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Revisiting the identification of distributed elastic coefficients in anisotropic media

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This article is dedicated to the 85th birthday of Professor Cristescu. We were fortunate, not only to have him between the inspirational teachers we had in mechanics during our undergraduate and graduate studies but also had the privilege to develop these early contacts into and stimulating dialogue over the years.

Abstract - The inverse problem discussed here is the identification of the distributed elastic moduli from overspecified boundary conditions. We present an extension of the classical result of Ikehata from the case of an isotropic material, for a class of anisotropic materials with cubic and orthotropic symmetry.

Key words and phrases : identification of elastic coefficients, anisotropy, Betty reciprocity formulation.

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1. Introduction

Linearized elasticity theory is one of the classical areas of mathematical physics. Several problem settings are well understood, existence and uniqueness results are available and closed-form or numerical solutions are ready for the applications. One of the recent areas in linearized elasticity is related to inverse problems. The main type are related to the determination of material parameters, boundary condition, geometrical details, damage, inclusions or cracks. They are usually motivated by practical engineering situations and the interest in these problems has recently blown up given the explosion of physical measuring technique, like the digital image correlation, which permit an easy access to an extended data field.

The inverse problem discussed next is the identification of the distributed elastic moduli from overspecified boundary conditions. This problem has already been addressed in the past (for a review see [5]) and several results have been obtained.

The fundamental tool is the Betti reciprocity equation which is a standard theorem in linear elasticity textbooks. It provided the entrance of the first uniqueness result given by Calderon for the electrical impedance tomography problem[8] and was equally used in inverse elasticity problems for the identification of cracks [4, 2, 7].

The main result for this problem has been given by Ikehata [16], who has proven the existence and the uniqueness of the solution in the case of an isotropic material. The advancement discussed here is an extension of the result of Ikehata for a small class of anisotropic materials with cubic and orthotropic symmetry.

The papers starts with a presentation of the direct and inverse elasticity problem. The second section recalls Ikehata's uniqueness result [16]. The next sections are dedicated to the discussion of different cases of anisotropic materials.

2. The direct and the inverse problem

Let us consider an elastic body Ω having the boundary $\partial\Omega$ under small strain assumption and in the absence of residual stresses in the reference configuration. The vector field of elastic displacements, denoted as $\mathbf{u}$ is the solution of the following system of elliptic partial differentail equations:

$$\operatorname{div}(\ \mathbf{c}(\mathbf{x})\nabla\mathbf{u}(x)\ ) = 0, \quad \mathbf{x} \in \Omega. \tag{2.1}$$

Here $\mathbf{c}$ denotes the forth order tensor of elastic moduli. $\mathbf{c}$ is is positive defined and exhibits the following symmetries expressed in terms of its coefficients using the cartesian coordinates:

$$c_{ijkl} = c_{klij} = c_{jikl}. \tag{2.2}$$

The preceding relations follow from the symmetry of strain and stress tensors and the existence of a strain energy, respectively. The constitutive equation is now written in either form:

$$\sigma = \mathbf{c}\varepsilon = \mathbf{c}\nabla\mathbf{u}, \tag{2.3}$$

where σ and ε are the stress and strain tensor, respectively.

The boundary conditions are defined either in terms of surface displacements, i.e. Dirichlet conditions: $\mathbf{u} = \mathbf{u}^D$ defined on $\mathcal{S}_u$, or in terms of surface tractions, i.e. Neumann conditions: $\sigma \cdot \mathbf{n} = \mathbf{t}^D$ defined on $\mathcal{S}_t$.

The *direct problem* provides a given boundary data pair $(\mathbf{u}^D, \mathbf{t}^D)$ on a complementary partition of the boundary $\mathcal{S}_u$ and $\mathcal{S}_t$. A solution always exists in this case and is unique if $\mathcal{S}_u \neq \emptyset$.

The *inverse problem*, discussed next, seeks to identify the heterogeneous tensor of elastic moduli $\mathbf{c} = \mathbf{c}(\mathbf{x}), \mathbf{x} \in \Omega$ from overspecified boundary data, i.e. $(\mathbf{u}^D, \mathbf{t}^D)$ on a partition of the boundary $\mathcal{S}_u$ and $\mathcal{S}_t$ such that $\mathcal{S}_u \cap \mathcal{S}_t \neq \emptyset$.

In the ideal case of the inverse problem all boundary conditions all perfectly known, which is equivalent to the knowledge of the Dirichlet-to-Neumann data map $\Lambda_{\mathbf{c}}$, which maps each boundary displacements into its corresponding boundary traction field:

$$\Lambda_{\mathbf{c}} : \mathbf{u}^D \to \mathbf{t}^D. \tag{2.4}$$

The standard difficulty for this problem can be illustrated by the following closed-form example for an elastic sphere (initially proposed in [9, 10]). Let us consider an elastic isotropic sphere of radius R. Under radial symmetry, the governing equations of the problem become:

$$\left(\frac{\lambda + 2\mu}{r^2}\left(r^2 u_r\right)_{,r}\right)_{,r} - \frac{\mu}{r}\, u_r = 0, \tag{2.5}$$

for $r \in [0, R]$. We recall that the isotropic elastic moduli are expressed as:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.6}$$

