{u}_\theta)\,(ma,\theta) = 0 \,,\; \forall\, \theta \in [0, 2\pi] \,. \end{cases} \tag{4.1}$$

Proposition 4.1. *If on the walls of the cavity,* $\Gamma_1 = \{(a,\theta) \,|\, \theta \in [0, 2\pi)\}$, *a pressure* p *is acting (due to various reasons and which may be constant or variable):*

$$\sigma^{\mathrm{S}}_{rr}(a,\theta) = p \,, \quad \sigma^{\mathrm{S}}_{r\theta}(a,\theta) = 0 \,, \quad \forall\, \theta \in [0, 2\pi) \,, \tag{4.2}$$

and on the external boundary of the domain Ω, $\Gamma_2 = \{(ma,\theta) \,|\, \theta \in [0, 2\pi)\}$, *we have:*

$$\begin{cases} \sigma^{\mathrm{S}}_{rr}(ma,\theta) = \dfrac{1}{2}\,(\sigma_h + \sigma_v) + \dfrac{1}{2}\,(\sigma_h - \sigma_v)\cos 2\theta \\[2ex] \sigma^{\mathrm{S}}_{\theta\theta}(ma,\theta) = \dfrac{1}{2}\,(\sigma_h + \sigma_v) - \dfrac{1}{2}\,(\sigma_h - \sigma_v)\cos 2\theta \\[2ex] \sigma^{\mathrm{S}}_{r\theta}(ma,\theta) = -\dfrac{1}{2}\,(\sigma_h - \sigma_v)\sin 2\theta \end{cases} , \tag{4.3}$$

then the stress state just after excavation is:

$$\begin{aligned}
\tilde{\sigma}^{\mathrm{S}}_{rr}(r,\theta) &= 2A_1 + B_1(2\ln r + 1) + \frac{C_1}{r^2} + \left(-2A_2 - \frac{6C_2}{r^4} - \frac{4D_2}{r^2}\right)\cos 2\theta \\
\tilde{\sigma}^{\mathrm{S}}_{\theta\theta}(r,\theta) &= 2A_1 + B_1(2\ln r + 3) - \frac{C_1}{r^2} + \left(2A_2 + 12B_2 r^2 + \frac{6C_2}{r^4}\right)\cos 2\theta \\
\tilde{\sigma}^{\mathrm{S}}_{r\theta}(r,\theta) &= \left(2A_2 + 6B_2 r^2 - \frac{6C_2}{r^4} - \frac{2D_2}{r^2}\right)\sin 2\theta \\
\tilde{\sigma}^{\mathrm{S}}_{zz}(r,\theta) &= \nu\left[4A_1 + B_1(2\ln r + 1) + \left(12B_2 r^2 - \frac{4D_2}{r^2}\right)\cos 2\theta\right] + \sigma_h
\end{aligned} \tag{4.4}$$

where A_1, B_1, C_1, A_2, B_2, C_2, D_2 are constants :

$$\begin{aligned}
A_1 &= \frac{1}{4}(\sigma_h + \sigma_v) + \frac{\ln(ma) + 1}{m^2 - 2\ln m + 1}\left(p - \frac{1}{2}(\sigma_h + \sigma_v)\right), \\
B_1 &= \frac{-1}{m^2 - 2\ln m + 1}\left(p - \frac{1}{2}(\sigma_h + \sigma_v)\right), \\
C_1 &= \frac{m^2 a^2}{m^2 - 1}\left(p - \frac{1}{2}(\sigma_h + \sigma_v)\right), \\
A_2 &= -\frac{m^4\left(m^4 + 1\right)}{4\left(m^4 - 2m^2 + 1\right)\left(m^4 - 2m^2 - 1\right)}(\sigma_h - \sigma_v), \\
B_2 &= -\frac{m^4}{6a^2\left(m^4 - 2m^2 + 1\right)\left(m^4 - 2m^2 - 1\right)}(\sigma_h - \sigma_v), \\
C_2 &= \frac{-a^4 m^4\left(3m^4 - 1\right)}{12\left(m^4 - 2m^2 + 1\right)\left(m^4 - 2m^2 - 1\right)}(\sigma_h - \sigma_v), \\
D_2 &= \frac{a^2 m^8}{2\left(m^4 - 2m^2 + 1\right)\left(m^4 - 2m^2 - 1\right)}(\sigma_h - \sigma_v).
\end{aligned} \tag{4.5}$$

