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Saint Venant's problem in the strain gradient theory of elasticity

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Dedicated to Professor Nicolaie D. Cristescu, on the occasion of his 85th birthday

Abstract - This paper is concerned with the linear theory of strain gradient elasticity. The deformation of an isotropic chiral cylinder subjected to resultant forces and moments on the ends is investigated. The three-dimensional problem is reduced to the study of some generalized plane strain problems. It is shown that the flexure of a chiral cylinder, in contrast with the case of achiral materials, is accompanied by extension and bending.

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1. Introduction

With a view toward describing the aim of the present article we recall first that Saint-Venant's problem consists in determining the equilibrium of an elastic cylinder that is subjected only to surface forces distributed over its plane ends. Saint-Venant's approach of the problem is based on a relaxed statement in which the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. The deformation of elastic cylinders continues to attract attention both from theoretical and technical point of view (see, e.g., [2,4-6] and references therein). In this paper we study Saint-Venant's problem for isotropic and homogeneous chiral elastic rods in the strain gradient theory. The mechanical behaviour of chiral materials is of interest for the investigation of carbon nanotubes, auxetic materials and bones. The chiral effects cannot be described within classical elasticity [8]. Recently, Papanicolopulos [11] has shown that the strain gradient theory of elasticity is adequate to describe the deformation of isotropic chiral elastic solids. The equations and the boundary conditions of the non-linear strain gradient theory of elastic solids were first established by Toupin [12,13]. The linear theory has been developed by Mindlin [9] and Mindlin and Eshel [10]. The interest in the gradient theory of elasticity is stimulated by the fact that this theory is used to investigate problems related to size effects and nanotechnology [1]. We note that the torsion of isotropic chiral elastic cylinders has been studied in various papers (see [7,11]). In the present article we study Saint-Venant's problem, where the cylinder is subjected to extension, bending, torsion and flexure. The problem is reduced to the study of some two-dimensional problems. It is shown that, in contrast with the case of achiral materials, the flexure of a chiral cylinder is accompanied by extension and bending.

2. Basic equations

In this section we present the equations and the boundary conditions in the linear strain-gradient theory of elastostatics. We consider a solid that in its undeformed state occupies the region B of Euclidean three-dimensional space and is bounded by the surface ∂B. We refer the deformation of the continuum to a fixed system of rectangular axes Ox_k, $(k = 1, 2, 3)$. Throughout we employ standard indicial notations: Latin subscripts (unless otherwise specified) are understood to range over the integers $(1, 2, 3)$, whereas Greek subscripts to the range $(1, 2)$, summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. We assume that B is occupied by a homogeneous and isotropic chiral elastic solid. The strain measures are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}, \tag{2.1}$$

where u_i is the displacement vector field. The constitutive equations for homogeneous and isotropic chiral elastic solids are [11]

$$\begin{aligned} \tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}), \\ \mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \\ &\quad + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} + \\ &\quad + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}), \end{aligned} \tag{2.2}$$

where τ_{ij} is the stress tensor, μ_{ijk} is the double stress tensor, δ_{ij} is the Kronecker delta, ε_{ijk} is the alternating symbol and λ, μ, α_m, $(m = 1, 2, \ldots, 5)$, and f are constitutive constants. For an achiral material the coefficient f is equal to zero. Let F_i be the body force per unit volume. The equilibrium equations are

$$\tau_{ji,j} - \mu_{sji,sj} + F_i = 0. \tag{2.3}$$

We assume that B is a bounded region with Lipschitz boundary ∂B, consisting of a finite number of smooth surfaces. Let Γ_p be the intersection of two adjoined smooth surfaces and $C = \cup\Gamma_p$. Mindlin [9] has introduced the notations

$$P_i = (\tau_{ki} - \mu_{ski,s})n_k - D_j(n_r\mu_{rji}) + (D_k n_k)n_s n_p \mu_{spi},$$
$$R_i = \mu_{rsi} n_r n_s, \quad Q_i =< \mu_{pji} n_p n_q > \varepsilon_{jrq} s_r, \tag{2.4}$$

where n_i are the components of the outward unit normal of ∂B, D_i are the components of the surface gradient, $D_i = (\delta_{ik} - n_i n_k)\partial/\partial x_k$, s_i are the components of the unit vector tangent to C, and $< g >$ denotes the difference of limits of g from both sides of C. The traction problem consists in finding a displacement field that satisfies the equations (2.1), the constitutive equations (2.2), and the equilibrium equations (2.3) on B, and the boundary conditions

$$P_i = \widetilde{P}_i, \; R_i = \widetilde{R}_i \text{ on } B\backslash C, \;\; Q_i = \widetilde{Q}_i \text{ on } \; C, \tag{2.5}$$

where F_i, $\widetilde{P}_i$, $\widetilde{R}_i$ and $\widetilde{Q}_i$ are prescribed functions. Throughout this paper we assume that the elastic potential is a positive definite quadratic form in the variables e_{ij} and κ_{ijk}.

3. Saint-Venant's problem

In this section we present the formulation of Saint-Venant's problem in the context of the strain gradient theory of elasticity. We suppose that the region B from here on refers to the interior of a right cylinder of length h with the cross-section Σ and the lateral boundary Π. The Cartesian coordinate frame is supposed to be chosen in such a way that x_3-axis is parallel to the generator of B and the x_1Ox_2 plane contains one of terminal cross-sections. We denote by Σ_1 and Σ_2 the cross-sections located at $x_3 = 0$ and $x_3 = h$, respectively. We denote by Γ the boundary of the cross-section Σ_1. We assume that the lateral surface Π is smooth, so that Q_i is equal to zero on Π. We assume that the cylinder is free from lateral loading. The conditions on the lateral surface are

$$P_i = 0, \quad R_i = 0 \;\text{ on } \;\; \Pi. \tag{3.1}$$

We shall use Saint-Venant's approach of the problem, in which the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. We assume that the body forces are absent and that the load on the cylinder is distributed over its ends, Σ_1 and Σ_2, in a way which fulfills the equilibrium conditions of the body. Let the loading applied on Σ_1 be statically equivalent to the force $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ and the moment $\boldsymbol{M} = (M_1, M_2, M_3)$. The conditions on the end located at $x_3 = 0$ are

$$\int_{\Sigma_1} P_\alpha da + \int_\Gamma Q_\alpha ds = \mathcal{F}_\alpha, \tag{3.2}$$

$$\int_{\Sigma_1} P_3 da + \int_\Gamma Q_3 ds = \mathcal{F}_3, \tag{3.3}$$

$$\int_{\Sigma_1} (x_\alpha P_3 + R_\alpha) da + \int_\Gamma x_\alpha Q_3 ds = \varepsilon_{\beta\alpha 3} M_\beta, \tag{3.4}$$

$$\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta da + \int_\Gamma \varepsilon_{\alpha\beta 3} x_\alpha Q_\beta ds = M_3. \tag{3.5}$$