We consider two families of solutions corresponding to two families of spheres. First, the solution of the unperturbed problem:

$$\begin{aligned} u_r &= 1, \\ \mu(r) &= \mu_0 + r^2, \\ \lambda(r) &= \frac{1}{2} - 2\mu(r), \end{aligned} \tag{2.7}$$

and second a series of solution of perturbed problems:

$$\begin{aligned} u_r^N &= 1 + \frac{1}{Nr}\sin Nr + \frac{1}{N^2 r^2}\cos Nr, \\ \mu^N(r) &= \mu_0 + \int_{r_1}^{r_2} \frac{r dr}{1 + \frac{1}{Nr}\sin Nr + \frac{1}{N^2 r^2}\cos Nr}, \\ \lambda^N(r) &= \frac{1}{2 + \cos Nr} - 2\mu^N(r), \end{aligned} \tag{2.8}$$

One can easily remark that the the solution of the perturbed problem converges in terms of displacements to the unperturbed solution: $u^N \to u$. However $\lambda^N \nrightarrow \lambda$!

This simple example illustrates that one can not easily expect uniqueness or continuity for this class of inverse problems.

2.1. Variational formulation and Betti-Reciprocity

An essential tool in solving the inverse problems was provided by the variational formulation of the problem and the Betti reciprocity principle.

Let us consider two bodies occupying the same domain Ω and having the the elastic moduli $\mathbf{c}$ and $\mathbf{c}^*$. If the two elastic solutions, corresponding to the two bodies, are denoted as $\mathbf{u}$ and $\mathbf{u}^*$, using the standard technique of the Betti Reciprocity Principle , i.e. crossing solutions and virtual displacement fields and integrating by parts, one obtains:

$$\int_{\partial\Omega} (\mathbf{u}\boldsymbol{\Lambda}_{\mathbf{c}}(\mathbf{u}^*) - \mathbf{u}^*\boldsymbol{\Lambda}_{\mathbf{c}^*}(\mathbf{u}))\mathrm{d}s = \int_{\partial\Omega} \nabla\mathbf{u} : (\mathbf{c}^* - \mathbf{c}) : \nabla\mathbf{u}^*\mathrm{d}s. \tag{2.9}$$

A classical argument, involving an infinitesimal perturbation of the coefficients $\mathbf{c}^*$, conducts to a linear form of the problem:

$$\begin{aligned} \mathbf{c}^* &= \mathbf{c} + \delta\mathbf{c}, \\ \mathbf{u} &= \mathbf{u}_0 + \delta\mathbf{u}, \\ \mathbf{u}^* &= \mathbf{u}_0^*, \end{aligned} \tag{2.10}$$

and leads to the linearized form of the Betti Reciprocity principle:

$$\mathcal{RB}(\mathbf{u}, \mathbf{u}^*) = \int_{\partial\Omega} (\mathbf{u}\Lambda_{\mathbf{c}^*}(\mathbf{u}^*) - \mathbf{u}^*\Lambda_{\mathbf{c}}(\mathbf{u}))\mathrm{d}v, \tag{2.11}$$

$$= \int_{\Omega} \nabla\mathbf{u}\delta\mathbf{c}\nabla\mathbf{u}^*\mathrm{d}v. \tag{2.12}$$

3. Ikehata's solution for isotropic elasticity

The method discussed for the identification of anisotropic elastic moduli is based on the technique proposed by Ikehata [16] for isotropic elasticity. For the purpose of clarity, we shall presented briefly the main steps of this method.

We recall that $\mathbf{c}$ takes the following form in the case of isotropic elasticity

$$\mathbf{c}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \tag{3.1}$$

where λ and μ denote the Lamé moduli.

The inverse problem may be stated in this case:

Can $\gamma = (\lambda, \mu)$ be complete determined from the knowledge of the Dirichlet-to-Neumann data map: $\Lambda_{\mathbf{c}}$?

The results obtained in [16] can be resumed as follows:

- The Dirichlet-to-Neumann data map:

$$\Lambda : L(\Omega) \ni \gamma \longmapsto \Lambda(\gamma) \in B(H^{\frac{1}{2}}(\partial\Omega, C^n), H^{-\frac{1}{2}}(\partial\Omega, C^n)) \tag{3.2}$$

  is twice continuously differentiable in the sense of Frêchet [6].

- Let $\gamma = (\lambda, \mu) \in L(\Omega)$ be the nonperturbed homogeneous distribution of elastic moduli. The Frêchet derivative:

$$\mathrm{d}\Lambda(\gamma) : L^\infty(\Omega) \times L^\infty(\Omega) \to B(H^{\frac{1}{2}}(\partial\Omega, C^n)) \tag{3.3}$$

is injective and its inverse is computed as follows.

Consider now a small perturbation of the moduli $\delta(\mathbf{x}) = (\delta\lambda(\mathbf{x}), \delta\mu(\mathbf{x})) \in L^\infty(\Omega) \times L^\infty(\Omega)$. For all $\mathbf{m} \in \mathbb{R}^n$, choose a vector $\mathbf{m}^\perp \in \mathbb{R}^n$ satisfying:

$$\begin{aligned} &\mathbf{m} \cdot \mathbf{m}^\perp = 0, \\ &|\mathbf{m}| = |\mathbf{m}^\perp| \end{aligned} \tag{3.4}$$

and define the following complex valued vectors:

$$\begin{aligned} \xi_1 &= \frac{\mathbf{m} + \mathbf{i}\mathbf{m}^\perp}{2}, \\ \xi_2 &= \frac{\mathbf{m} - \mathbf{i}\mathbf{m}^\perp}{2}, \end{aligned} \tag{3.5}$$

