Annals of the University of Bucharest (mathematical series)
5 (LXIII) (2014), 279–297

A variational analysis of a class of dynamic problems with slip dependent friction

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Dedicated to Professor Nicolaie Cristescu on the occasion of his eighty fifth birthday

Abstract - This work is concerned with the extension of some recent existence results proved for a class of nonsmooth dynamic frictional contact problem, to the case of a coefficient of friction depending on the slip velocity. Based on existence and approximation results for some general implicit variational inequalities, which are established by using Ky Fan's fixed point theorem, several estimates and compactness arguments, relaxed unilateral conditions with slip dependent friction between two viscoelastic bodies of Kelvin-Voigt type are analyzed.

Key words and phrases : Implicit inequalities, set-valued mapping, dynamic problems, pointwise conditions, Coulomb friction.

Mathematics Subject Classification (2010) **:** 35Q74, 49J40, 74A55, 74D05, 74H20.

1. Introduction

This paper deals with the analysis of a dynamic contact problem with some relaxed unilateral contact conditions, adhesion, and slip dependent pointwise friction, between two Kelvin-Voigt viscoelastic bodies.

The quasistatic unilateral contact problems with local Coulomb friction have been studied in [1, 29, 30] and adhesion laws based on the evolution of intensity of adhesion were investigated in [28, 10]. Also, the normal compliance model, which can be seen as a particular regularization of the Signorini's conditions, has been considered by several authors, see e.g. [18, 16, 31] and references therein.

A recent unified approach including unilateral and bilateral contact with nonlocal friction, or normal compliance conditions, in the quasistatic case and for a nonlinear elastic behavior, has been proposed in [2].

In the dynamic case, viscoelastic contact problems with nonlocal friction laws were considered in [17, 20, 21, 14, 6, 11] and the corresponding problems with normal compliance laws have been analyzed in [23, 18, 19, 5, 24]. Dynamic frictionless problems with adhesion have been studied by several authors, see, e.g [33] and references therein, and dynamic viscoelastic problems coupling unilateral contact, recoverable adhesion and nonlocal friction have been investigated in [12, 8]. Using the Clarke subdifferential, various variational contact problems can be analyzed by using the theory of hemivariational inequalities, which represent a broad generalization of the variational inequalities to locally Lipschitz functions, see [24, 25, 26] and references therein.

Based on new abstract formulations and on Ky Fan's fixed point theorem, a static contact problem with relaxed unilateral conditions and pointwise Coulomb friction was studied in [27]. The extension of this interesting approach to an elastic quasistatic contact problem was considered in [7] and to a dynamic viscoelastic contact problem with slip independent coefficient of friction was investigated in [9].

This paper extends the results presented in [9] to the case of a coefficient of friction depending on the slip velocity, which enables to treat more realistic situations.

The paper is organized as follows. In Section 2 the classical formulation of the dynamic contact problem is presented. In Section 3 two variational formulations are given as a two-field problem. In Section 4 a more general evolution implicit variational inequality is considered and some auxiliary results are proved. Section 5 is devoted to the study of a fixed point problem, which is equivalent to the previous variational inequality. Using the Ky Fan's theorem, the existence of a fixed point is proved. In Section 6 this abstract result is used to prove the existence of a variational solution of the dynamic contact problem with slip dependent friction.

The applications presented in this paper concern the contact between two linear viscoelastic bodies but these results can be extended to more general constitutive laws, as, for example, the ones characterizing some elastoviscoplastic materials investigated in [13].

2. Classical formulation

Let Ω^α be the reference domains of $\mathbb{R}^d$, $d = 2$ or 3, occupied by two viscoelastic bodies, characterized by a Kelvin-Voigt constitutive law. Suppose that the bodies have Lipschitz boundaries $\Gamma^\alpha = \partial\Omega^\alpha$, $\alpha = 1, 2$.

Let Γ_U^α, Γ_F^α and Γ_C^α be three open disjoint sufficiently smooth parts of Γ^α such that $\Gamma^\alpha = \overline{\Gamma}_U^\alpha \cup \overline{\Gamma}_F^\alpha \cup \overline{\Gamma}_C^\alpha$ and, to simplify the estimates, $\text{meas}(\Gamma_U^\alpha) > 0$, $\alpha = 1, 2$. We shall assume the small deformation hypothesis and we shall use Cartesian coordinate representations.

