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Quasi-static and dynamic analysis for viscoelastic beams with the constitutive equation in a hereditary integral form

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Dedicated to Professor Nicolaie Cristescu on the occasion of his Birthday

Abstract - An isotropic beam with simply supported boundary conditions is subjected to a uniform distributed creep load. Using the correspondence principle, an exact solution for the quasi-static case is determined. A variational method and Laplace transform techniques will be approached in the dynamic analysis of viscoelastic beam. Accuracy of the two algorithms is verified by a numerical example.

Key words and phrases : viscoelastic beam, Euler-Bernoulli beam theory, Galerkin method, Laplace transform, the constitutive law in hereditary integral form, correspondence principle.

Mathematics Subject Classification (2010) **:** 74D05, 45K05, 65D17.

1. Introduction

Engineering design of the past thirty years are frequently present the structures with viscoelastic components due to their ability to dampen out the vibrations. The metals at elevated temperatures, rubbers, polymers that have the characteristic of both elastic solids as well as viscous solids are examples of viscoelastic materials. There are remarkable theoretical studies on these materials of Cristescu (see [6]), Cristescu and Suliciu (see [7]), Dinca (see [8]), Flugge (see [12]), Chriatensen (see [4]), Payette and Reddy (see [18]), Kennedy (see [14]). These are complemented by the works dealing with the analysis of viscoelastic structures from both mathematical and engineering points of views: [1], [2], [11], [13], [17], [21]. Using the Euler-Bernoulli beam theory, we present in our paper the governing equation for a simply supported viscoelastic beam under a uniform distributed load, see [19]. This equation is accompanied by a constitutive law presented in a hereditary integral form. In order to obtain the quasi-static exact solution (i.e. the solution ignoring inertia effects), we will use the correspondence principle (see [16], [23]). This principle relates mathematically the solution of a linear, viscoelastic boundary value problem to an analogous problem of an elastic body of the same geometry and under the same initial boundary conditions. Mention that not all problems can be solved by this principle, but only those for which the boundary conditions do not vary with the time. Then, applying the principle of d'Alembert, this structure will be analyzed in the dynamic case with a mixed algorithm that is based on the Galerkin's method for the spatial domain and the Laplace transform for the time domain. Numerical results for both quasi-static and dynamic analysis are presented and these will be accompanied of the comparative studies made with the help of graphical representations.

2. Beam governing equation

The differential equation of the transverse oscillations of a beam, which is subjected to uniformly distributed forces $\overline{p}$, is obtained from the dynamic equilibrium of an element having the length dx.

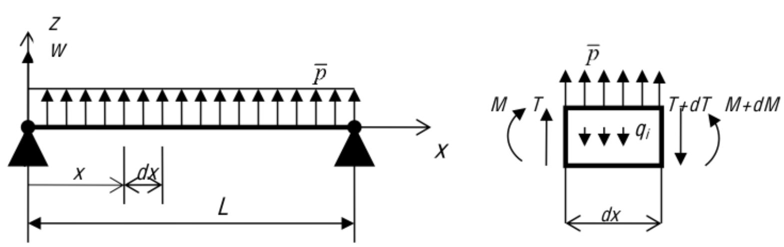

Figure 1.

If all the forces acting on the beam element are projected on the axis Oz, we find:

$$\overline{p}dx + T - q_i dx - T - \frac{\partial T}{\partial x}dx = 0 \tag{2.1}$$

so

$$\overline{p} - q_i = \frac{\partial T}{\partial x}, \tag{2.2}$$

where T is the shearing force and q_i are the inertial force. Accordance with the d'Alembert's principle, (2.2) becomes

$$\frac{\partial T}{\partial x} = \overline{p} - \rho A \frac{\partial^2 w}{\partial t^2} \tag{2.3}$$

with the following notations: ρ − density of material; A − cross sectional area; w − transverse displacement of the beam at section x and t − time.

Let us now consider the moments in relation to section $x + dx$:

$$M - M - \frac{\partial M}{\partial x} dx + T dx - \overline{p} \frac{(dx)^2}{2} = 0 \tag{2.4}$$

and approximating $(dx)^2 \approx 0$, is obtained

$$\frac{\partial M}{\partial x} = T \tag{2.5}$$

and (2.2) becomes the differential equation:

$$\frac{\partial^2 M}{\partial x^2} = \overline{p} - \rho A \frac{\partial^2 w}{\partial t^2}. \tag{2.6}$$

Next, we will study the deformation of the beam element. The strain on the bottom of the deformed element is tensile (positive) and the strain on the top is compressive (negative). Because the strain is continuous throughout the cross section, will exist an axis where the strain is zero. This is named neutral axis and is rs in the Figure 2.