Proof. The components of stress are obtained from equilibrium equation using the Airy function and the constants result from conditions (4.2) and (4.3). □

Remark 4.1. It is easy to observe that in the case of infinitely domain, when $m \to \infty$, the stress, deformation and displacement components are the same like in papers of Cristescu [1], Paraschiv and Cristescu [10], because, when $m \to \infty$, we have:

$$A_1 \to \frac{1}{4}(\sigma_h + \sigma_v)\ ,\ B_1 \to 0\,,\ C_1 \to a^2\left(p - \frac{1}{2}(\sigma_h + \sigma_v)\right)$$
$$A_2 \to -\frac{1}{4}(\sigma_h - \sigma_v)\ ,\ B_2 \to 0\,,\ C_2 \to -\frac{a^4}{4}(\sigma_h - \sigma_v)\ ,\ D_2 \to \frac{a^2}{2}(\sigma_h - \sigma_v)\,.$$

So, this result proves in one way that taking $m \geq m_0 \geq 5$ it is acceptable for moving the infinitely condition on the boundary $r = ma$.

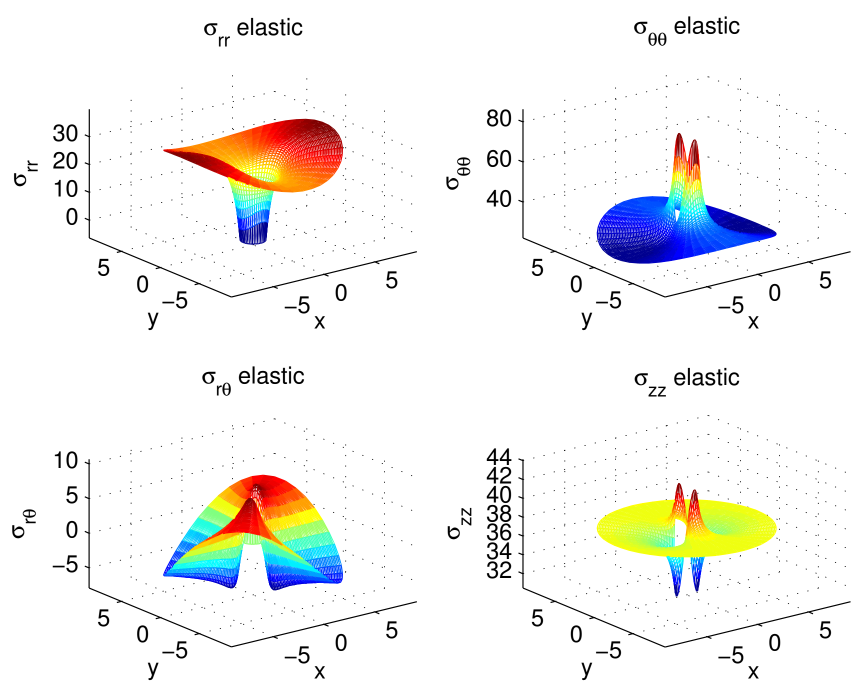

Figure 2. The stress state.

From (4.4) it is easy to obtain:

Proposition 4.2. *The components of deformation corresponding stress state*

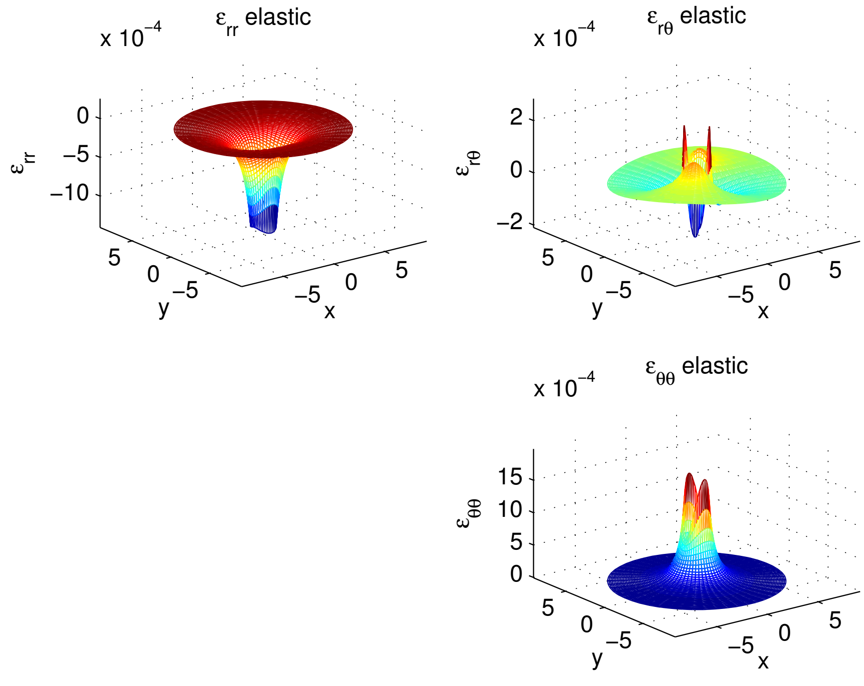

Figure 3. The deformation state.

(4.4) *are:*

$$
\begin{aligned}
\tilde{\varepsilon}_{rr}(r,\theta) = \frac{1+\nu}{E}\Bigg\{ & 2(1-2\nu)\left(A_1-\frac{\sigma_h+\sigma_v}{4}\right)+\\
& [2(1-2\nu)\ln r+(1-4\nu)]\,B_1+\frac{C_1}{r^2}+\\
& \left[-2\left(A_2+\frac{\sigma_h-\sigma_v}{4}\right)-12\nu B_2 r^2-\frac{6C_2}{r^4}-\frac{4D_2}{r^2}(1-\nu)\right]\cos 2\theta\Bigg\}\,,\\
\tilde{\varepsilon}_{\theta\theta}(r,\theta) = \frac{1+\nu}{E}\Bigg\{ & 2(1-2\nu)\left(A_1-\frac{\sigma_h+\sigma_v}{4}\right)+\\
& [2(1-2\nu)\ln r+(3-4\nu)]\,B_1-\frac{C_1}{r^2}+\\
& \left[2\left(A_2+\frac{\sigma_h-\sigma_v}{4}\right)+12(1-\nu)B_2 r^2+\frac{6C_2}{r^4}+\frac{4D_2}{r^2}\nu\right]\cos 2\theta\Bigg\}\,,\\
\tilde{\varepsilon}_{r\theta}(r,\theta) = \frac{1+\nu}{E} & \left(2\left(A_2+\frac{\sigma_h-\sigma_v}{4}\right)+6B_2 r^2-\frac{6C_2}{r^4}-\frac{2D_2}{r^2}\right)\sin 2\theta\,,
\end{aligned}
\tag{4.6}
$$

and the components of the displacement are:

$$
\begin{aligned}
\tilde{u}_r(r,\theta) = \frac{1+\nu}{E}\Bigg\{ & 2(1-2\nu)\left(A_1 - \frac{\sigma_h+\sigma_v}{4}\right) + \\
& [2(1-2\nu)(\ln r - 1) + (1-4\nu)]\, rB_1 - \frac{C_1}{r} + \\
& \left[-2\left(A_2 + \frac{\sigma_h - \sigma_v}{4}\right) r - 4\nu B_2 r^3 + \frac{2C_2}{r^3} + \frac{4D_2}{r}(1-\nu)\right]\cos 2\theta \Bigg\},
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\tilde{u}_\theta(r,\theta) = \frac{1+\nu}{E}\Bigg\{ & 4(1-\nu)B_1 r\theta + \frac{1}{2}\Bigg[4\left(A_2 + \frac{\sigma_h-\sigma_v}{4}\right) r \\
& +4(3-2\nu)B_2 r^3 + \frac{4C_2}{r^3} + \frac{4D_2}{r}(2\nu - 1)\Bigg]\sin 2\theta \Bigg\}.
\end{aligned}
$$