Let $\boldsymbol{y}^\alpha(\boldsymbol{x}^\alpha, t)$ denote the position at time $t \in [0, T]$, where $0 < T < +\infty$, of the material point represented by $\boldsymbol{x}^\alpha$ in the reference configuration, and $\boldsymbol{u}^\alpha(\boldsymbol{x}^\alpha, t) := \boldsymbol{y}^\alpha(\boldsymbol{x}^\alpha, t) - \boldsymbol{x}^\alpha$ denote the displacement vector of $\boldsymbol{x}^\alpha$ at time t, with the Cartesian coordinates $u^\alpha = (u_1^\alpha, ..., u_d^\alpha) = (\bar{u}^\alpha, u_d^\alpha)$. Let $\boldsymbol{\varepsilon}^\alpha$, with the Cartesian coordinates $\varepsilon^\alpha = (\varepsilon_{ij}(u^\alpha))$, and $\boldsymbol{\sigma}^\alpha$, with the Cartesian coordinates $\sigma^\alpha = (\sigma_{ij}^\alpha)$, be the infinitesimal strain tensor and the stress tensor, respectively, corresponding to Ω^α, $\alpha = 1, 2$. The usual summation convention will be used for $i, j, k, l = 1, \ldots, d$.

Assume that the displacement $\boldsymbol{U}^\alpha = \boldsymbol{0}$ on $\Gamma_U^\alpha \times (0, T)$, $\alpha = 1, 2$, and that the densities of both bodies are equal to 1. Let $\boldsymbol{f}_1 = (\boldsymbol{f}_1^1, \boldsymbol{f}_1^2)$ denote the given body forces in $\Omega^1 \cup \Omega^2$ and $\boldsymbol{f}_2 = (\boldsymbol{f}_2^1, \boldsymbol{f}_2^2)$ denote the tractions on $\Gamma_F^1 \cup \Gamma_F^2$. The initial displacements and velocities of the bodies are denoted by $\boldsymbol{u}_0 = (\boldsymbol{u}_0^1, \boldsymbol{u}_0^2)$, $\boldsymbol{u}_1 = (\boldsymbol{u}_1^1, \boldsymbol{u}_1^2)$.

Suppose that the solids can be in contact between the potential contact surfaces Γ_C^1 and Γ_C^2 which are parametrized by two C^1 functions, φ^1, φ^2, defined on an open and bounded subset Ξ of $\mathbb{R}^{d-1}$, such that $\varphi^1(\xi) - \varphi^2(\xi) \geq 0 \ \forall \xi \in \Xi$ and each Γ_C^α is the graph of φ^α on Ξ that is $\Gamma_C^\alpha = \{ (\xi, \varphi^\alpha(\xi)) \in \mathbb{R}^d ; \xi \in \Xi\}$, $\alpha = 1, 2$. Define the initial normalized gap between the two contact surfaces by

$$g_0(\xi) = \frac{\varphi^1(\xi) - \varphi^2(\xi)}{\sqrt{1 + |\nabla\varphi^1(\xi)|^2}} \quad \forall \xi \in \Xi$$

and suppose that this initial gap is sufficiently small. Let $\boldsymbol{n}^\alpha$ denote the unit outward normal vector to Γ^α, $\alpha = 1, 2$. We shall use the following notations for the normal and tangential components of a displacement field $\boldsymbol{v}^\alpha$, $\alpha = 1, 2$, of the relative displacement corresponding to $\boldsymbol{v} := (\boldsymbol{v}^1, \boldsymbol{v}^2)$ and of the stress vector $\boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha$ on Γ_C^α:

$$\begin{aligned}
&\boldsymbol{v}^\alpha(\xi, t) := \boldsymbol{v}^\alpha(\xi, \varphi^\alpha(\xi), t),\ v_N^\alpha(\xi, t) := \boldsymbol{v}^\alpha(\xi, t) \cdot \boldsymbol{n}^\alpha(\xi),\\
&v_N(\xi, t) := v_N^1(\xi, t) + v_N^2(\xi, t),\ [v_N](\xi, t) := v_N(\xi, t) - g_0(\xi),\\
&\boldsymbol{v}_T^\alpha(\xi, t) := \boldsymbol{v}^\alpha(\xi, t) - v_N^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi),\ \boldsymbol{v}_T(\xi, t) := \boldsymbol{v}_T^1(\xi, t) - \boldsymbol{v}_T^2(\xi, t),\\
&\sigma_N^\alpha(\xi, t) := (\boldsymbol{\sigma}^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi)) \cdot \boldsymbol{n}^\alpha(\xi),\\
&\boldsymbol{\sigma}_T^\alpha(\xi, t) = \boldsymbol{\sigma}^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi) - \sigma_N^\alpha(\xi, t)\boldsymbol{n}^\alpha(\xi),
\end{aligned}$$

for all $\xi \in \Xi$ and for all $t \in [0, T]$. Let $g := -[u_N] = g_0 - u_N^1 - u_N^2$ be the gap corresponding to the solution $\boldsymbol{u} := (\boldsymbol{u}^1, \boldsymbol{u}^2)$. Using a similar method as the one presented in [3] (see also [11], [8]) we obtain the following unilateral contact condition at time t in the set Ξ: $[u_N](\xi, t) = -g(\xi, t) \leq 0 \ \forall \xi \in \Xi$. Let $\boldsymbol{\mathcal{A}}^\alpha$, $\boldsymbol{\mathcal{B}}^\alpha$ denote two fourth-order tensors, the elasticity tensor and the viscosity tensor corresponding to Ω^α, with the components $\mathcal{A}^\alpha = (\mathcal{A}_{ijkl}^\alpha)$ and $\mathcal{B}^\alpha = (\mathcal{B}_{ijkl}^\alpha)$, respectively. Assume that these components satisfy the following classical symmetry and ellipticity conditions: $\mathcal{C}_{ijkl}^\alpha = \mathcal{C}_{jikl}^\alpha = \mathcal{C}_{klij}^\alpha \in L^\infty(\Omega^\alpha)$, $\forall i, j, k, l = 1, \ldots, d$, $\exists \alpha_{\mathcal{C}^\alpha} > 0$ such that $\mathcal{C}_{ijkl}^\alpha \tau_{ij}\tau_{kl} \geq \alpha_{\mathcal{C}^\alpha} \tau_{ij}\tau_{ij}$ $\forall \tau = (\tau_{ij})$ verifying $\tau_{ij} = \tau_{ji}, \forall i, j = 1, \ldots, d$, where $\mathcal{C}_{ijkl}^\alpha = \mathcal{A}_{ijkl}^\alpha$, $\mathcal{C}^\alpha = \mathcal{A}^\alpha$ or $\mathcal{C}_{ijkl}^\alpha = \mathcal{B}_{ijkl}^\alpha$, $\mathcal{C}^\alpha = \mathcal{B}^\alpha$ $\forall i, j, k, l = 1, \ldots, d$, $\alpha = 1, 2$.

Let $\mu = \mu(\xi, \dot{\boldsymbol{u}}_T)$ be the slip rate dependent coefficient of friction and assume that $\mu : \Xi \times \mathbb{R}^d \to \mathbb{R}_+$ is a bounded function such that for a.e.

$\xi \in \Xi$ $\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant, denoted by C_μ, independent of ξ, and for every $v \in \mathbb{R}^d$ $\mu(\cdot, v)$ is measurable.

Let $\underline{\kappa}$, $\overline{\kappa} : \mathbb{R} \to \mathbb{R}$ be two mappings with $\underline{\kappa}$ lower semicontinuous and $\overline{\kappa}$ upper semicontinuous, satisfying the following conditions:

$$\underline{\kappa}(s) \leq \overline{\kappa}(s) \text{ and } 0 \notin (\underline{\kappa}(s), \overline{\kappa}(s)) \ \ \forall\, s \in \mathbb{R}, \tag{2.1}$$

$$\exists\, r_0 \geq 0 \text{ such that } \max(|\underline{\kappa}(s)|, |\overline{\kappa}(s)|) \leq r_0 \ \ \forall\, s \in \mathbb{R}. \tag{2.2}$$

We consider the following dynamic viscoelastic contact problem.