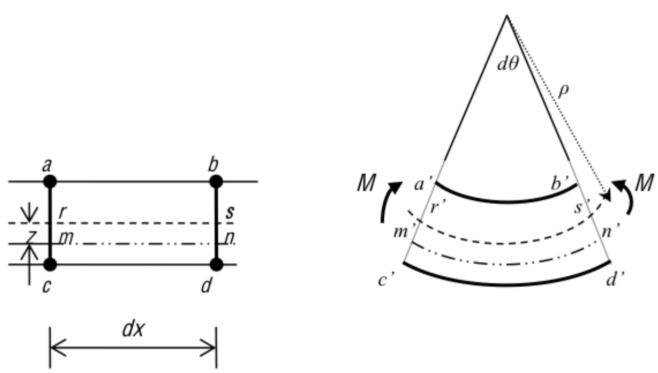

Figure 2.

If line mn there is at a distance z from the neutral axis, then the strain corresponding to mn is defined as:

$$\varepsilon = \frac{(\rho + z) d\theta - \rho d\theta}{dx} = \frac{z d\theta}{\rho d\theta} = \frac{z}{\rho}, \tag{2.7}$$

where ρ is the curvature radius of the neutral axis and θ is the angle subtended by the deformed element.

The sections ac and bd remain plane and normal on the deformed axis rs of the beam after deformation, according to Bernoulli's hypothesis. In differential geometry, the expression of the curvature radius is the following:

$$\frac{1}{\rho} = \frac{\dfrac{\partial^2 w}{\partial x^2}}{\left[1 + \left(\dfrac{\partial w}{\partial x}\right)^2\right]}. \tag{2.8}$$

Practical applications show that $\frac{\partial w}{\partial x}$ is very small with respect to unity and so we can consider

$$\frac{1}{\rho} = \frac{\partial^2 w}{\partial x^2}. \tag{2.9}$$

Introducing (2.9) in (2.7), we get

$$\varepsilon(x,t) = z\frac{\partial^2 w(x,t)}{\partial x^2}. \tag{2.10}$$

Let us consider a constitutive law in the hereditary integral form, see [22]:

$$\sigma(x,t) = E(0)\varepsilon(x,t) - \int_0^t \frac{dE(t-\tau)}{d\tau}\varepsilon(x,\tau)d\tau, \tag{2.11}$$

where E is the relaxation modulus for the beam material and σ the stress corresponding to the strain ε.

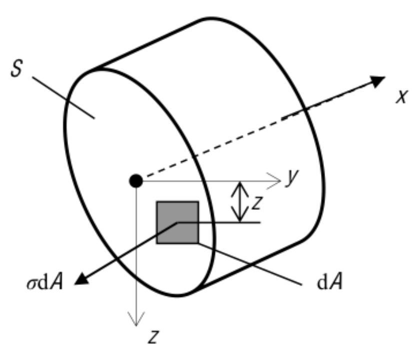

Figure 3.

The bending moment M at the beam cross section x may be expressed in terms of stress in the form of an integral:

$$M = \int\int_S \sigma(x,t)zdydz \tag{2.12}$$

and using (2.11) this becomes:

$$M(x,t) = E(0)I\frac{\partial^2 w(x,t)}{\partial x^2} - I\int_0^t \frac{dE(t-\tau)}{d\tau}\frac{\partial^2 w(x,\tau)}{\partial x^2}d\tau, \qquad (2.13)$$

where I is the moment of inertia of the section S with respect to the axis Oy. Thus, after substitution of M in (2.6), we obtain the following integro - differential equation

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2} + E(0)I\frac{\partial^4 w(x,t)}{\partial x^4} - I\int_0^t \frac{dE(t-\tau)}{d\tau}\frac{\partial^4 w(x,\tau)}{\partial x^4}d\tau = \overline{p}. \qquad (2.14)$$

Solving of equation (2.14) will lead to the finding of the transverse displacements for any boundary and initial conditions given for the viscoelastic beam subjected to a uniformly distributed loading p.