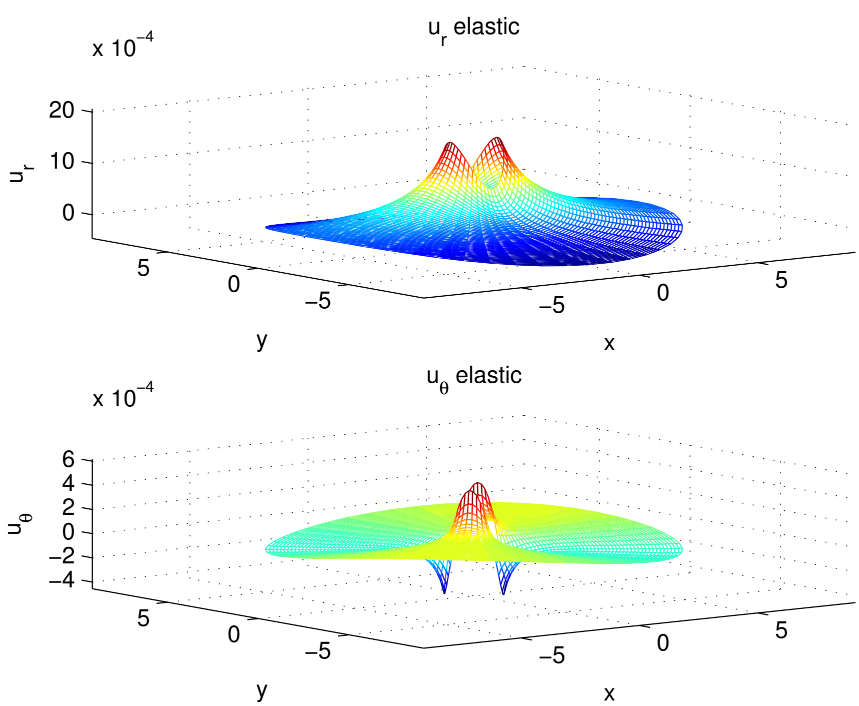

Figure 4. The components of displacement.

Elastic solution is used as initial data for the integration in long time intervals using FEM. In Figures 2, 3 and 4 are being represented the stress state, the deformation state and the displacement components corresponding to the instantaneous response.

5. Elasto-viscoplastic creep. Simplified solution

In case of elasto-viscoplastic creep (see [3], [4], [11], [16]) we assume that in [0,T] the stress components are constants equal with the instantaneous response given by

$$\boldsymbol{\sigma}^{\mathrm{R}}(t) = \tilde{\boldsymbol{\sigma}}^{\mathrm{R}}, \quad (\forall)\, t \in [0, T], \tag{5.1}$$

where $\tilde{\boldsymbol{\sigma}}^{\mathrm{R}}$ is given by (4.4).
Of (2.1) is obtained:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{I}} = k_{\mathrm{T}} \left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \right\rangle \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) + k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}), \tag{5.2}$$

where $\boldsymbol{\sigma}^{\mathrm{S}} = \tilde{\boldsymbol{\sigma}}^{\mathrm{R}} + \boldsymbol{\sigma}^{\mathrm{P}}$.

Of (5.2) it results the following equation:

$$\dot{W}^{\mathrm{I}} = \left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \right\rangle \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}} + k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}, \tag{5.3}$$

from where using the initial condition $W^{\mathrm{I}}(0) = H(\boldsymbol{\sigma}^{\mathrm{P}}) \equiv W^{\mathrm{IP}}$, is obtained:

$$\begin{gathered} 1 - \frac{W^{\mathrm{I}}(t)}{H(\boldsymbol{\sigma}^{\mathrm{S}})} = -\frac{k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}}{k_{\mathrm{T}} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}} + \\ \left[\left\langle 1 - \frac{W^{\mathrm{IP}}}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \right\rangle + \frac{k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}}{k_{\mathrm{T}} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}} \right] \exp\left[-\frac{k_{\mathrm{T}}}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}} t \right]. \end{gathered} \tag{5.4}$$