Problem $\boldsymbol{P_c}$: Find $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$ and, for all $t \in (0, T)$,

$$\ddot{\boldsymbol{u}}^\alpha - \operatorname{div} \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{f}_1^\alpha \ \text{ in } \ \Omega^\alpha, \tag{2.3}$$

$$\boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{\mathcal{A}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + \boldsymbol{\mathcal{B}}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) \ \text{ in } \ \Omega^\alpha, \tag{2.4}$$

$$\boldsymbol{u}^\alpha = \boldsymbol{0} \ \text{ on } \ \Gamma_U^\alpha, \ \boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha = \boldsymbol{f}_2^\alpha \ \text{ on } \ \Gamma_F^\alpha, \ \alpha = 1, 2, \tag{2.5}$$

$$\boldsymbol{\sigma}^1 \boldsymbol{n}^1 + \boldsymbol{\sigma}^2 \boldsymbol{n}^2 = \boldsymbol{0} \ \text{ in } \ \Xi, \tag{2.6}$$

$$\underline{\kappa}([u_N]) \leq \sigma_N \leq \overline{\kappa}([u_N]) \ \text{ in } \ \Xi, \tag{2.7}$$

$$|\boldsymbol{\sigma}_T| \leq \mu(\dot{\boldsymbol{u}}_T)\, |\sigma_N| \ \text{ in } \ \Xi \ \text{ and } \tag{2.8}$$

$$\begin{aligned} &|\boldsymbol{\sigma}_T| < \mu(\dot{\boldsymbol{u}}_T)\, |\sigma_N| \Rightarrow \dot{\boldsymbol{u}}_T = \boldsymbol{0}, \\ &|\boldsymbol{\sigma}_T| = \mu(\dot{\boldsymbol{u}}_T)\, |\sigma_N| \ \Rightarrow \exists\, \vartheta \geq 0, \ \dot{\boldsymbol{u}}_T = -\vartheta \boldsymbol{\sigma}_T, \end{aligned}$$

where $\boldsymbol{\sigma}^\alpha = \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha)$, $\alpha = 1, 2$, $\sigma_N := \sigma_N^1$ and $\boldsymbol{\sigma}_T := \boldsymbol{\sigma}_T^1$.

Some contact and friction conditions, corresponding to particular $\underline{\kappa}$ and $\overline{\kappa}$, with a general coefficient of friction, are presented in the following examples.

Example 1. (Adhesion and friction conditions) Let $s_0 \geq 0$, $M \geq 0$ be constants, $k : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $k \geq 0$ with $k(0) = 0$ and define

$$\underline{\kappa}(s) = \begin{cases} 0 & \text{if } \ s \leq -s_0, \\ k(s) & \text{if } \ -s_0 < s < 0, \\ -M & \text{if } \ s \geq 0, \end{cases} \quad \overline{\kappa}(s) = \begin{cases} 0 & \text{if } \ s < -s_0, \\ k(s) & \text{if } \ -s_0 \leq s \leq 0, \\ -M & \text{if } \ s > 0. \end{cases}$$

Example 2. (Friction condition)

In Example 1 we set $k = s_0 = 0$ and define

$$\underline{\kappa}_M(s) = \begin{cases} 0 & \text{if } \ s < 0, \\ -M & \text{if } \ s \geq 0, \end{cases} \quad \overline{\kappa}_M(s) = \begin{cases} 0 & \text{if } \ s \leq 0, \\ -M & \text{if } \ s > 0. \end{cases}$$

The classical Signorini's conditions correspond formally to $M = +\infty$.

Example 3. (General normal compliance conditions)

Various normal compliance conditions, friction and adhesion laws can be obtained from the previous general formulation if one considers $\underline{\kappa} = \overline{\kappa} = \kappa$, where $\kappa : \mathbb{R} \to \mathbb{R}$ is some bounded Lipschitz continuous function with $\kappa(0) = 0$, so that σ_N is given by the relation $\sigma_N = \kappa([u_N])$, see e.g. [8], where the intensity of adhesion was also considered.