In view of (2.4) we obtain

$$P_i = -\tau_{3i} + 2\mu_{\alpha 3i,\alpha} + \mu_{33i,3},\ R_i = \mu_{33i} \text{ on } \Sigma_1,\ Q_i = -2\mu_{\alpha 3i} n_\alpha \text{ on } \ \Gamma, \tag{3.6}$$

where $(n_1, n_2, 0)$ are the direction cosines of the exterior normal to Π. On Σ_2 there are tractions that satisfy the equilibrium conditions of the body. The equilibrium equations become

$$\tau_{ji,j} - \mu_{pji,pj} = 0. \tag{3.7}$$

Saint-Venant's problem consists in finding the functions u_i of class $C^4(B) \cap C^3(\overline{B})$ which satisfy the equations (2.1), (2.2) and (3.7) on B, the conditions (2.6) on the lateral surface, and the conditions (3.2)-(3.5) on the end Σ_1, when the constants $\mathcal{F}_j$, M_j, and the constitutive coefficients are prescribed. If $\mathcal{F}_3 = 0$ and $M_j = 0$, then the problem reduces to the flexure problem.

4. Two-dimensional problems

In this section we assume that the cylinder B is subjected to the external loading $(F_i, \widetilde{P}_i, \widetilde{R}_i, \widetilde{Q}_i)$ where F_i, $\widetilde{P}_i$ and $\widetilde{R}_i$ are independent of the axial coordinate and $\widetilde{Q}_i = 0$ on C. A state of generalized plane strain in cylinder B is characterized by the fact that the displacement vector is independent of the axial coordinate,

$$u_i = u_i(x_1, x_2),\ \ (x_1, x_2) \in \Sigma_1. \tag{4.1}$$

The relations (2.1), (2.2) and (4.1) imply that $e_{ij}, \kappa_{ijk}, \tau_{ij}$ and μ_{ijk} are all independent of the axial coordinate. In the case of a generalized plane strain, the strain measures reduce to

$$2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha},\ 2e_{\alpha 3} = u_{3,\alpha},\ \kappa_{\alpha\beta k} = u_{k,\alpha\beta}, \tag{4.2}$$

and $e_{33}=0$, $\kappa_{i3j}=0$. The constitutive equations can be written as

$$\begin{aligned}
\tau_{\alpha\beta} &= \lambda e_{\rho\rho}\delta_{a\beta}+2\mu e_{\alpha\beta}+f(\varepsilon_{\alpha\rho3}\kappa_{\beta\rho3}+\varepsilon_{\beta\rho3}\kappa_{\alpha\rho3}),\\
\tau_{\alpha3} &= 2\mu e_{\alpha3}+f\varepsilon_{\rho\beta3}\kappa_{\alpha\rho\beta},\\
\mu_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\kappa_{\rho\rho\alpha}\delta_{\beta\gamma}+2\kappa_{\gamma\rho\rho}\delta_{\alpha\beta}+\kappa_{\rho\rho\beta}\delta_{\alpha\gamma})+\\
&\quad+\alpha_2(\kappa_{\alpha\rho\rho}\delta_{\beta\gamma}+\kappa_{\beta\rho\rho}\delta_{\alpha\gamma})+2\alpha_3\kappa_{\rho\rho\gamma}\delta_{\alpha\beta}+\\
&\quad+2\alpha_4\kappa_{\alpha\beta\gamma}+\alpha_5(\kappa_{\gamma\beta\alpha}+\kappa_{\gamma\alpha\beta})+f(\varepsilon_{\alpha\gamma3}e_{\beta3}+\varepsilon_{\beta\gamma3}e_{\alpha3}),\\
\mu_{\alpha\beta3} &= 2\alpha_3\kappa_{\rho\rho3}\delta_{\alpha\beta}+2\alpha_4\kappa_{\alpha\beta3}+f(\varepsilon_{\rho\alpha3}e_{\beta\rho}+\varepsilon_{\rho\beta3}e_{\alpha\rho}),
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
&\tau_{33}=\lambda e_{\rho\rho},\ \mu_{3\alpha\beta}=\frac{1}{2}\alpha_1\kappa_{\rho\rho3}\delta_{\alpha\beta}+\alpha_5\kappa_{\beta\alpha3}+f\varepsilon_{\beta\rho3}e_{\alpha\rho},\\
&\mu_{3\alpha3}=\frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha}+\alpha_2\kappa_{\alpha\rho\rho}+f\varepsilon_{\rho\alpha3}e_{3\rho},\\
&\mu_{33\alpha}=\alpha_1\kappa_{\alpha\rho\rho}+2\alpha_3\kappa_{\rho\rho\alpha}+2f\varepsilon_{\alpha\rho3}e_{3\rho},\\
&\mu_{333}=(\alpha_1+2\alpha_3)\kappa_{\rho\rho3}.
\end{aligned}\tag{4.4}$$

In the context of generalized plane strain, the equations of equilibrium become

$$\tau_{\alpha i,\alpha}-\mu_{\beta\nu i,\beta\nu}+F_i=0\ \text{ on } \Sigma_1.\tag{4.5}$$

The functions P_i and R_i associated to the lateral surface are given by

$$\begin{aligned}
&P_i=(\tau_{\beta i}-\mu_{\rho\beta i,\rho})n_\beta-D_\rho(n_\beta\mu_{\beta\rho i})+(D_\rho n_\rho)n_\beta n_\nu\mu_{\beta\nu i},\\
&R_i=\mu_{\rho\nu i}n_\rho n_\nu.
\end{aligned}\tag{4.6}$$

The conditions on the lateral surface become

$$P_i=\widetilde{P}_i,\ R_i=\widetilde{R}_i\ \text{ on } \Gamma.\tag{4.7}$$

The generalized plane strain problem consists in finding the displacement field which satisfies the equations (4.2), the constitutive equations (4.3) and the equilibrium equations (4.5) on Σ_1, and the boundary conditions (4.7) on Γ. With the help of (4.2) and (4.3), the equations of equilibrium (4.5) can be expressed in the form

$$\begin{aligned}
&\mu\Delta u_\alpha+(\lambda+\mu)u_{\beta,\beta\alpha}-2(\alpha_3+\alpha_4)\Delta\Delta u_\alpha-\\
&-2(\alpha_1+\alpha_2+\alpha_5)\Delta u_{\beta,\beta\alpha}+2f\varepsilon_{\alpha\beta3}\Delta u_{3,\beta}+F_\alpha=0,\\
&\mu\Delta u_3-2(\alpha_3+\alpha_4)\Delta\Delta u_3+2f\varepsilon_{\rho\nu3}\Delta u_{\nu,\rho}+F_3=0,\ \text{ on } \Sigma_1.
\end{aligned}\tag{4.8}$$