We consider now the two families of solutions of the non perturbed problem:

For the first family defined by the displacement field: $\mathbf{u} = \nabla(e^{-\xi_i \mathbf{x}})$, using the notation $\phi_i(\mathbf{x}) = \nabla(e^{-\xi_i \mathbf{x}})|_{\partial\Omega}$. For the trace on the boundary of the displacement we obtain the representation:

$$< \phi_1, \mathrm{d}\Lambda(\gamma)(\delta)\phi_2 >= \frac{|\mathbf{m}|^4}{4} \int_\Omega 2\ \delta\mu(\mathbf{x})\ e^{-\mathbf{i}\mathbf{m}\cdot\mathbf{x}} \mathrm{d}v \tag{3.6}$$

For the second family defined using a Galerkin potential under the form:

$$\begin{aligned} g_{ij} &= -\frac{1}{2}|\xi_i|^{-2}(\mathbf{x}\bar{\xi_i})e^{-\xi_i\cdot\mathbf{x}}\xi_j, \\ \mathbf{u}_{ij} &= (\lambda + 2\mu)\ \Delta g_{ij} - (\lambda + \mu)\ \nabla \cdot (\nabla g_{ij}), \end{aligned} \tag{3.7}$$

and denoting the boundary values by:

$$\begin{aligned} \phi_i &= \mathbf{u}_{ij}|_{\partial\Omega}, \\ \phi_j &= \mathbf{u}_{ji}|_{\partial\Omega}. \end{aligned} \tag{3.8}$$

we obtain the representation:

$$\begin{aligned} &< \phi_1, \mathrm{d}\Lambda(\gamma)(\delta)\phi_2 >= \mu^2 \frac{|\mathbf{m}|^4}{4} \int_\Omega 2\ (\delta\lambda(\mathbf{x}) + \delta\mu(\mathbf{x}))\ e^{-\mathbf{i}\mathbf{m}\mathbf{x}} \mathrm{d}v \\ &+ (\lambda + \mu)^2 \frac{|\mathbf{m}|^4}{8} (\xi_1 \cdot \xi_2) \int_\Omega 2\ \delta\mu(\mathbf{x})\ \mathbf{x}^2\ e^{-\mathbf{i}\mathbf{m}\mathbf{x}} \mathrm{d}v \end{aligned} \tag{3.9}$$

The result implies that the perturbation of the modulus δ is uniquely determined as an inverse Fourier transform of the first order approximation $\Lambda(\gamma) + \mathrm{d}\Lambda(\gamma)(\delta)$ of $\Lambda(\gamma+\delta)$. However, this representation does not directly imply that $\Lambda(\gamma+\delta)$ determines uniquely or not δ. This was obtained using additional lemma's proven by Sylvester and Uhlmann [19].

We can conclude expressing the perturbation of the Lamé moduli in terms of the inverse Fourier transforms of the Betti-Reciprocity computed on the boundary using the two family of solutions and the "measured" Dirichlet-to-Neumann data map:

$$\begin{aligned}
&\mathcal{RB}(\mathbf{u}_1, \mathbf{u}_2; \delta c) = \frac{|\mathbf{m}|^4}{4} \int_\Omega 2\ \delta\mu(\mathbf{x})\ e^{-i(\mathbf{m}^\perp \mathbf{x})} \mathrm{d}v,\\
&\mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{21}; \delta c) = \frac{|\mathbf{m}|^4}{4}\mu^2 \int_\Omega 2\ (\delta\lambda(\mathbf{x}) + \delta\mu(\mathbf{x}))\ e^{-i(\mathbf{m}^\perp \mathbf{x})} \mathrm{d}v\\
&\quad + \frac{|\mathbf{m}|^4}{8}(\lambda+\mu)^2(\xi_1 \cdot \xi_2) \int_\Omega 2\ \delta\mu(\mathbf{x})\ \mathbf{x}^2\ e^{-i(\mathbf{m}^\perp \mathbf{x})} \mathrm{d}v.
\end{aligned} \tag{3.10}$$

4. Anisotropic case

Starting from the preceding technique, Bonnet and Constantinescu [5] proposed the following method to obtain a similar result for different classes of anisotropy.

The consider a particular representation of the auxiliary vectors $\mathbf{m}$ si $\mathbf{m}^\perp$, in an orthogonal coordinate system $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ defined by:

$$\begin{aligned}
\mathbf{m} &= m\ \mathbf{c},\\
\mathbf{m}^\perp &= m\ (\mathbf{a}\ \cos(\nu) + \mathbf{b}\ \sin(\nu)).
\end{aligned} \tag{4.1}$$

As a consequence, the algebraic expansion of the term within the integral in the Betti Reciprocity conducts to:

$$\nabla \mathbf{u}_{\xi_1}(\mathbf{x}) \delta \mathbf{c}(\mathbf{x}) \nabla \mathbf{u}_{\xi_2}(\mathbf{x}) = [A_0 + \sum_{k=1}^{2} (A_k\ \cos(2k\nu) + B_k\ \sin(2k\nu))]\ \delta \mathbf{c}(\mathbf{x}). \tag{4.2}$$

As a consequence, provided the coordinate system $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is chosen appropriately, one can expect to obtain explicit expression for which the Fourier transform can be computed easily.

Using this technique, next we shall present several solutions in particular cases of material anisotropy, especially for cubic anisotropy.