3. Variational formulations

We shall consider two different variational formulations of problem P_c. We adopt the following notations:

$$\begin{aligned}
&\boldsymbol{H}^s(\Omega^\alpha) := H^s(\Omega^\alpha;\mathbb{R}^d),\ \alpha=1,2,\ \boldsymbol{H}^s := \boldsymbol{H}^s(\Omega^1)\times\boldsymbol{H}^s(\Omega^2),\\
&\langle \boldsymbol{v},\boldsymbol{w}\rangle_{-s,s} = \langle \boldsymbol{v}^1,\boldsymbol{w}^1\rangle_{\boldsymbol{H}^{-s}(\Omega^1)\times\boldsymbol{H}^s(\Omega^1)} + \langle \boldsymbol{v}^2,\boldsymbol{w}^2\rangle_{\boldsymbol{H}^{-s}(\Omega^2)\times\boldsymbol{H}^s(\Omega^2)}\\
&\forall\, \boldsymbol{v}=(\boldsymbol{v}^1,\boldsymbol{v}^2)\in\boldsymbol{H}^{-s},\ \forall\,\boldsymbol{w}=(\boldsymbol{w}^1,\boldsymbol{w}^2)\in\boldsymbol{H}^s,\quad \forall\, s\in\mathbb{R}.
\end{aligned}$$

Define the Hilbert spaces $(\boldsymbol{H},\ |.|)$ with the associated inner product denoted by $(.\,,.)$, $(\boldsymbol{V},\ \|.\|)$ with the associated inner product (of $\boldsymbol{H}^1$) denoted by $\langle .\,,.\rangle$, and the closed convex cones $L^2_+(\Xi)$, $L^2_+(\Xi\times(0,T))$ as follows:

$$\begin{aligned}
&\boldsymbol{H} := \boldsymbol{H}^0 = L^2(\Omega^1;\mathbb{R}^d)\times L^2(\Omega^2;\mathbb{R}^d),\ \boldsymbol{V} := \boldsymbol{V}^1\times\boldsymbol{V}^2,\ \text{ where}\\
&\boldsymbol{V}^\alpha = \{\boldsymbol{v}^\alpha\in\boldsymbol{H}^1(\Omega^\alpha);\ \boldsymbol{v}^\alpha=\boldsymbol{0}\ \text{ a.e. on } \Gamma_U^\alpha\},\ \alpha=1,2,\\
&L^2_+(\Xi) := \{\delta\in L^2(\Xi);\ \delta\ge 0\ \text{ a.e. in }\ \Xi\},\\
&L^2_+(\Xi\times(0,T)) := \{\eta\in L^2(0,T;L^2(\Xi));\ \eta\ge 0\ \text{ a.e. in }\ \Xi\times(0,T)\}.
\end{aligned}$$

Let a, b be two bilinear, continuous and symmetric mappings defined on $\boldsymbol{H}^1\times\boldsymbol{H}^1\to\mathbb{R}$ by

$$\begin{aligned}
&a(\boldsymbol{v},\boldsymbol{w}) = a^1(\boldsymbol{v}^1,\boldsymbol{w}^1)+a^2(\boldsymbol{v}^2,\boldsymbol{w}^2),\ b(\boldsymbol{v},\boldsymbol{w}) = b^1(\boldsymbol{v}^1,\boldsymbol{w}^1)+b^2(\boldsymbol{v}^2,\boldsymbol{w}^2)\\
&\forall\,\boldsymbol{v}=(\boldsymbol{v}^1,\boldsymbol{v}^2),\ \boldsymbol{w}=(\boldsymbol{w}^1,\boldsymbol{w}^2)\in\boldsymbol{H}^1,\ \text{where, for }\ \alpha=1,2,
\end{aligned}$$

$$a^\alpha(\boldsymbol{v}^\alpha,\boldsymbol{w}^\alpha) = \int_{\Omega^\alpha}\boldsymbol{\mathcal{A}}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\,dx,\ b^\alpha(\boldsymbol{v}^\alpha,\boldsymbol{w}^\alpha) = \int_{\Omega^\alpha}\boldsymbol{\mathcal{B}}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\,dx.$$