The necessary and sufficient conditions for the existence of a solution to the generalized plane strain problem are [3]

$$\begin{aligned}
&\int_\Sigma F_k da+\int_\Gamma \widetilde{P}_k ds=0,\\
&\int_\Sigma \varepsilon_{3\alpha\beta}x_\alpha F_\beta da+\int_\Gamma \varepsilon_{3\alpha\beta}(x_\alpha\widetilde{P}_\beta+n_\alpha\widetilde{R}_\beta)ds=0.
\end{aligned}\tag{4.9}$$

In the next section we will use four special problems of generalized plane strain, denoted by $A^{(k)}$, $(k = 1, 2, 3, 4)$. Let us denote by $u_i^{(k)}$, $e_{ij}^{(k)}$, $\kappa_{irs}^{(k)}$, $\tau_{ij}^{(k)}$ and $\mu_{irs}^{(k)}$ the displacement, the strain measures, the stress tensor and the double stress tensor in the problem $A^{(k)}$, respectively. The problem $A^{(k)}$ is associated to the body forces $F_i^{(k)}$ and to the tractions $\widetilde{P}_i^{(k)}$ and $\widetilde{R}_i^{(k)}$, defined by

$$
\begin{aligned}
&F_i^{(1)} = \lambda\delta_{1i},\ \widetilde{P}_1^{(1)} = -\lambda x_1 n_1 + (\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1 n_2)_{,\nu} n_\alpha,\\
&\widetilde{P}_2^{(1)} = -\lambda x_1 n_2 + \frac{1}{2}(\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1^2 - n_2^2)_{,\alpha} n_\nu,\\
&\widetilde{P}_3^{(1)} = 2 f n_2,\ \widetilde{R}_1^{(1)} = 2\alpha_3 - \alpha_1 + (\alpha_1 - 2\alpha_2)n_1^2,\\
&\widetilde{R}_2^{(1)} = (\alpha_1 - 2\alpha_2)n_1 n_2,\ \widetilde{R}_3^{(1)} = 0.\\
&F_i^{(2)} = \lambda\delta_{2i},\ \widetilde{P}_1^{(2)} = -\lambda x_2 n_1 + \frac{1}{2}(\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1^2 - n_2^2)_{,\alpha} n_\nu,\\
&\widetilde{P}_2^{(2)} = -\lambda x_2 n_2 + (\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1 n_2)_{,\nu} n_\alpha,\ \widetilde{P}_3^{(2)} = -2 f n_1, \qquad (4.10)\\
&\widetilde{R}_1^{(2)} = (\alpha_1 - 2\alpha_2)n_1 n_2,\ \widetilde{R}_2^{(2)} = 2\alpha_3 - \alpha_1 + (\alpha_1 - 2\alpha_2)n_2^2,\ \widetilde{R}_3^{(2)} = 0,\\
&F_i^{(3)} = 0,\ \widetilde{P}_\alpha^{(3)} = -\lambda n_\alpha,\ \widetilde{P}_3^{(3)} = 0,\ \widetilde{R}_i^{(3)} = 0,\\
&F_i^{(4)} = 0,\ \widetilde{P}_1^{(4)} = \frac{1}{2} f[5n_1 + D_1(x_2 n_2) + D_2(x_2 n_1 - 2x_1 n_2) -\\
&- 2(x_2 n_1 n_2 - x_1 n_2^2)(D_\rho n_\rho)],\\
&\widetilde{P}_2^{(4)} = \frac{1}{2} f[5n_2 + D_1(x_1 n_2 - 2x_2 n_1) + D_2(x_1 n_1) -\\
&- 2(x_1 n_1 n_2 - x_2 n_1^2)(D_\rho n_\rho)],\quad \widetilde{P}_3^{(4)} = \mu\varepsilon_{3\beta\rho} x_\rho n_\beta,\\
&\widetilde{R}_1^{(4)} = f(x_1 n_2^2 - x_2 n_1 n_2),\ \widetilde{R}_2^{(4)} = f(x_2 n_1^2 - x_1 n_1 n_2),\ \widetilde{R}_3^{(4)} = 0.
\end{aligned}
$$

We introduce the notations

$$
\begin{aligned}
&P_i^{(k)} = (\tau_{\beta i}^{(k)} - \mu_{\rho\beta i,\rho}^{(k)})n_\beta - D_\rho(n_\beta \mu_{\beta\rho i}^{(k)}) + (D_\rho n_\rho)\mu_{\beta\nu i}^{(k)} n_\beta n_\nu,\\
&R_i^{(k)} = \mu_{\rho\nu i}^{(k)} n_\rho n_\nu. \qquad (4.11)
\end{aligned}
$$

The functions $u_i^{(k)}$, $e_{ij}^{(k)}$, $\kappa_{irs}^{(k)}$, $\tau_{ij}^{(k)}$ and $\mu_{irs}^{(k)}$ satisfy the equations

$$
2e_{\alpha\beta}^{(k)} = u_{\alpha,\beta}^{(k)} + u_{\beta,\alpha}^{(k)},\ 2e_{\alpha 3}^{(k)} = u_{3,\alpha}^{(k)},\ \kappa_{\alpha\beta j}^{(k)} = u_{j,\alpha\beta}^{(k)}, \qquad (4.12)
$$

the constitutive equations

$$\begin{aligned}
\tau^{(k)}_{\alpha\beta} &= \lambda e^{(k)}_{\rho\rho}\delta_{\alpha\beta} + 2\mu e^{(k)}_{\alpha\beta} + f(\varepsilon_{\alpha\rho 3}\kappa^{(k)}_{\beta\rho 3} + \varepsilon_{\beta\rho 3}\kappa^{(k)}_{\alpha\rho 3}),\\
\tau^{(k)}_{\alpha 3} &= 2\mu e^{(k)}_{\alpha 3} + f\varepsilon_{\rho\beta 3}\kappa^{(k)}_{\alpha\rho\beta},\\
\mu^{(k)}_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\kappa^{(k)}_{\rho\rho\alpha}\delta_{\beta\gamma} + 2\kappa^{(k)}_{\gamma\rho\rho}\delta_{\alpha\beta} + \kappa^{(k)}_{\rho\rho\beta}\delta_{\alpha\gamma}) +\\
&+ \alpha_2(\kappa^{(k)}_{\alpha\rho\rho}\delta_{\beta\gamma} + \kappa^{(k)}_{\beta\rho\rho}\delta_{\alpha\gamma}) + 2\alpha_3\kappa^{(k)}_{\rho\rho\gamma}\delta_{\alpha\beta} +\\
&+ 2\alpha_4\kappa^{(k)}_{\alpha\beta\gamma} + \alpha_5(\kappa^{(k)}_{\gamma\beta\alpha} + \kappa^{(k)}_{\gamma\alpha\beta}) + f(\varepsilon_{\alpha\gamma 3}e^{(k)}_{\beta 3} + \varepsilon_{\beta\gamma 3}e^{(k)}_{\alpha 3}),\\
\mu^{(k)}_{\alpha\beta 3} &= 2\alpha_3\kappa^{(k)}_{\rho\rho 3}\delta_{\alpha\beta} + 2\alpha_4\kappa^{(k)}_{\alpha\beta 3} + f(\varepsilon_{\rho\alpha 3}e^{(k)}_{\beta\rho} + \varepsilon_{\rho\beta 3}e^{(k)}_{\alpha\rho}),
\end{aligned} \tag{4.13}$$