4.1. Case of cubic material symmetry

Let us consider that the tensor of elastic moduli is in the case of cubic material symmetry [11, 12, 13, 14]. Its representation under the Voigt notation is given as:

$$\mathbf{c} = \begin{pmatrix} \lambda+2(\mu-\beta) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2(\mu-\beta) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2(\mu-\beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix},$$

where λ, μ, β are the elastic moduli.

In the following, we shall consider an elastic body $\Omega \in \mathbb{R}^3$. Its moduli are defined as the sum between a pair of homogenous isotropic moduli and a triplet of heterogeneous moduli of cubic material symmetry. Let $\gamma = (\lambda, \mu, \beta) \in L(\Omega)^3$ denote the homogeneous unperturbed moduli and $\delta = (\delta\lambda,\ \delta\mu,\ \delta\beta) \in L^\infty(\Omega)^3$ the perturbed moduli.

As already discussed previously in the general case, one can write the Betti-Reciprocity equality:

$$\mathcal{RB}(\mathbf{u}, \mathbf{u}^*, \delta\mathbf{c}) =< \mathbf{u}|_{\partial\Omega}, \mathrm{d}\Lambda(\gamma)(\delta)\mathbf{u}^*|_{\partial\Omega} > \tag{4.3}$$

where $\mathbf{u} \in H^1(\Omega)$ and $\mathbf{u}^* \in H^1(\Omega)$ are the solutions of the perturbed and the adjoint unpertubed problems and $\Lambda(\gamma)$ is the Dirichlet-to-Neumann data map.

As in the isotropic case, we shall choose families of solutions for the auxiliary problems. The first family is defined as:

$$\begin{aligned} \mathbf{u}_1(\mathbf{x}) &= \nabla e^{-\mathrm{i}\,\xi_1\mathbf{x}}, \\ \mathbf{u}_2(\mathbf{x}) &= \nabla e^{+\mathrm{i}\,\xi_2\mathbf{x}}. \end{aligned} \tag{4.4}$$

The second family is based on the Galerkin representation of the displacements field defined as follows:

$$\mathbf{u}_{jk} = (\lambda+2\mu)\,\Delta g_{jk} - (\lambda+\mu)\,\nabla\cdot(\nabla g_{jk}), \tag{4.5}$$

where

$$g_{jk} = -\frac{1}{2}\,(\mathbf{x}\cdot\bar{\xi_j})\,e^{-\,\xi_j\cdot\mathbf{x}}\xi_k, \tag{4.6}$$

with $j, k = \bar{1,2}$. The functions g_{ij} are harmonic, i.e.

$$\Delta\,\Delta\,g_{jk} = 0\ ,\ \forall j,k \in \bar{1,2}. \tag{4.7}$$

In the preceding computations we have considered that:

$$\xi_1 = \frac{1}{2}(\mathbf{m}+\mathrm{i}\mathbf{m}^\perp), \qquad \xi_2 = \frac{1}{2}(-\mathbf{m}+\mathrm{i}\mathbf{m}^\perp) \tag{4.8}$$

with $\mathbf{m}, \mathbf{m}^\perp \in \mathcal{R}^3$, such that $\mathbf{m}^2 = (\mathbf{m}^\perp)^2$, $\mathbf{m} \cdot \mathbf{m}^\perp = 0$ and $\mathbf{m}^2 = 1$, in order to simplify computations.

Starting with the method proposed in [5] and the preliminary results of expression (4.2), we obtain after a change of frame, from cartesian to spherical coordinates:

$$\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\} \to \{\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi\} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \tag{4.9}$$

and further:

$$\begin{aligned}(m_1, m_2, m_3) &= (\cos\varphi_{\mathbf{m}} \sin\theta_{\mathbf{m}}, \sin\varphi_{\mathbf{m}} \sin\theta_{\mathbf{m}}, \cos\theta_{\mathbf{m}}),\\ (m_1^\perp, m_2^\perp, m_3^\perp) &= (\cos a \cos\varphi_{\mathbf{m}} \cos\theta_{\mathbf{m}} - \sin a \sin\varphi_{\mathbf{m}},\\ &\qquad \cos\varphi_{\mathbf{m}} \sin a + \cos a \cos\theta_{\mathbf{m}} \sin\varphi_{\mathbf{m}}, -\cos a \sin\theta_{\mathbf{m}}).\end{aligned} \tag{4.10}$$

Using the first set of functions for the isotropic case (4.4), together with (4.10) and using an appropriate value of a, the Betti reciprocity equation writes finally in the form:

$$\mathcal{RB}(\mathbf{u}_1, \mathbf{u}_2, \delta\mathbf{c}) = \frac{1}{32} \int_\Omega (7 + \cos 4\varphi_{\mathbf{m}})\delta\mu(\mathbf{x}) + (9 - \cos 4\varphi_{\mathbf{m}})\delta\beta(\mathbf{x}))\ e^{-\mathbf{imx}} \mathrm{d}v \tag{4.11}$$