3. Rheological model

The stretching - relaxation process is analyzed using a Poynting model that contains two spring elements and one damping and consists in a serial connection of a spring and a Kelvin - Voigt model (Figure 4). Now, formulating equilibrium and taking the kinematics of the rheological model into account, we get the following equalities:

$$\begin{aligned} \sigma &= k_1\varepsilon_1 & \varepsilon &= \varepsilon_1 + \varepsilon_2 \\ \sigma &= k_2\varepsilon_2 + \eta\dot{\varepsilon_2} & \dot{\varepsilon} &= \dot{\varepsilon_1} + \dot{\varepsilon_2} \end{aligned} \qquad (3.1)$$

where k_1, k_2 are the elastic modulus of the springs and η is the coefficient of viscosity of the dashpot.

These equations (3.1) lead to the differential equation

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{k_1} + \frac{\sigma - k_2\left(\varepsilon - \dfrac{\sigma}{k_1}\right)}{\eta} \qquad (3.2)$$

or

$$(k_1 + k_2)\sigma + \eta\dot{\sigma} = k_1k_2\varepsilon + k_1\eta\dot{\varepsilon}, \qquad (3.3)$$

where σ and ε depend on the time t.

Let us consider that the material of beam is in its relaxation phase, so, under constant strain $\varepsilon = \varepsilon_0$ the stress will decrease. In this case, the equation (3.3) becomes

$$(k_1 + k_2)\sigma + \eta\dot{\sigma} = k_1k_2\varepsilon_0. \qquad (3.4)$$

For the condition: $\sigma(0) = k_1\varepsilon_0$, the solution σ of equation (3.4) will be of the form

$$\sigma(t) = \frac{k_1k_2\varepsilon_0}{k_1+k_2}\left(1 - e^{-\dfrac{t}{\tau_1}}\right) + k_1\varepsilon_0 e^{-\dfrac{t}{\tau_1}}, \qquad (3.5)$$

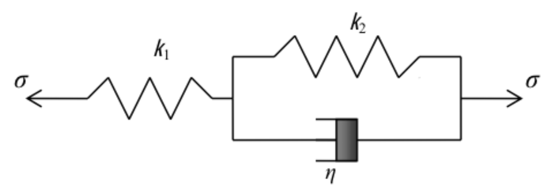

Figure 4.

where the relaxation time is

$$\tau_1 = \frac{\eta}{k_1 + k_2}. \tag{3.6}$$

Using the definition of the relaxation modulus $E(t)$, we have in this case:

$$\sigma(t) = E(t)\varepsilon_0. \tag{3.7}$$

From (3.5), we get

$$E(t) = \frac{k_1 k_2}{k_1 + k_2}\left(1 - e^{-\frac{t}{\tau_1}}\right) + k_1 e^{-\frac{t}{\tau_1}}, \tag{3.8}$$

which is a function that depends on the beam material.

4. Interconversion equation for relaxation modulus and creep compliance

The linear viscoelasticity interconversion equation allows computational obtaining of the relaxation modulus from experimentally derived estimates of the creep compliance and vice versa. In this paper we present a theoretical analysis for an exponential model, in which the interconversion from the relaxation to creep is always stable, whereas that from the creep to relaxation may, under appropriate circumstance, be unstable.

The rheological characteristics of a linear viscoelastic material are modeled as a function of the time t, in terms of the relaxation $E(t)$ and the creep, $D(t)$.

In literature, there is a considerable interest in computational techniques to obtain $E(t)$ when is known $D(t)$ and alternatively, $D(t)$, via the interconversion equation. The choice of the models for $D(t)$ and $E(t)$ will be getting so the following constraints to be guaranteed, see [6]

$$\int_0^t E(t-\tau)D(\tau)d\tau = t \tag{4.1}$$

or, equivalently,

$$\int_0^t E(\tau)D(t-\tau)d\tau = t \tag{4.2}$$

and

$$E(0)D(0) = E(\infty)D(\infty) = 1. \tag{4.3}$$

Numerous algorithms have been proposed for numerical solution of interconversion problem, (see [10], [18], [23]). Without losing generality, it can be assumed that the relaxation function is of the form

$$E(t) = 1 + \alpha e^{-\frac{t}{\tau}} \tag{4.4}$$

and the creep function is

$$D(t) = 1 - \beta e^{-\frac{t}{\lambda}}. \tag{4.5}$$

We remark that the condition $E(\infty)D(\infty) = 1$ is automatically satisfied. Substitution of $t = 0$ into equations (4.4) and (4.5) yields in conjunction with the condition $E(0)D(0) = 1$, that

$$(1+\alpha)(1-\beta) = 1. \tag{4.6}$$

Using the Laplace transform of the functions (4.4) and (4.5) in convolution product (4.1), $E(t) * D(t) = t$, we get

$$\left(\frac{1}{p} + \frac{\alpha}{p + \frac{1}{\tau}}\right)\left(\frac{1}{p} - \frac{\beta}{p + \frac{1}{\lambda}}\right) = \frac{1}{p^2}. \tag{4.7}$$