From (5.2) and (5.4) for $\boldsymbol{\varepsilon}^{\mathrm{I}}$ is obtained the following equation:

$$\begin{gathered} \dot{\boldsymbol{\varepsilon}}^{\mathrm{I}} = k_{\mathrm{T}} \left\{ -\frac{k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}}{k_{\mathrm{T}} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}} + \left(\left\langle 1 - \frac{W^{\mathrm{IP}}}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \right\rangle \right) + \frac{k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}}{k_{\mathrm{T}} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}}} \right) \\ \exp\left[-\frac{k_{\mathrm{T}}}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) \cdot \boldsymbol{\sigma}^{\mathrm{S}} t \right] \Bigg\} \frac{\partial F}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}) + k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{\mathrm{S}}), \end{gathered} \tag{5.5}$$

that integrate with the condition, $\boldsymbol{\varepsilon}^{\mathrm{I}}(0, r, \theta) = \tilde{\boldsymbol{\varepsilon}}(r, \theta)$, corresponding with the instantaneous response given by (4.6). Denote $\boldsymbol{\varepsilon}_0(r, \theta) = \tilde{\boldsymbol{\varepsilon}}(r, \theta)$. By integration we obtain

$$
\begin{aligned}
\boldsymbol{\varepsilon}^{\mathrm{I}}(t,r,\theta) = \boldsymbol{\varepsilon}_0(r,\theta) + \left(-\frac{k_{\mathrm{S}}\dfrac{\partial S}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}}{\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}} \frac{\partial F}{\partial \boldsymbol{\sigma}} + k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}} \right) t + \\
\frac{H\dfrac{\partial F}{\partial \boldsymbol{\sigma}}}{\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}} \left(\left\langle 1 - \frac{W^{\mathrm{IP}}}{H} \right\rangle + \frac{k_{\mathrm{S}}\dfrac{\partial S}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}}{k_{\mathrm{T}}\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}} \right) \left\{ 1 - \exp\left[-\frac{k_{\mathrm{T}}}{H} \frac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma} t \right] \right\} ,
\end{aligned} \tag{5.6}
$$

for $0 \le t \le t_{\mathrm{S}}$, and

$$
\begin{aligned}
\boldsymbol{\varepsilon}^{\mathrm{I}}(t,r,\theta) = \boldsymbol{\varepsilon}_0(r,\theta) + \left(-\frac{k_{\mathrm{S}}\dfrac{\partial S}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}}{\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}} \frac{\partial F}{\partial \boldsymbol{\sigma}} + k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}} \right) t_{\mathrm{S}} + \\
\frac{H\dfrac{\partial F}{\partial \boldsymbol{\sigma}}}{\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}} \left(\left\langle 1 - \frac{W^{\mathrm{IP}}}{H} \right\rangle + \frac{k_{\mathrm{S}}\dfrac{\partial S}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}}{k_{\mathrm{T}}\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}} \right) \left\{ 1 - \exp\left[-\frac{k_{\mathrm{T}}}{H} \frac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma} t_{\mathrm{S}} \right] \right\} + \\
k_{\mathrm{S}} \frac{\partial S}{\partial \boldsymbol{\sigma}} (t - t_{\mathrm{S}}) , \quad \text{for } t \ge t_{\mathrm{S}} ,
\end{aligned} \tag{5.7}
$$

where t_{S} is the time of creep stabilization which is obtain for $W^{\mathrm{I}} \longrightarrow H(\boldsymbol{\sigma}^{\mathrm{S}})$, so

$$
t_{\mathrm{S}} = -\frac{H}{k_{\mathrm{T}}\dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^{\mathrm{S}}} \ln \frac{k_{\mathrm{S}}\dfrac{\partial S}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^{\mathrm{S}}}{\left\langle 1 - \dfrac{W^{\mathrm{IP}}}{H(\boldsymbol{\sigma}^{\mathrm{S}})} \right\rangle \left(k_{\mathrm{T}} \dfrac{\partial F}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^{\mathrm{S}} \right) + k_{\mathrm{S}} \dfrac{\partial S}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}^{\mathrm{S}}} . \tag{5.8}
$$

Pentru deplasarea radială u avem

$$
u(t,r) = r\varepsilon_{\theta\theta}(t,r) . \tag{5.9}
$$

For displacement components we have to integrate the equations (3.1) using (5.7).

6. The numerical approach

For the problem (3.2)-(3.6) we determine a numerical solution based on some results presented by Ionescu and Sofonea [5] using a complete implicit method for integration in time (see Paraschiv-Munteanu [11]).