Similarly, from the second set of functions (4.5) and relation (4.3), one obtains:

$$\begin{aligned}\mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{21}, \delta\mathbf{c}) &= \int_\Omega (\delta\lambda(\mathbf{x}) + \delta\mu(\mathbf{x}))\mathrm{E}_1^2(\lambda, \mu, \mathbf{x}, \xi_\mathbf{1}) e^{-\mathbf{imx}} \mathrm{d}v\\ &+ \int_\Omega (\delta\beta(\mathbf{x}) + \delta\mu(\mathbf{x}))\mathrm{E}_2^2(\lambda, \mu, \mathbf{x}, \xi_\mathbf{1}) e^{-\mathbf{imx}} \mathrm{d}v,\end{aligned} \tag{4.12}$$

and

$$\mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{11}, \delta\mathbf{c}) = \int_\Omega (\delta\beta(\mathbf{x}) - \delta\mu(\mathbf{x}))\mathrm{E}^3(\lambda, \mu, \mathbf{x}, \xi_\mathbf{1}) e^{-\mathbf{m}^\perp\mathbf{x}} e^{-\mathbf{imx}} \mathrm{d}v. \tag{4.13}$$

Here, E_1^2, E_2^2 and E^3 are polinomial expressions in the corresponding variables. Furthermore, under the small strain assumptions we have: $e^{-\mathbf{m}^\perp\mathbf{x}} \cong 1$.

Next we present three particular cases of theses expressions for particular values of $\mathbf{m}$ where computations can be obtained in a closed form.

1. $\mathbf{m}$ and $\mathbf{m}^\perp$ are variable and coplanar,
2. $\mathbf{m}^\perp = \mathbf{e_z}$, $\mathbf{m}$ varies within a plane such that $\mathbf{m}^\perp \perp \mathrm{plan}(\mathbf{m})$,
3. $\mathbf{m} = -\mathbf{e_z}$, $\mathbf{m}^\perp$ varies within a plane such that $\mathbf{m} \perp \mathrm{plan}(\mathbf{m}^\perp)$.

4.1.1. First case of cubic elasticity

The first case assumes coplanar vectors $\mathbf{m}$ and $\mathbf{m}^{\perp}$, i.e. we assume that $\varphi_{\mathbf{m}} = \frac{\pi}{2}$ and $a = 0$. We shall further apply the relations in spherical coordinates (4.10) and

$$\begin{aligned} \mathbf{m} &= (0, \cos\theta_{\mathbf{m}}, \sin\theta_{\mathbf{m}}), \\ \mathbf{m}^{\perp} &= (0, \sin\theta_{\mathbf{m}}, -\cos\theta_{\mathbf{m}}). \end{aligned} \tag{4.14}$$

For the first set of functions (4.4) the reciprocity writes:

$$\mathcal{RB}(\mathbf{u}_1, \mathbf{u}_2, \delta\mathbf{c}) = \frac{1}{4}\int_{\Omega} (\delta\mu(\mathbf{x}) + \delta\beta(\mathbf{x}))e^{-\mathrm{i}\mathbf{m}\mathbf{x}}\mathrm{d}v, \tag{4.15}$$

while for the second set of functions (4.5) we obtain:

$$\begin{aligned} \mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{21}, \delta\mathbf{c}) &= -\frac{1}{4}\mu^2 \int_{\Omega} (\delta\lambda(\mathbf{x}) + \delta\mu(\mathbf{x}))e^{-\mathrm{i}\mathbf{m}\mathbf{x}}\mathrm{d}v \\ &+ \frac{1}{64}(\lambda+\mu)^2 \int_{\Omega} (\delta\beta(\mathbf{x}) + \delta\mu(\mathbf{x}))(y^2+z^2)e^{-\mathrm{i}\mathbf{m}\mathbf{x}}\mathrm{d}v \end{aligned} \tag{4.16}$$

and

$$\mathcal{RB}(\mathbf{u}_{11}, \mathbf{u}_{12}, \delta\mathbf{c}) = \frac{1}{32}(\lambda+3\mu)(\lambda+\mu)e^{-3\mathrm{i}\theta_{\mathbf{m}}}\int_{\Omega}(\delta\beta(\mathbf{x}) - \delta\mu(\mathbf{x}))(-\mathrm{i}y+z)e^{-\mathrm{i}\mathbf{m}\mathbf{x}}\mathrm{d}v. \tag{4.17}$$

As a consequence, using similar arguments as in the isotropic case, we have been able to reduce the identification of the perturbed moduli $(\delta\lambda, \delta\mu, \delta\beta)$ to the solution of an algebraic system of equations and an inverse Fourier transform.

Remarks

Computations performed with different starting planes: $\mathbf{m} = (0, m2, m3)$, $\mathbf{m} = (m1, 0, m3)$, $\mathbf{m} = (m1, m2, 0)$ with coplanar $\mathbf{m}^{\perp}$ and $\mathbf{m}$, or linear combinations of these vectors, implying the two families of functions proposed in the isotropic case have conducted to different Fourier representations without providing a general relation between these representations.

As a consequence of cubic anisotropy we do not have a result combining the 2-dimensional formulae into a complet coherent 3-dimensional one.

If we further reduce the dimension of the problem, to a 1-dimensional one, where $\mathbf{m} = (0, 0, m)$, then the equations (4.15), (4.16) and (4.17), are transformed into a nonlinear system of integral-differential equations.