and the equilibrium equations

$$\tau^{(k)}_{\beta j,\beta} - \mu^{(k)}_{\rho\nu j,\rho\nu} + F^{(k)}_j = 0, \tag{4.14}$$

on Σ_1, and the boundary conditions

$$P^{(k)}_i = \widetilde{P}^{(k)}_i, \quad R^{(k)}_i = \widetilde{R}^{(k)}_i \quad \text{on } \Gamma. \tag{4.15}$$

The functions $F^{(k)}_i$, $\widetilde{P}^{(k)}_i$ and $\widetilde{R}^{(k)}_i$ which appear in (4.14) and (4.15) are defined by (4.10). Let us note that the necessary and sufficient conditions (4.9) for the existence of solution are satisfied for each boundary value problem $A^{(k)}$. The problems $A^{(k)}$ have been introduced in [7] in order to study the torsion problem.

5. Solution of the problem

In this section we investigate Saint-Venant's problem for homogeneous and isotropic chiral rods. We seek the solution in the form [6]

$$\begin{aligned}
u_\alpha &= -\frac{1}{2}a_\alpha x_3^2 - \frac{1}{6}b_\alpha x_3^3 - (a_4 + \frac{1}{2}b_4 x_3)\varepsilon_{3\alpha\beta}x_\beta x_3 +\\
&+ \sum_{k=1}^{4}(a_k + b_k x_3)u^{(k)}_\alpha + w_\alpha(x_1, x_2),\\
u_3 &= (a_1x_1 + a_2x_2 + a_3)x_3 + \frac{1}{2}(b_1x_1 + b_2x_2 + b_3)x_3^2 + w_3(x_1, x_2),
\end{aligned} \tag{5.1}$$

where $u^{(k)}_i$ are the displacements in the plane problems $A^{(k)}$, w_j are unknown functions and a_k and b_k are unknown constants. Let us consider a generalized plane strain of the cylinder B in which the components of the displacement vector are the functions w_j. We denote by γ_{ij} and $\eta_{\alpha\beta j}$ the strain measures corresponding to the displacements w_j,

$$\gamma_{\alpha\beta} = \frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha}),\ 2\gamma_{\alpha 3} = w_{3,\alpha},\ \eta_{\alpha\beta k} = w_{k,\alpha\beta}. \tag{5.2}$$

From the equations (2.1) and (5.1) we obtain

$$
\begin{aligned}
e_{\alpha\beta} &= \gamma_{\alpha\beta} + \sum_{k=1}^{4}(a_k + b_k x_3)e^{(k)}_{\alpha\beta},\\
e_{\alpha 3} &= \frac{1}{2}\varepsilon_{3\beta\alpha}(a_4 + b_4 x_3)x_\beta + \sum_{k=1}^{4}(a_k + b_k x_3)e^{(k)}_{\alpha 3} + \gamma_{\alpha 3} + \frac{1}{2}\sum_{k=1}^{4} b_k u^{(k)}_\alpha,\\
e_{33} &= a_1 x_1 + a_2 x_2 + a_3 + (b_1 x_1 + b_2 x_2 + b_3)x_3 + \sum_{k=1}^{4} b_k u^{(k)}_3,\\
\kappa_{\alpha\beta\gamma} &= \eta_{\alpha\beta\gamma} + \sum_{k=1}^{4}(a_k + b_k x_3)\kappa^{(k)}_{\alpha\beta\gamma},\\
\kappa_{\alpha\beta 3} &= \eta_{\alpha\beta 3} + \sum_{k=1}^{4}(a_k + b_k x_3)\kappa^{(k)}_{\alpha\beta 3},\\
\kappa_{\beta 3\alpha} &= \varepsilon_{3\beta\alpha}(a_4 + b_4 x_3) + \sum_{k=1}^{4} b_k u^{(k)}_{\alpha,\beta},\\
\kappa_{\alpha 33} &= a_\alpha + b_\alpha x_3 + \frac{1}{2}\sum_{k=1}^{4} b_k e^{(k)}_{\alpha 3},\\
\kappa_{33\alpha} &= -a_\alpha - b_\alpha x_3 + \varepsilon_{3\beta\alpha} b_4 x_\beta,\ \kappa_{333} = b_1 x_1 + b_2 x_2 + b_3,
\end{aligned}
\tag{5.3}
$$

where $e^{(k)}_{\alpha j}$ and $\kappa^{(k)}_{\alpha\beta j}$ are given by (4.12). The stress tensor and the double stress tensor associated to the strain measures γ_{ij} and $\eta_{\alpha\beta j}$ are defined by

$$
\begin{aligned}
t_{\alpha\beta} &= \lambda\gamma_{\rho\rho}\delta_{\alpha\beta} + 2\mu\gamma_{\alpha\beta} + f(\varepsilon_{\alpha\rho 3}\eta_{\beta\rho 3} + \varepsilon_{\beta\rho 3}\eta_{\alpha\rho 3}),\\
t_{\alpha 3} &= 2\mu\gamma_{\alpha 3} + f\varepsilon_{\rho\beta 3}\eta_{\alpha\rho 3},\ t_{33} = \lambda\gamma_{\rho\rho},\\
\nu_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\eta_{\rho\rho\alpha}\delta_{\beta\gamma} + 2\eta_{\gamma\rho\rho}\delta_{\alpha\beta} + \eta_{\rho\rho\beta}\delta_{\alpha\gamma}) +\\
&\quad + \alpha_2(\eta_{\alpha\rho\rho}\delta_{\beta\gamma} + \eta_{\beta\rho\rho}\delta_{\alpha\gamma}) + 2\alpha_3\eta_{\rho\rho\gamma}\delta_{\alpha\beta} +\\
&\quad + 2\alpha_4\eta_{\alpha\beta\eta} + \alpha_5(\eta_{\gamma\beta\alpha} + \eta_{\gamma\alpha\beta}) +\\
&\quad + f(\varepsilon_{a\gamma 3}\gamma_{\beta 3} + \varepsilon_{\beta\gamma 3}\gamma_{\alpha 3}),\\
\nu_{\alpha\beta 3} &= 2\alpha_3\eta_{\rho\rho 3}\delta_{\alpha\beta} + 2\alpha_4\eta_{\alpha\beta 3} + f(\varepsilon_{\rho\alpha 3}\gamma_{\beta\rho} + \varepsilon_{\rho\beta 3}\gamma_{\alpha\rho}),\\
\nu_{\alpha 33} &= \frac{1}{2}\alpha_1\eta_{\rho\rho\alpha} + \alpha_2\eta_{\alpha\rho\rho} + f\varepsilon_{3\rho\alpha}\gamma_{3\rho},\\
\nu_{33\alpha} &= \alpha_1\eta_{\alpha\rho\rho} + 2\alpha_3\eta_{\rho\rho\alpha} + 2f\varepsilon_{3\alpha\rho}\gamma_{3\rho},\\
\nu_{\alpha 3\beta} &= \frac{1}{2}\alpha_1\eta_{\rho\rho 3} + \alpha_5\eta_{\beta\alpha 3} + f\varepsilon_{3\beta\rho}\gamma_{\alpha\rho},\\
\nu_{333} &= (\alpha_1 + 2\alpha_3)\eta_{\rho\rho 3}.
\end{aligned}
\tag{5.4}
$$