Since the polynomial identity

$$p^2\left[(1+\alpha)(1-\beta) - 1\right] + p\left(\frac{\alpha}{\lambda} - \frac{\beta}{\tau}\right) = 0, \quad \forall p > 0 \tag{4.8}$$

must hold for all positive p, it follows that

$$\alpha\tau = \beta\lambda. \tag{4.9}$$

Finally, the equalities (4.6) and (4.9) will lead to the below results.

Interconversion from relaxation to the creep

In view of (4.6)

$$\beta = \frac{\alpha}{1+\alpha} \tag{4.10}$$

and using (4.9)

$$\lambda = \tau(1+\alpha). \tag{4.11}$$

Interconversion from creep to relaxation

$$\alpha = \frac{\beta}{1-\beta} \quad \text{and} \quad \tau = \lambda(1-\beta). \tag{4.12}$$

5. Mixed method for solving integro differential equation (2.14)

To solve the equation (2.14) associated with the boundary and initial conditions, we express the transverse deflection w by an expansion of the form

$$w_n(x,t) = \sum_{j=1}^{n} a_j(t)\varphi_j(x), \tag{5.1}$$

where $\varphi_j(x)$ is the j-th shape function and $a_j(t)$ is the corresponding time-dependent amplitude. For the spatial domain will be used the Galerkin's method and then, the techniques of the Laplace transform for the time domain. The shape functions are chosen to be linearly independent, orthonormal

$$\int_0^L \varphi_i(x)\varphi_j(x)dx = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \tag{5.2}$$

and must satisfy all boundary condition for the convergence of Galerkin's method. Although w_n satisfies the boundary conditions, it generally, does not satisfy equation (2.14). If the expansion (5.1) is substituted into (2.14) will result the residual function

$$\overline{R}_n(x,t) = \rho A\left(\sum_{i=1}^{n} \ddot{a}_i(t)\varphi_i(x)\right) + E(0)I\left(\sum_{i=1}^{n} a_i(t)\varphi_i^{(4)}(x)\right) - \\ -I\int_0^t \frac{dE(t-\tau)}{d\tau}\left(\sum_{i=1}^{n} a_i(\tau)\varphi_i^{(4)}(x)\right)d\tau - \overline{p}. \tag{5.3}$$

Let us now consider the shape functions of the form

$$\varphi_j(x) = \sqrt{\frac{2}{L}}\sin\frac{j\pi x}{L},\ j = 1,2,\ldots,n$$

and

$$\varphi_j^{(4)}(x) = \left(\frac{j\pi}{L}\right)^4 \varphi_j(x) = \lambda_j\varphi_j(x).$$

The Galerkin's method requires that the residual to be orthogonal to each of the chosen shape functions, so

$$\int\int_\Omega \overline{R}_n(x,t)\varphi_j(x)dxdt = 0,\ i = 1,2,\ldots,n, \tag{5.4}$$

where $\Omega = [0,L]\times[0,t]$. This leads to n equations verified by the functions $a_j(t)$:

$$\rho A\ddot{a}_j(t) - \lambda_j I\int_0^t \frac{dE(t-\tau)}{d\tau}a_j(\tau)d\tau + \lambda_j IE(0)a_j(t) = \int_0^L \overline{p}\varphi_j(x)dx. \tag{5.5}$$

Using the Laplace transform techniques and the initial conditions

$$\frac{d^k a_j}{dt^k}|_{t=0} = \int_0^L \frac{\partial^k w(x,0)}{\partial t^k}\varphi_j(x)dx,\ k = 0,1,2,\dots. \tag{5.6}$$

the functions $a_j(t)$ are determined independent of one another. Finally, an approximate value of the transverse deflection $w(x,t)$ will be found by (5.1).