Using the simplifying notations:

$$\begin{aligned} \mathcal{RB}_1(m) &= \mathcal{RB}(\mathbf{u}_1, \mathbf{u}_2, \delta\mathbf{c}) \\ \mathcal{RB}_2(m) &= \mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{21}, \delta\mathbf{c}) \\ \mathcal{RB}_3(m) &= \mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{11}, \delta\mathbf{c}) \end{aligned} \tag{4.18}$$

the system (4.15), (4.16) and (4.17) is transformed into:

$$\begin{aligned} \widehat{\delta\mu}(m) + \widehat{\delta\beta}(m) &= \frac{1}{4}\,\mathcal{RB}_1(m), \\ \widehat{\delta\lambda}(m) + \widehat{\delta\mu}(m) &= C_1\,\mathcal{RB}_2(m) + C_2\,\frac{\mathrm{d}^2\mathcal{RB}_1(m)}{\mathrm{d}m^2}, \\ \widehat{\delta\mu}(m) - \widehat{\delta\beta}(m) &= C_3 \int \mathcal{RB}_3(m)\,\mathrm{d}m, \end{aligned} \tag{4.19}$$

where C_1, C_2 şi C_3 multiplicative real constants. Here $\widehat{\delta\mu}(m)$, $\widehat{\delta\lambda}(m)$ and $\widehat{\delta\beta}(m)$ denote the Fourier transforms of the perturbed moduli: $\delta\mu(m)$, $\delta\lambda(m)$ şi $\delta\beta(m)$.

Finally the expressions of the Fourier transform are:

$$\begin{aligned} \widehat{\delta\mu}(m) &= \frac{1}{8}\,\mathcal{RB}_1(m) + \frac{C_3}{2}\int \mathcal{RB}_3(m)\,\mathrm{d}m, \\ \widehat{\delta\lambda}(m) &= C_1\,\mathcal{RB}_2(m) + C_2\,\frac{\mathrm{d}^2\mathcal{RB}_1(m)}{\mathrm{d}m^2} - \frac{1}{8}\,\mathcal{RB}_1(m) - \frac{C_3}{2}\int \mathcal{RB}_3(m)\,\mathrm{d}m, \\ \widehat{\delta\beta}(m) &= \frac{1}{8}\,\mathcal{RB}_1(m) - \frac{C_3}{2}\int \mathcal{RB}_3(m)\,\mathrm{d}m. \end{aligned} \tag{4.20}$$

4.1.2. Second case of cubic elasticity

For the second state, we consider $\theta_{\mathbf{m}} = \frac{\pi}{2}$ şi $a = 0$, i.e. we chose $\mathbf{m}^{\perp} = -\mathbf{e_z}$, $\mathbf{m}$ variable within a plane such that $\mathbf{m}^{\perp} \perp \mathrm{plan}(\mathbf{m})$.

As before, we shall now use the transformations (4.10), which leads to:

$$\begin{aligned} \mathbf{m} &= (\cos\varphi_{\mathbf{m}}, \sin\varphi_{\mathbf{m}}, 0), \\ \mathbf{m}^{\perp} &= (0, 0, -1). \end{aligned} \tag{4.21}$$

The first set of functions (4.4) conducts to the following expressions of the Betti reciprocity:

$$\mathcal{RB}(\mathbf{u}_1, \mathbf{u}_2, \delta\mathbf{c}) = \frac{1}{32}\int_{\Omega} (7 + \cos 4\varphi_{\mathbf{m}})\delta\mu(\mathbf{x}) + (9 - \cos 4\varphi_{\mathbf{m}})\delta\beta(\mathbf{x}))\; e^{-\mathbf{imx}}\mathrm{d}v \tag{4.22}$$

wether the second set of functions (4.5) leads to:

$$\begin{aligned} \mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{21}, \delta\mathbf{c}) = &\int_{\Omega} (\delta\lambda(\mathbf{x}) + \delta\mu(\mathbf{x}))\mathrm{E}_1^2(\lambda, \mu, \mathbf{x}, \xi_{\mathbf{1}})|_{\theta_{\mathbf{m}}=\frac{\pi}{2}}\, e^{-\mathbf{im}^{\perp}\mathbf{x}}\mathrm{d}v \\ &+ \int_{\Omega} (\delta\beta(\mathbf{x}) + \delta\mu(\mathbf{x}))\mathrm{E}_2^2(\lambda, \mu, \mathbf{x}, \xi_{\mathbf{1}})|_{\theta_{\mathbf{m}}=\frac{\pi}{2}}\, e^{-\mathbf{im}^{\perp}\mathbf{x}}\mathrm{d}v \end{aligned} \tag{4.23}$$

and

$$\mathcal{RB}(\mathbf{u}_{11},\mathbf{u}_{12},\delta\mathbf{c}) = \frac{1}{256}(\lambda+3\mu)\int_{\Omega}(\delta\beta(\mathbf{x})-\delta\mu(\mathbf{x}))\,[(\lambda+\mu)(7+\cos 4\varphi_{\mathbf{m}}) \cdot (\mathrm{i}z+\mathbf{mx}) - 8\mu\sin(2\varphi_{\mathbf{m}})^2]\, e^{-\mathrm{i}\mathbf{m}^{\perp}\mathbf{x}}\mathrm{d}v. \tag{4.24}$$

4.1.3. Third case of cubic elasticity

In the third case, we consider: $\theta_{\mathbf{m}} = 0$ and $a = 0$. Our choice is then of variable vectors in plane $\mathbf{m} = m\mathbf{e_z}$, such that: $\mathbf{m}^{\perp}$ $\mathbf{m} \perp \mathrm{plan}(\mathbf{m}^{\perp})$.