From the constitutive equations and the relations (5.3) and (5.4) we find the following form of the stress tensor τ_{ij},

$$\begin{aligned}\tau_{\alpha\beta} &= t_{\alpha\beta} + \{\lambda[a_1x_1 + a_2x_2 + a_3 + (b_1x_1 + b_2x_2 + b_3)x_3] - \\ &\quad - 2f(a_4 + b_4x_3)\}\delta_{\alpha\beta} + T_{\alpha\beta} + \sum_{k=1}^{4}(a_k + b_kx_3)\tau^{(k)}_{\alpha\beta}, \\ \tau_{\alpha 3} &= t_{\alpha 3} + 2f\varepsilon_{\alpha\rho 3}(a_\rho + b_\rho x_3) + \mu\varepsilon_{3\beta\alpha}(a_4 + b_4x_3)x_\beta + \\ &\quad + T_{\alpha 3} + \sum_{k=1}^{4}(a_k + b_kx_3)\tau^{(k)}_{\alpha 3}, \\ \tau_{33} &= t_{33} + (\lambda + 2\mu)[a_1x_1 + a_2x_2 + a_3 + (b_1x_1 + b_2x_2 + b_3)x_3] + \\ &\quad + 4f(a_4 + b_4x_3) + T_{33} + \lambda\sum_{k=1}^{4}(a_k + b_kx_3)u^{(k)}_{\rho,\rho},\end{aligned} \tag{5.5}$$

where we have used the notations

$$\begin{aligned}T_{\alpha\beta} &= f\sum_{k=1}^{4} b_k(\varepsilon_{3\rho\alpha}u^{(k)}_{\rho,\beta} + \varepsilon_{3\rho\beta}u^{(k)}_{\rho,\alpha}), \\ T_{\alpha 3} &= \mu\sum_{k=1}^{4} b_k u^{(k)}_\alpha - f(b_4x_\alpha + \sum_{k=1}^{4} b_k\varepsilon_{\rho\alpha 3}u^{(k)}_{3,\rho}), \\ T_{33} &= (\lambda + 2\mu)\sum_{k=1}^{4} b_k u^{(k)}_3 + 2f\varepsilon_{3\alpha\beta}\sum_{k=1}^{4} b_k u^{(k)}_{\alpha,\beta}.\end{aligned} \tag{5.6}$$

The components of the double tensor are given by

$$\begin{aligned}\mu_{111} &= \nu_{111} + 2(\alpha_2 - \alpha_3)(a_1 + b_1x_3) + \sum_{k=1}^{4}(a_k + b_kx_3)\mu^{(k)}_{111} + N_{111}, \\ \mu_{222} &= \nu_{222} + 2(\alpha_2 - \alpha_3)(a_2 + b_2x_3) + \sum_{k=1}^{4}(a_k + b_kx_3)\mu^{(k)}_{222} + N_{222}, \\ \mu_{221} &= \nu_{221} + (\alpha_1 - 2\alpha_3)(a_1 + b_1x_3) - f(a_4 + b_4x_3)x_1 + \\ &\quad + \sum_{k=1}^{4}(a_k + b_kx_3)\mu^{(k)}_{221} + N_{221}, \\ \mu_{112} &= \nu_{112} + (\alpha_1 - 2\alpha_3)(a_2 + b_2x_3) - f(a_4 + b_4x_3)x_2 + \\ &\quad + \sum_{k=1}^{4}(a_k + b_kx_3)\mu^{(k)}_{112} + N_{112},\end{aligned}$$

$$
\begin{aligned}
\mu_{121} &= \nu_{121} + \frac{1}{2}(2\alpha_2 - \alpha_1)(a_2 + b_2 x_3) + \frac{1}{2} f(a_4 + b_4 x_3) x_2 + \\
&\quad + \sum_{k=1}^{4} (a_k + b_k x_3) \mu_{121}^{(k)} + N_{121}, \\
\mu_{122} &= \nu_{122} + \frac{1}{2}(2\alpha_2 - \alpha_1)(a_1 + b_1 x_3) + \frac{1}{2} f(a_4 + b_4 x_3) x_1 + \\
&\quad + \sum_{k=1}^{4} (a_k + b_k x_3) \mu_{122}^{(k)} + N_{122}, \\
\mu_{\rho 33} &= \nu_{\rho 33} + \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)(a_\rho + b_\rho x_3) - \frac{1}{2} f(a_4 + b_4 x_3) x_\rho + \\
&\quad + \sum_{k=1}^{4} (a_k + b_k x_3) \mu_{\rho 33}^{(k)} + N_{\rho 33}, \\
\mu_{33\rho} &= \nu_{33\rho} + (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5)(a_\rho + b_\rho x_3) + \qquad (5.7) \\
&\quad + f(a_4 + b_4 x_3) x_\rho + \sum_{k=1}^{4} (a_k + b_k x_3) \mu_{33\rho}^{(k)} + N_{33\rho}, \\
\mu_{\alpha 3\beta} &= \nu_{\alpha 3\beta} + (2\alpha_4 - \alpha_5)(a_4 + b_4 x_3) \varepsilon_{\alpha\beta 3} + \\
&\quad + f[a_1 x_1 + a_2 x_2 + a_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3] \varepsilon_{\alpha\beta 3} + \\
&\quad + \sum_{k=1}^{4} (a_k + b_k x_3) \mu_{\alpha 3\beta}^{(k)} + N_{\alpha 3\beta}, \\
\mu_{\alpha\beta 3} &= \nu_{\alpha\beta 3} + \sum_{k=1}^{4} (a_k + b_k x_3) \mu_{\alpha\beta 3}^{(k)} + N_{\alpha\beta 3}, \\
\mu_{333} &= \nu_{333} + (\alpha_1 + 2\alpha_3) \sum_{k=1}^{4} (a_k + b_k x_3) \kappa_{333}^{(k)} + N_{333}.
\end{aligned}
$$