6. Numerical example

Let us consider a simply supported beam under the uniform distributed load $\overline{p} = 4$ N/m, which is applied as a creep load at $t = 0$ (the load is applied suddenly at $t = 0$ and then maintained constant). The length of the beam is $L = 4$ m, width $b = 0.08$ m and height $h = 0.23$ m. These input data lead to the moment of inertia of the rectangular section: $I = bh^3/12 = 8 \cdot 10^{-5}$ cm^4. The material is taken to have the density of 1200 kg/m^3. For this example, we employ the three-parameter solid model (Poynting model) with the relaxation modulus expressed by (4.4)-(4.5), [10], [18], where

$$k_1 = 9.8 \cdot 10^7 \text{N/m}^2,\quad k_2 = 2.45 \cdot 10^7 \text{N/m}^2,\quad \eta = 2.74 \cdot 10^8 \text{N}\cdot \sec/\text{m}^2 \tag{6.1}$$

So, the relaxation modulus will be

$$E(t) = 1.96 \cdot 10^7 + 7.84 \cdot 10^7 e^{-t/2.24} \text{N/m}^2 \tag{6.2}$$

with t in seconds. To get the creep compliance $D(t)$, we divide (6.2) by the first term on right and obtain (4.4) for $E_1(t)$

$$E_1(t) = \frac{E(t)}{1.96 \cdot 10^7} = 1 + 4e^{-\frac{t}{2.24}},\quad \alpha = 4, \tau = 2.24. \tag{6.3}$$

Further, using (4.10)-(4.11), we have

$$\beta = \frac{\alpha}{1+\alpha} = 0.8 \text{ and } \lambda = \tau(1+\alpha) = 11.2 \tag{6.4}$$

and (4.5) becomes

$$D_1(t) = D(t) \cdot 1.96 \cdot 10^7 = 1 - 0.8e^{-\frac{t}{11.2}}.$$

Finally, the creep compliance that corresponds to the relaxation modulus (6.2), will be of the form:

$$D(t) = 0.51 \cdot 10^{-7} - 0.408 \cdot 10^{-7} e^{-\frac{t}{11.2}}. \tag{6.5}$$

The solving of the problem (2.14) will be accompanied by the appropriate boundary conditions for the simply supported beams:

$$w(0,t) = w(L,t) = 0 \text{ and } M(0,t) = M(L,t) = 0. \tag{6.6}$$

Quasi - static case

In the quasi-static case the inertial forces are ignored. An exact solution for the creep loading applied at $t = 0$ can be computed using the correspondence principle. We get the transverse displacements of the following form:

$$w(x,t) = \overline{w}(x)D(t), \tag{6.7}$$

where $D(t)$ is given in (6.5) and $\overline{w}$ is the solution for a similar elastic structure that has the modulus of elasticity $E = 1$, see [19]:

$$\overline{w}(x) = \frac{\overline{p}x(x^3 - 2Lx^2 + L^3)}{24I}. \tag{6.8}$$

The variation of w with respect to t at $x = 2$ m is presented in Figure 5, where the displacement has been noted by we. The value of w at $t = 6$ seconds is: we(2, 6) = 0.00452 m.

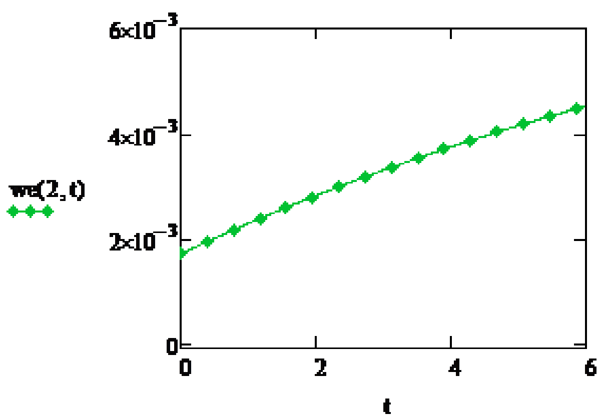

Figure 5.