As before, we shall now use the transformations (4.10), which leads to:

$$\begin{aligned}\mathbf{m} &= (0,0,1),\\ \mathbf{m}^{\perp} &= (\cos\varphi_{\mathbf{m}}, \sin\varphi_{\mathbf{m}}, 0).\end{aligned} \tag{4.25}$$

The first set of functions (4.4) conducts to the following expressions of the Betti reciprocity:

$$\mathcal{RB}(\mathbf{u}_1,\mathbf{u}_2,\delta\mathbf{c}) = \frac{1}{32}\int_{\Omega}(7+\cos 4\varphi_{\mathbf{m}})\delta\mu(\mathbf{x}) + (9-\cos 4\varphi_{\mathbf{m}})\delta\beta(\mathbf{x}))\, e^{-\mathrm{i}\mathbf{mx}}\mathrm{d}v \tag{4.26}$$

wether the second set of functions (4.5) conducts to:

$$\begin{aligned}\mathcal{RB}(\mathbf{u}_{12},\mathbf{u}_{21},\delta\mathbf{c}) &= \int_{\Omega}(\delta\lambda(\mathbf{x})+\delta\mu(\mathbf{x}))\mathrm{E}_1^2(\lambda,\mu,\mathbf{x},\xi_{\mathbf{1}})|_{\theta_{\mathbf{m}}=0}e^{-\mathrm{i}\mathbf{m}^{\perp}\mathbf{x}}\mathrm{d}v\\ &+ \int_{\Omega}(\delta\beta(\mathbf{x})+\delta\mu(\mathbf{x}))\mathrm{E}_2^2(\lambda,\mu,\mathbf{x},\xi_{\mathbf{1}})|_{\theta_{\mathbf{m}}=0}e^{-\mathrm{i}\mathbf{m}^{\perp}\mathbf{x}}\mathrm{d}v\end{aligned} \tag{4.27}$$

and

$$\mathcal{RB}(\mathbf{u}_{11},\mathbf{u}_{12},\delta\mathbf{c}) = \frac{1}{256}(\lambda+3\mu)\int_{\Omega}(\delta\beta(\mathbf{x})-\delta\mu(\mathbf{x}))\,[(\lambda+\mu)(7+\cos 4\varphi_{\mathbf{m}}) \cdot (z-\mathrm{i}\mathbf{m}^{\perp}\mathbf{x}) - 8\mu\sin(2\varphi_{\mathbf{m}})^2]\, e^{-\mathrm{i}\mathbf{m}^{\perp}x}\mathrm{d}v. \tag{4.28}$$

4.2. Orthotropic material symmetry

We shall now proceed with similar computations in the case of cubic symmetry. For this material symmetry expressions and the Fourier transform get more complex as before.

Let us consider an tensor of elastic moduli in the case of a particular case of orthotropic material symmetry [11, 12, 13, 14], where the representation under the Voigt notation is given as (see also [18]):

$$\mathbf{c} = \begin{pmatrix} c_{11} & \nu\sqrt{c_{11}c_{22}} & \nu\sqrt{c_{11}c_{33}} & 0 & 0 & 0 \\ \nu\sqrt{c_{11}c_{22}} & c_{22} & \nu\sqrt{c_{22}c_{33}} & 0 & 0 & 0 \\ \nu\sqrt{c_{11}c_{33}} & \nu\sqrt{c_{22}c_{33}} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2}\sqrt{c_{22}c_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2}\sqrt{c_{11}c_{33}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-\nu}{2}\sqrt{c_{11}c_{22}} \end{pmatrix},$$

There are 4 moduli in this representation: c_{11}, c_{22}, c_{33} and ν, instead of the standard 6 moduli of orthotropic materials. The mechanical motivation for this class is to be found in [18], were a direct transformation is defined between this class and general isotropy. This remark opens the question of the possibility to transpose directly the isotropic proof of Ikehata [16] in the case of materials with this class of perturbations.

The computations lead in a similar way as before, to the following expressions of the Betti reciprocity:

$$\mathcal{RB}(\mathbf{u}_1, \mathbf{u}_2; \delta\mathbf{c}) = \frac{1}{16}\int_\Omega (\delta c_{22} + \delta c_{33} + 2\sqrt{\delta c_{22}\delta\ c_{33}}(1 - 2\delta N))e^{-\mathrm{i}(\mathbf{m}^\perp \mathbf{x})}\mathrm{d}v. \tag{4.29}$$

and:

$$\begin{aligned} \mathcal{RB}(\mathbf{u}_{12}, \mathbf{u}_{21}; \delta\mathbf{c}) &= -\frac{1}{16}\mu^2\int_\Omega (\delta c_{22} + \delta c_{33})e^{-\mathrm{i}(\mathbf{m}^\perp \mathbf{x})}\mathrm{d}v \\ &+ \frac{1}{256}(\lambda+\mu)^2\int_\Omega (\mathbf{x}\cdot\mathbf{x})(\sqrt{\delta c_{22}} + \sqrt{\delta c_{33}})^2 e^{-\mathrm{i}(\mathbf{m}^\perp \mathbf{x})}\mathrm{d}v \\ &+ \frac{1}{128}\int_\Omega [-8\mu^2 - (\lambda+\mu)^2(\mathbf{x}\cdot\mathbf{x})]\delta N\sqrt{\delta c_{22}\delta c_{33}}e^{-\mathrm{i}(\mathbf{m}^\perp \mathbf{x})}\mathrm{d}v \\ &+ \frac{1}{32}\mu(\lambda+\mu)\int_\Omega \mathrm{i}\ (y\cos\theta + z\sin\theta)(\delta c_{22} - \delta c_{33})e^{-\mathrm{i}(\mathbf{m}^\perp \mathbf{x})}\mathrm{d}v \end{aligned} \tag{4.30}$$

Remarks One can observe that the system is still incomplete, as one will still need at least two equations, which can however be obtained from the same sets of functions using particular forms. Nonetheless, it is straightforward that the complexity of the system was increased in a non-trivial manner.