Here we have used the following notations

$$
\begin{aligned}
N_{111} &= -(\alpha_1 + 2\alpha_3) b_4 x_2 + (\alpha_1 + 2\alpha_2) \sum_{k=1}^{4} b_k u_{3,1}^{(k)}, \\
N_{222} &= (\alpha_1 + 2\alpha_3) b_4 x_1 + (\alpha_1 + 2\alpha_2) \sum_{k=1}^{4} b_k u_{3,2}^{(k)}, \\
N_{221} &= -2\alpha_3 b_4 x_2 + \alpha_1 \sum_{k=1}^{4} b_k u_{3,1}^{(k)} - f \sum_{k=1}^{4} b_k u_2^{(k)},
\end{aligned}
$$

$$N_{112} = 2\alpha_3 b_4 x_1 + \sum_{k=1}^{4} b_k(\alpha_1 u_{3,2}^{(k)} + 2\alpha_3 u_1^{(k)} + f u_1^{(k)}),$$

$$N_{121} = \frac{1}{2}\alpha_1 b_4 x_1 + \sum_{k=1}^{4} b_k(\alpha_2 u_{3,2}^{(k)} - \frac{1}{2} f u_1^{(k)}),$$

$$N_{122} = -\frac{1}{2}\alpha_1 b_4 x_2 + \sum_{k=1}^{4} b_k(\alpha_2 u_{3,1}^{(k)} + \frac{1}{2} f u_2^{(k)}),$$

$$\begin{aligned} N_{\rho 33} &= \frac{1}{2}(\alpha_1 + 2\alpha_5)\varepsilon_{3\beta\rho} b_4 x_\beta + \sum_{k=1}^{4} b_k[(\alpha_2 + 2\alpha_4 + \alpha_5)u_{3,\rho}^{(k)} + \\ &+ \frac{1}{2} f \varepsilon_{3\rho\beta} u_\beta^{(k)}], \end{aligned} \tag{5.8}$$

$$N_{33\rho} = 2(\alpha_3 + \alpha_4)\varepsilon_{3\beta\rho} b_4 x_\beta + \sum_{k=1}^{4} b_k[(\alpha_1 + 2\alpha_5)u_{3,\rho}^{(k)} + f\varepsilon_{3\rho\beta} u_\beta^{(k)}],$$

$$\begin{aligned} N_{\alpha 3\beta} &= \frac{1}{2}(\alpha_1 + 2\alpha_2)(b_1 x_1 + b_2 x_2 + b_3)\delta_{a\beta} + \sum_{k=1}^{4} b_k(2\alpha_4 u_{\beta,\alpha}^{(k)} + \\ &+ \alpha_5 u_{\alpha,\beta}^{(k)} + \alpha_2 \delta_{\alpha\beta} u_{\rho,\rho}^{(k)} + f\varepsilon_{\alpha\beta 3} u_3^{(k)}), \end{aligned}$$

$$N_{\alpha\beta 3} = (\alpha_1 + 2\alpha_3)\delta_{\alpha\beta}(b_1 x_1 + b_2 x_2 + b_3) + \sum_{k=1}^{4} b_k(\alpha_1 \delta_{\alpha\beta} u_{\rho,\rho}^{(k)} + 2\alpha_5 e_{\alpha\beta}^{(k)}),$$

$$N_{333} = 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)(b_1 x_1 + b_2 x_2 + b_3).$$

In view of (4.10), (4.14), (5.5) and (5.7), the equations of equilibrium become

$$t_{\alpha i,\alpha} - \nu_{\rho\alpha i,\rho\alpha} + G_i = 0 \ \text{ on } \Sigma_1, \tag{5.9}$$

where

$$\begin{aligned} G_\alpha &= T_{\beta\alpha,\beta} - N_{\rho\eta\alpha,\rho\eta} - \sum_{k=1}^{4} b_k(2\mu_{3\rho\alpha,\rho}^{(k)} - \tau_{\alpha 3}^{(k)}) + \\ &+ 4f\varepsilon_{3\alpha\beta} b_\beta + \mu\varepsilon_{3\rho\alpha} x_\rho b_4, \end{aligned} \tag{5.10}$$

$$\begin{aligned} G_3 &= T_{\beta 3,\beta} - N_{\rho\eta 3,\rho\eta} + \sum_{k=1}^{4} b_k(\lambda u_{\rho,\rho}^{(k)} - 2\mu_{\rho 33,\rho}^{(k)}) + \\ &+ (\lambda + 2\mu)(b_1 x_1 + b_2 x_2 + b_3) + 6 f b_4. \end{aligned}$$

We denote

$$\Pi_j = (t_{\beta j} - \nu_{\rho\beta j,\rho})n_\beta - D_\eta(n_\rho \nu_{\rho\eta j}) + (D_\nu n_\nu)n_\rho n_\eta \nu_{\rho\eta j}, \Lambda_j = \nu_{\rho\eta j} n_\rho n_\eta, \tag{5.11}$$

and introduce the notations

$$H_j = (T_{\beta j} - N_{\rho\beta j,\rho})n_\beta - D_\eta(n_\rho N_{\rho\eta j}) + (D_\nu n_\nu)n_\rho n_\eta N_{\rho\eta j}, L_j = N_{\rho\eta j} n_\rho n_\eta. \tag{5.12}$$

It follows from (2.4), (5.5) and (5.7) that the conditions on the lateral surface (2.6) become

$$\Pi_j = \widetilde{\Pi}_j, \quad \Lambda_j = \widetilde{\Lambda}_j \quad \text{on} \quad \Gamma, \tag{5.13}$$

where

$$\begin{aligned} \widetilde{\Pi}_\alpha &= 2\varepsilon_{\rho\alpha 3} n_\rho [(2\alpha_4 - \alpha_5) b_4 + f(b_1 x_1 + b_2 x_2 + b_3)] - H_\alpha + \\ &+ 2n_\rho \sum_{k=1}^{4} b_k \mu^{(k)}_{3\rho\alpha}, \qquad (5.14) \\ \widetilde{\Pi}_3 &= 2n_\rho [\frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4) b_\rho - \frac{1}{2} f b_4 x_\rho + \sum_{k=1}^{4} b_k \mu^{(k)}_{\rho 33}] - H_3, \quad \widetilde{\Lambda}_j = -L_j. \end{aligned}$$