If $(u, \boldsymbol{\sigma}^{\mathrm{R}}, W^{\mathrm{I}})$, where $u = (u_r, u_\theta)$, is the solution of the problem (3.2)-(3.6) then we determine:

$$\overline{u} = u - \tilde{u}\,, \quad \overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{\mathrm{R}} - \tilde{\boldsymbol{\sigma}}^{\mathrm{R}}\,, \tag{6.1}$$

such that

$$\begin{gathered} \overline{u}(t, ma, \theta) = 0\,, \quad (\forall)\, t > 0\,,\, \theta \in [0, 2\pi)\,, \\ \mathrm{Div}\, \overline{\boldsymbol{\sigma}}(t, r, \theta) = 0\,, \quad (\forall)\, t > 0 \text{ and } (r, \theta) \in \Omega\,, \\ \overline{\boldsymbol{\sigma}}(t, a, \theta)\mathbf{n} = 0\,, \quad (\forall)\, t > 0\,,\, \theta \in [0, 2\pi). \end{gathered} \tag{6.2}$$

So, we have to solve the problem:

to determine the displacement function $\overline{u} : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}^2$*, the stress function* $\overline{\boldsymbol{\sigma}} : \mathbf{R}_+ \times \Omega \longrightarrow \mathcal{S}_3$ *and the irreversible stress work function* $W^{\mathrm{I}} : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}$ *such that:*

$$\begin{aligned} \dot{\overline{\boldsymbol{\sigma}}} \;=\;& 2G\boldsymbol{\varepsilon}(\dot{\overline{u}}) + (3K - 2G)\,\varepsilon(\dot{\overline{u}})\mathbf{1} + k_{\mathrm{T}}\left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})}\right\rangle \\ & \left[\frac{(2G-3K)}{3}\frac{\partial F}{\partial \sigma}(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\mathbf{1} - 2G\frac{\partial F}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\right] + \\ & k_{\mathrm{S}}\left[\frac{(2G-3K)}{3}\frac{\partial S}{\partial \sigma}(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\mathbf{1} - 2G\frac{\partial S}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\right] \\ & \textit{in } \mathbf{R}_+ \times \Omega\,, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \dot{W}^{\mathrm{I}} \;=\;& k_{\mathrm{T}}\left\langle 1 - \frac{W^{\mathrm{I}}(t)}{H(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})}\right\rangle \frac{\partial F}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}}) \cdot (\overline{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}}) \\ & \textit{in } \mathbf{R}_+ \times \Omega\,, \end{aligned} \tag{6.4}$$

$$\left\{\begin{array}{l} \overline{\boldsymbol{\sigma}}(0, r, \theta) = 0 \\ \overline{u}(0, r, \theta) = 0 \\ W^{\mathrm{I}}(0, r, \theta) = H(\boldsymbol{\sigma}^{\mathrm{P}}(\theta)) \end{array}\right. \,, \qquad (\forall)\, (r, \theta) \in \Omega\,. \tag{6.5}$$

In order to determine a numerical approach of the solution of the problem (6.3)-(6.5) we consider an interval $[0, T]$, $T > 0$.

Let us note

$$\mathbf{V}_1 = \left\{v = (v_1, v_2, 0) \,|\, v_i \in \mathrm{L}^2(\Omega),\, v_i = v_i(r, \theta),\, v_i(ma, \theta) = 0\; i = 1, 2\right\}, \tag{6.6}$$

$$\mathcal{V}_2 = \left\{\boldsymbol{\sigma} \in \left[\mathrm{L}^2(\Omega)\right]_S^{3\times 3} \,|\, \boldsymbol{\sigma} = \boldsymbol{\sigma}(r)\,,\, \mathrm{Div}\, \boldsymbol{\sigma} = 0 \text{ în } \Omega\,,\, \boldsymbol{\sigma}(a, \theta)\mathbf{n} = 0\right\}. \tag{6.7}$$

From (6.2) result that the solution $(\overline{u}, \overline{\boldsymbol{\sigma}}, W^{\mathrm{I}})$ of the problem (6.3)-(6.5) has the properties:

$$\overline{u} \in \mathbf{V}_1\,, \quad \overline{\boldsymbol{\sigma}} \in \mathcal{V}_2\,. \tag{6.8}$$

Let $M \in \mathbf{N}$, $M \geq 2$, $\Delta t = \dfrac{T}{M}$ be the step time and

$$t_0 = 0\,, \quad t_{n+1} = t_n + \Delta t\,, \quad n = \overline{0, M-1}\,. \tag{6.9}$$

Let us consider $\mathbf{V}_h \subset \mathbf{V}_1$ a finite-dimensional subspace constructed using the finite element method. We determine $\left(\overline{u}_h^n, \overline{\boldsymbol{\sigma}}_h^{n+1}, \left(W^{\mathrm{I}}\right)_h^{n+1}\right)$ approach of the solution $\left(\overline{u}, \overline{\boldsymbol{\sigma}}, W^{\mathrm{I}}\right)$ on the moment t_n.