Let us also notice that the 1-dimensional technique could be applied in a similar way in this case.

5. Conclusion and Perspectives

The present paper discussed some possible extension of the Calderon method to show that an anisotropic perturbation of a isotropic homogeneous field of elastic moduli can be recovered by means of the Betti-Reciprocity principle from the Dirichlet-to-Neumann boundary data map. The method exposed uses as adjoint (auxiliary function) the families already proposed by Ikehata in [16].

The presented results cover some special case of cubic and orthotropic material symmetry. However, the complete problem of anisotropic material symmetry remains still open.

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References

[1] S. Andrieux and A. Ben Abda, Identification of planar cracks by complete over-determined data: inversion formula, *Inverse Problems*, **12**, *3* (1996), 553-563, http://dx.doi.org/10.1088/0266-5611/12/5/002.

[2] S. Andrieux, A. Ben Abda and H.D. Bui, Reciprocity principle and crack identification, *Inverse Problems*, **15** (1999), 59-65, http://dx.doi.org/10.1088/0266-5611/15/1/010.

[3] T. Bannour, A. Ben Abda and M. Jaoua, A semi-explicit algorithm for the reconstruction of 3d planar cracks, *Inverse Problems*, **13**, *4* (1997), 899-917, http://dx.doi.org/10.1088/0266-5611/13/4/002.

[4] A. Ben Abda, Hend Ben Ameur and Mohamed Jaoua, Identification of 2D cracks by elastic boundary measurements, *Inverse Problems*, **15** (1999), 67-77, http://dx.doi.org/10.1088/0266-5611/15/1/011.

[5] M. Bonnet and A. Constantinescu, Inverse problems in elasticity, *Inverse Problems*, **21** (2005), R1-R50, http://dx.doi.org/10.1088/0266-5611/21/2/R01.

[6] H. Brezis, *Analyse functionnelle*, Masson, Paris, 1983.

[7] H.D. Bui, A. Constantinescu and H. Maigre, Difraction acoustique inverse de fissure plane: solution explicite pour un solide borné, *C. R. Math. Acad. Sci. Paris, série II*, **327** (1999), 971-976, http://dx.doi.org/10.1016/s1287-4620(00)87006-4.

[8] A. Calderon, On an inverse boundary value problem, in: Rio de Janeiro Soc. Brasiliera de Matématica (Ed.), *Seminar on Numerical analysis and its Application to Continuum Physics*, pp. 65-73, 1980.

[9] A. Constantinescu, *On the identification of elastic moduli*, PhD Thesis, Ecole Polytechnique, 1994.

[10] A. Constantinescu, On the identification of elastic moduli from displacement-force boundary measurements, *Int. J. Inverse Probl. Eng.*, **1** (1995), 293-315, http://dx.doi.org/10.1080/174159795088027587.

[11] A. Constantinescu and A.M. Korsunsky, *Elasticity with Mathematica*, Cambridge University Press, 2008, http://dx.doi.org/10.1017/cbo9780511546167.

[12] S. C. Cowin and M. Mehrabadi, Eigentensors of linear elastic materials, *Quart. J. Mech. Appl. Math.*, **43** (1990), 15-41, http://dx.doi.org/10.1093/qjmam/43.1.15.

[13] S. C. Cowin and M. Mehrabadi, The Structure of the linear anisotropic symmetries, *J. Mech. Phys. Solids*, **40**, *7* (1992), 1459-1471, http://dx.doi.org/10.1016/0022-5096(92)90029-2.

[14] N.D. Cristescu, E.M. Craciun and E. Soós, *Mechanics of elastic composites*, CRC Press, Boca Raton, 2003, http://dx.doi.org/10.1201/9780203502815.

[15] M.E. Gurtin, The linear theory of elasticity, in *Mechanics of Solids*, Vol. 2, Springer, 1973, http://dx.doi.org/10.1007/978-3-662-39776-3_1.

[16] M. Ikehata, Inversion formulas for the linearized problem for an inverse boundary value problem in elastic prospection, *SIAM J. Math. Anal.*, **50** (1990), 1635-1644, http://dx.doi.org/10.1137/0150097.

[17] M. Ikehata, An inverse problem for the plate in the Love Kirchhoff theory, *SIAM J. Appl. Math.*, **53** (1993), 942-970, http://dx.doi.org/10.1137/0153047.

[18] A. Pouya and A. Zaoui, A Transformation of Elastic Boundary Value Problems with Application to Anisotropic Behaviour, *Int. J. Solids Struct.*, **43** (2006), 4937-4956, http://dx.doi.org/10.1016/j.ijsolstr.2005.06.046.

[19] J. Sylvester and G. Uhlman, A global uniqueness theorem for an inverse boundary diffusion equation, *Ann. of Math.*, **125** (1987), 153-169, http://dx.doi.org/10.2307/1971291.

[20] J. Sylvester and G. Uhlman, A uniqueness theorem for an inverse boundary value problem in electrical prospection, *Comm. Pure Appl. Math.*, **39** (1986), 91-112, http://dx.doi.org/10.1002/cpa.3160390106.

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