Thus, the functions w_i are the components of the displacement vector in the generalized plane strain problem characterized by the equations (5.2), the constitutive equations (5.4) and the equilibrium equations (5.9) on Σ_1 and the boundary conditions (5.13) on Γ. The necessary and sufficient conditions to solve this problem are

$$\int_{\Sigma_1} G_i da + \int_\Gamma \widetilde{\Pi}_i ds = 0, \ \int_{\Sigma_1} \varepsilon_{3\alpha\beta} x_\alpha G_\beta da + \int_\Gamma \varepsilon_{3\alpha\beta} (x_\alpha \widetilde{\Pi}_\beta + n_\alpha \widetilde{\Lambda}_\beta) ds = 0. \tag{5.15}$$

By using the divergence theorem we obtain

$$\begin{aligned} &\int_{\Sigma_1} (T_{\beta j,\beta} - N_{\rho\eta j,\rho\eta}) da + \int_\Gamma \Pi_j ds = 0, \\ &\int_{\Sigma_1} \varepsilon_{3\alpha\beta} x_\alpha (T_{\rho\beta,\rho} - N_{\nu\eta\beta,\nu\eta}) da + \int_\Gamma \varepsilon_{3\alpha\beta} (x_\alpha \Pi_\beta + n_\alpha \Lambda_\beta) ds = 0. \end{aligned} \tag{5.16}$$

In view of (5.4), (5.10), (5.14) and (5.16) we find that

$$\int_{\Sigma_1} G_\alpha da + \int_\Gamma \widetilde{\Pi}_\alpha ds = \int_{\Sigma_1} \tau_{\alpha 3,3} da. \tag{5.17}$$

With the help of the equilibrium equations (3.7) we can write

$$\begin{aligned} \tau_{\alpha 3} &= \tau_{\alpha 3} + x_\alpha (\tau_{j3,j} - \mu_{rs3,rs}) = \\ &= [x_\alpha (\tau_{\beta 3} - \mu_{\beta\nu 3,\nu})]_{,\beta} + \mu_{\alpha\nu 3,\nu} + x_\alpha (\tau_{33} - \mu_{333,33} - 2\mu_{3\beta 3,3\beta}). \end{aligned} \tag{5.18}$$

We note that

$$(D_k n_k) n_s n_p \mu_{spi} - D_j (n_r \mu_{rji}) = [(\mu_{pji} n_p n_r - \mu_{pri} n_p n_j)_{,r}] n_j,$$

so that, the condition $P_3 = 0$ on Π can be expressed as

$$[\tau_{\beta 3} - \mu_{\beta\nu 3,\nu} + (\mu_{\rho\beta 3} n_\rho n_\eta - \mu_{\rho\eta 3} n_\rho n_\beta)_{,\eta} - (\mu_{\rho 33} n_\rho n_\beta)_{,3} - \mu_{3\beta 3,3}] n_\beta = 0 \ \text{ on } \Pi. \tag{5.19}$$

From (5.18) and (5.19) we find

$$\int_{\Sigma_1} \tau_{\alpha 3} da = \int_{\Gamma} x_\alpha \{[(\mu_{\rho\nu 3} n_\rho n_\beta - \mu_{\rho\beta 3} n_\rho n_\nu)_{,\nu}] n_\beta +$$
$$+ 2\mu_{\rho 33,3} n_\rho\} ds + \int_{\Sigma_1} [\mu_{\alpha\nu 3,\nu} + x_\alpha(\tau_{33,3} - \mu_{333,33} - 2\mu_{3\alpha 3,3\alpha})] da. \quad (5.20)$$

Since $R_3 = 0$ on Π, we have

$$\int_{\Gamma} x_\alpha [(\mu_{\rho\nu 3} n_\rho n_\beta - \mu_{\rho\beta 3} n_\rho n_\nu)_{,\nu}] n_\beta ds = -\int_{\Sigma_1} \mu_{\alpha\nu 3,\nu} da.$$

Thus, from (5.20) we get

$$\int_{\Sigma_1} \tau_{\alpha 3} da = \int_{\Sigma_1} [x_\alpha(\tau_{33,3} - \mu_{333,33}) + 2\mu_{\alpha 33,3}] da. \quad (5.21)$$

It follows from (5.5), (5.7), (5.8) and (5.21) that

$$\int_{\Sigma_1} \tau_{\alpha 3,3} da = 0. \quad (5.22)$$

By (5.17) and (5.22) we conclude that the first two conditions from (5.15) are satisfied. Let us introduce the notations

$$D_{\alpha k} = \int_{\Sigma_1} (x_\alpha \pi_{33}^{(k)} + 2q_{\alpha 33}^{(k)} - q_{33\alpha}^{(k)}) da,$$
$$D_{3k} = \int_{\Sigma_1} \pi_{33}^{(k)} da, \ D_{4k} = \int_{\Sigma_1} \varepsilon_{3\alpha\beta}(x_\alpha \pi_{3\beta}^{(k)} + 2q_{\alpha 3\beta}^{(k)}) da, \quad (5.23)$$

and

$$\pi_{33}^{(\alpha)} = \lambda u_{\rho,\rho}^{(\alpha)} + (\lambda + 2\mu) x_\alpha, \pi_{33}^{(3)} = \lambda u_{\rho,\rho}^{(k)} + \lambda + 2\mu,$$
$$\pi_{33}^{(4)} = \lambda u_{\rho,\rho}^{(4)} + 4f, \ \pi_{3\beta}^{(\alpha)} = \tau_{\beta 3}^{(\alpha)} + 2f\varepsilon_{3\beta\rho} c_\rho,$$
$$\pi_{3\beta}^{(3)} = \tau_{\beta 3}^{(3)}, \ \pi_{3\beta}^{(4)} = \tau_{\beta 3}^{(4)} + \mu\varepsilon_{\rho\beta 3} x_\rho, \quad (5.24)$$
$$q_{\alpha 33}^{(j)} = \frac{1}{2}(2a_2 - \alpha_1 + 4\alpha_4)\delta_{j\alpha} + \mu_{\alpha 33}^{(j)}, \ (j = 1, 2, 3),$$
$$q_{\alpha 33}^{(4)} = -\frac{1}{2} f x_\alpha + \mu_{\alpha 33}^{(4)}, q_{33\alpha}^{(j)} = (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5)\delta_{j\alpha} + \mu_{33\alpha}^{(j)},$$
$$q_{33\alpha}^{(4)} = f x_\alpha + \mu_{33\alpha}^{(4)}, \ q_{\alpha 3\beta}^{(\rho)} = \varepsilon_{3\alpha\beta} f x_\rho + \mu_{\alpha 3\beta}^{(\rho)},$$
$$q_{\alpha 3\beta}^{(3)} = \varepsilon_{3\alpha\beta} f + \mu_{\alpha 3\beta}^{(3)}, q_{\alpha 3\beta}^{(4)} = \varepsilon_{3\alpha\beta}(2\alpha_4 - \alpha_5) + \mu_{\alpha 3\beta}^{(4)}.$$