Let $\mathcal{B} = \{\varphi_1, \ldots, \varphi_{\mathrm{I}}\} \subset \mathbf{V}_h$ be a base of $\mathbf{V}_h$, dim $\mathbf{V}_h = \mathrm{I}$. Taking $\overline{u}_h^0 = 0$ we determine $\overline{u}_h^{n+1} \in \mathbf{V}_h$, $n \geq 0$, such that:

$$\overline{u}_h^{n+1} = \sum_{j=1}^{\mathrm{I}} \alpha_j^{n+1} \varphi_j\,, \tag{6.10}$$

where the constants α_j^{n+1}, $j = \overline{1, I}$ are the solution of a linear system. For the stress approach and irreversible stress work approach we consider $\overline{\boldsymbol{\sigma}}_h^0 = 0$ and $\left(W^{\mathrm{I}}\right)_h^0 = H(\boldsymbol{\sigma}^{\mathrm{P}})$ and we determine $\overline{\boldsymbol{\sigma}}_h^{n+1}$ and $\left(W^{\mathrm{I}}\right)_h^{n+1}$, $n \geq 0$, using the following implicit scheme:

$$\begin{aligned}
&\overline{\boldsymbol{\sigma}}_h^{n+1} = \overline{\boldsymbol{\sigma}}_h^n + 2\mathrm{G}\left[\boldsymbol{\varepsilon}(\overline{u}_h^{n+1}) - \boldsymbol{\varepsilon}(\overline{u}_h^n)\right] - (3\mathrm{K} - 2\mathrm{G})\left[\varepsilon(\overline{u}_h^{n+1}) - \varepsilon(\overline{u}_h^n)\right]\mathbf{1} + \\
&\Delta t\; k_{\mathrm{T}} \left\langle 1 - \frac{\left(W^{\mathrm{I}}\right)_h^{n+1}}{H(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})} \right\rangle \\
&\left[\frac{(2\mathrm{G} - 3\mathrm{K})}{3}\frac{\partial F}{\partial \sigma}(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\mathbf{1} - 2\mathrm{G}\frac{\partial F}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\right] + \\
&\Delta t\; k_{\mathrm{S}} \left[\frac{(2\mathrm{G} - 3\mathrm{K})}{3}\frac{\partial S}{\partial \sigma}(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\mathbf{1} - 2\mathrm{G}\frac{\partial S}{\partial \boldsymbol{\sigma}}(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\right],
\end{aligned} \tag{6.11}$$

and, respectively,

$$\begin{aligned}
\left(W^{\mathrm{I}}\right)_h^{n+1} \quad = \quad & \left(W^{\mathrm{I}}\right)_h^n + \Delta t\; k_{\mathrm{T}} \left\langle 1 - \frac{\left(W^{\mathrm{I}}\right)_h^{n+1}}{H(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})} \right\rangle \\
& \frac{\partial F}{\partial \sigma}(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})(\overline{\boldsymbol{\sigma}}_h^{n+1} + \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{\mathrm{P}})\,.
\end{aligned} \tag{6.12}$$

For numerical solution we use the scheme proposed by Paraschiv-Munteanu [11] using FEM for spatial integration and a completely implicit method for integration in time. For short period of time the creep solution and the numerical solution are very close. Thus deformation by creep and stress variation can simultaneously be described. The similar results for deep boreholes are obtained by Paraschiv-Munteanu [11], [14], [13].

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Some of the results presented in this paper are based on my PhD thesis in mathematics ([11]) made under the scientific leadership of Professor Nicolaie Cristescu and several scientific articles published before and after the thesis sustaining ([10], [12], [14], [13], [16], [15]).

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