Dynamic analysis

For above input data, the equations (5.5) become:

$$28.8\ddot{a}_j(t) - 1065.4j^4 \int_0^t e^{-(t-\tau)/2.24} a_j(\tau)d\tau + 2983j^4 a_j(t) = \frac{2.7}{j}\left[(-1)^{j+1} + 1\right]$$

or

$$\ddot{a}_j(t) - 37j^4 \int_0^t e^{-(t-\tau)/2.24} a_j(\tau)d\tau + 104j^4 a_j(t) = \frac{0.1}{j}\left[(1)^{j+1} + 1\right]. \tag{6.9}$$

We remark from the form of (6.9) that $a_j(t)$ are zero for even values of j. Hence, only odd values of j need to be considered in the finding of the Galerkin's solution. These equations will have exact solutions determined with the techniques of Laplace transform. Since the initial conditions on w and its derivates are zero, then, and the conditions on a_j and its derivates are also zero. If $A_j(p)$ is Laplace transform $(\xrightarrow{L})$ of $a_j(t)$, using the Multiplication Theorem, we get

$$\int_0^t e^{-\frac{t-\tau}{2.24}} a_j(\tau)d\tau \xrightarrow{L} \frac{A_j(p)}{p+(1/2.24)} \tag{6.10}$$

and (6.9) becomes:

$$A_j(p) = \frac{0.2}{j} \cdot \frac{p+0.45}{p(p^3+0.45p^2+104j^4p+9.43j^4)}.$$

Using the Newton iterative formula, we find a root of a polynomial that there is into parenthesis: $p_1 = 0.091$. Then, the partial fraction decomposition will lead to the solution $a_j(t)$ of (6.9). Finally, the transverse displacements w are approximated by (5.1) and for $n = 9$. We present in Figure 6 the variation of w for $x = 0.5$ m, $x = 1$ m, $x = 2$ m and t between zero and 6 seconds. The calculations and graphical representations were made with MathCAD.

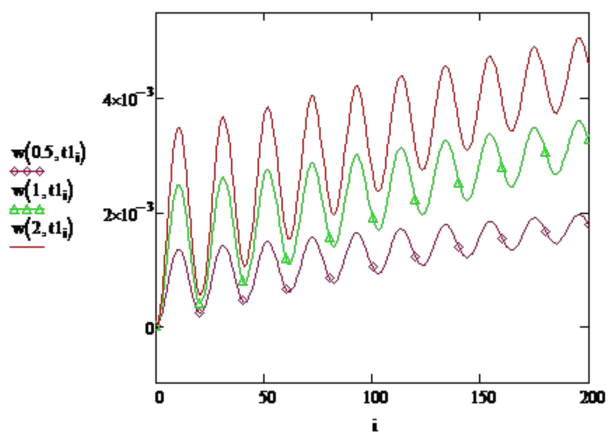

Figure 6.

The Figure 7 shows how the dynamic results oscillate around of the quasi-static results for the midpoint transverse deflection ($x = 2$ m). It may be noted that the amplitude of vibration decreases with increasing time, due to the presence of the viscoelastic damping. In general, the amplitude of the oscillations depends upon the material density, the rate at which the loading is applied and the amount of the viscoelastic damping.

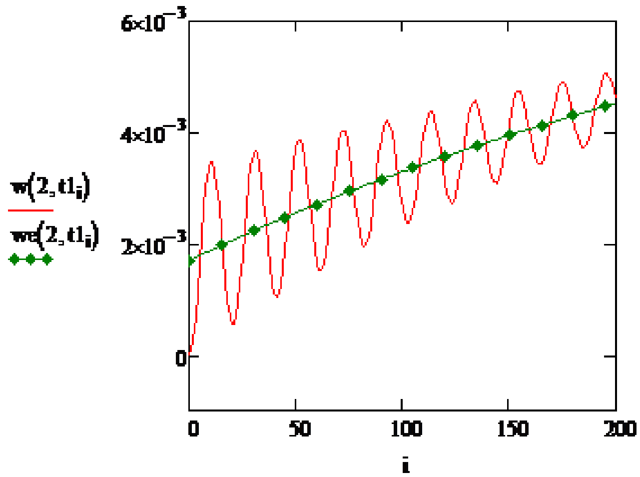

Figure 7.

7. Conclusions

In this paper is presented an efficient method for numerical simulation of a quasi-static and dynamic response of a linear viscoelastic beam, represented by a three parameter solid model. It has been obtained the interconversion equations that allow finding of the relaxation modulus from experimentally derived estimates of creep compliance and vice versa. Using the Laplace transform to solve the differential equation (5.5), we avoid the appearance of the errors that accompany algorithms based on the discretization of time interval. The numerical results confirm the accuracy and computational efficiency of the proposed method, which can be easily extended to analyze the complex viscoelastic structures.

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