In view of (5.10), (5.14) and (5.16) we obtain

$$\int_{\Sigma_1} G_3 da + \int_{\Gamma} \widetilde{\Pi}_3 ds = \sum_{j=1}^{4} D_{3j} b_j,$$
$$\int_{\Sigma_1} \varepsilon_{3\alpha\beta} x_\alpha G_\beta da + \int_{\Gamma} \varepsilon_{3\alpha\beta}(x_\alpha \widetilde{\Pi}_\beta + n_\alpha \widetilde{\Lambda}_\beta) ds = \sum_{j=1}^{4} D_{4j} b_j, \quad (5.25)$$

where D_{jk} are defined in (5.23). Thus, the last two conditions (5.15) can be written in the form

$$\sum_{k=1}^{4} D_{3k} b_k = 0, \quad \sum_{k=1}^{4} D_{4k} b_k = 0. \tag{5.26}$$

Let us impose now the conditons (3.2). We note that in view of (3.6) we obtain

$$\int_{\Sigma_1} P_\alpha da + \int_\Gamma Q_\alpha ds = -\int_{\Sigma_1} (\tau_{3\alpha} - \mu_{33\alpha,3}) da.$$

By using (5.21) we find

$$\int_{\Sigma_1} P_\alpha da + \int_\Gamma Q_\alpha ds = -\int_{\Sigma_1} [x_\alpha(\tau_{33,3} - \mu_{333,33}) + 2\mu_{\alpha 33,3} - \mu_{33a,3}] da. \tag{5.27}$$

It follows from (5.5), (5.7), (5.23) and (5.27) that

$$\int_{\Sigma_1} P_\alpha da + \int_\Gamma Q_\alpha ds = -\sum_{k=1}^{4} D_{\alpha k} b_k.$$

The conditions (3.2) reduce to

$$\sum_{k=1}^{4} D_{\alpha k} b_k = -\mathcal{F}_\alpha. \tag{5.28}$$

It is known [7] that

$$\det(D_{rs}) \neq 0, \quad D_{rs} = D_{sr}. \tag{5.29}$$

We conclude that the constants b_1, b_2, b_3 and b_4 are determined by the system (5.26), (5.28). We note that the necessary and sufficient conditions for the existence of the functions w_j are satisfied. In what follows we shall assume that these functions are known. Let us investigate the conditions (3.3)-(3.5). We can write

$$\begin{aligned}
&\int_{\Sigma_1} P_3 da + \int_\Gamma Q_3 ds = -\int_{\Sigma_1} (\tau_{33} - \mu_{333,3}) ds,\\
&\int_{\Sigma_1} (x_\alpha P_3 + R_\alpha) da + \int_\Gamma x_\alpha Q_3 ds = -\int_{\Sigma_1} [x_\alpha(\tau_{33} - \mu_{333,3}) + 2\mu_{\alpha 33} - \mu_{33\alpha}] da,\\
&\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta da + \int_\Gamma \varepsilon_{\alpha\beta 3} x_\alpha Q_\beta ds =\\
&= \int_{\Sigma_1} [\varepsilon_{\alpha\beta 3} x_\alpha (\mu_{33\beta,3} - \tau_{33}) - 2\varepsilon_{\alpha\beta 3}\mu_{\alpha 3\beta}] da.
\end{aligned} \tag{5.30}$$

From (5.5), (5.7), (5.23) and (5.30) we obtain

$$
\begin{aligned}
&\int_{\Sigma_1} P_3 da + \int_\Gamma Q_3 ds = -\sum_{k=1}^4 D_{3k} a_k - \widehat{\mathcal{F}}_3,\\
&\int_{\Sigma_1} (x_\alpha P_3 + R_\alpha) da + \int_\Gamma x_\alpha Q_3 ds = -\sum_{k=1}^4 D_{\alpha k} a_k - \varepsilon_{3\alpha\beta}\widehat{\mathcal{M}}_\beta,\\
&\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta da + \int_\Gamma \varepsilon_{\alpha\beta 3} x_\alpha Q_\beta ds = -\sum_{k=1}^4 D_{4k} a_k - \widehat{\mathcal{M}}_3,
\end{aligned}
\tag{5.31}
$$

where

$$
\begin{aligned}
&\widehat{\mathcal{F}}_3 = \int_{\Sigma_1} [t_{33} + T_{33} - (\alpha_1 + 2\alpha_3)\sum_{k=1}^4 b_k \kappa_{333}^{(k)}] da,\\
&\widehat{\mathcal{M}}_\alpha = \varepsilon_{3\alpha\beta}\int_{\Sigma_1} \{x_\beta [t_{33} + T_{33} - (\alpha_1 + 2\alpha_3)\sum_{k=1}^4 b_k \kappa_{333}^{(k)}] +\\
&+ 2\nu_{\beta 33} + 2N_{\beta 33} - \nu_{33\beta} - N_{33\beta}\} da,\\
&\widehat{\mathcal{M}}_3 = \int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha [t_{\beta 3} + T_{\beta 3} - (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5) b_\beta -\\
&- f b_4 x_\beta - \sum_{k=1}^4 b_k \mu_{33\beta}^{(k)}] + 2\varepsilon_{\alpha\beta 3}(\nu_{\alpha 3\beta} + N_{\alpha 3\beta})\} da.
\end{aligned}
\tag{5.32}
$$

On the basis of (5.31), the conditions (3.3)-(3.5) become

$$
\begin{aligned}
&\sum_{k=1}^4 D_{\alpha k} a_k = \varepsilon_{3\alpha\beta}(M_\beta + \widehat{\mathcal{M}}_\beta),\\
&\sum_{k=1}^4 D_{3k} a_k = -\mathcal{F}_3 - \widehat{\mathcal{F}}_3,\ \sum_{k=1}^4 D_{4k} a_k = -M_3 - \widehat{\mathcal{M}}_3.
\end{aligned}
\tag{5.33}
$$

The system (5.33) uniquely determines the constants a_1, a_2, a_3 and a_4. We conclude that the solution of Saint-Venant's problem is given by (5.1), where the functions w_j are the displacements in the plane strain problem characterized by (5.2), (5.4), (5.9), (5.13), and the constants a_k and b_k are determined by the systems (5.33) and (5.28), (5.26), respectively. First, we have to find the solutions of the problems $A^{(k)}$, $(k = 1, 2, 3, 4)$, and the constants D_{rs} defined in (5.23). From (5.28), (5.32) and (5.33) we see that, in contrast with the classical elasticity, the flexure of chiral cylinders produces extension and bending effects. Torsion of a chiral circular cylinder has been studied in [7]. The solution presented in this paper can be used to investigate the problem of extension, bending and flexure.

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