Assume $\boldsymbol{f}_1^\alpha\in W^{1,\infty}(0,T;L^2(\Omega^\alpha;\mathbb{R}^d))$, $\boldsymbol{f}_2^\alpha\in W^{1,\infty}(0,T;L^2(\Gamma_F^\alpha;\mathbb{R}^d))$, $\alpha=1,2$, $\boldsymbol{u}_0$, $\boldsymbol{u}_1\in\boldsymbol{V}$, $g_0\in L^2_+(\Xi)$, and define the following mappings:

$$\begin{aligned}
&J: L^2(\Xi)\times(\boldsymbol{H}^1)^2\to\mathbb{R},\ J(\delta,\boldsymbol{v},\boldsymbol{w}) = \int_\Xi \mu(\boldsymbol{v}_T)\,|\delta|\,|\boldsymbol{w}_T|\,d\xi\\
&\quad\forall\,\delta\in L^2(\Xi),\ \forall\,\boldsymbol{v}=(\boldsymbol{v}^1,\boldsymbol{v}^2),\ \boldsymbol{w}=(\boldsymbol{w}^1,\boldsymbol{w}^2)\in\boldsymbol{H}^1,\\
&\boldsymbol{f}\in W^{1,\infty}(0,T;\boldsymbol{H}^1),\ \langle\boldsymbol{f},\boldsymbol{v}\rangle = \textstyle\sum_{\alpha=1,2}\displaystyle\int_{\Omega^\alpha}\boldsymbol{f}_1^\alpha\cdot\boldsymbol{v}^\alpha\,dx + \sum_{\alpha=1,2}\int_{\Gamma_F^\alpha}\boldsymbol{f}_2^\alpha\cdot\boldsymbol{v}^\alpha\,ds\\
&\quad\forall\,\boldsymbol{v}=(\boldsymbol{v}^1,\boldsymbol{v}^2)\in\boldsymbol{H}^1,\ \forall\, t\in[0,T].
\end{aligned}$$

Assume the following compatibility conditions: $[u_{0N}]\le 0$, $\overline{\kappa}([u_{0N}])=0$ a.e. in Ξ and $\exists\,\boldsymbol{p}_0\in\boldsymbol{H}$ such that

$$(\boldsymbol{p}_0,\boldsymbol{w}) + a(\boldsymbol{u}_0,\boldsymbol{w}) + b(\boldsymbol{u}_1,\boldsymbol{w}) = \langle\boldsymbol{f}(0),\boldsymbol{w}\rangle\quad\forall\,\boldsymbol{w}\in\boldsymbol{V}. \tag{3.1}$$

For every $\zeta \in L^2(0,T;L^2(\Xi)) = L^2(\Xi \times (0,T))$, define the following sets:

$$\Lambda(\zeta) = \{\eta \in L^2(0,T;L^2(\Xi)); \underline{\kappa} \circ \zeta \leq \eta \leq \overline{\kappa} \circ \zeta \ \text{ a.e. in } \ \Xi \times (0,T)\,\},$$
$$\Lambda_+(\zeta) = \{\eta \in L^2_+(\Xi \times (0,T)); \underline{\kappa}_+ \circ \zeta \leq \eta \leq \overline{\kappa}_+ \circ \zeta \ \text{ a.e. in } \ \Xi \times (0,T)\,\},$$
$$\Lambda_-(\zeta) = \{\eta \in L^2_+(\Xi \times (0,T)); \overline{\kappa}_- \circ \zeta \leq \eta \leq \underline{\kappa}_- \circ \zeta \ \text{ a.e. in } \ \Xi \times (0,T)\,\},$$

where, for each $r \in \mathbb{R}$, $r_+ := \max(0,r)$ and $r_- := \max(0,-r)$ denote the positive and negative parts, respectively.

For each $\zeta \in L^2(0,T;L^2(\Xi))$ the sets $\Lambda(\zeta)$, $\Lambda_+(\zeta)$ and $\Lambda_-(\zeta)$ are clearly closed, convex and nonempty, because the bounding functions belong to the respective set. Since $\text{meas}(\Xi) < \infty$ and $\underline{\kappa}$, $\overline{\kappa}$ satisfy (2.2), it follows that for all $\zeta \in L^2(0,T;L^2(\Xi))$ these three sets are also bounded in norm in $L^\infty(\Xi \times (0,T))$ by r_0, and in $L^2(0,T;L^2(\Xi))$ by $r_1 = r_0\, T^{1/2}\, \text{meas}(\Xi)^{1/2}$.

A first variational formulation of the problem P_c is the following.

Problem $\boldsymbol{P_v^1}$: Find $\boldsymbol{u} \in C^1([0,T];\boldsymbol{H}^{-\iota}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$, $\lambda \in \Lambda([u_N])$ and

$$\begin{aligned}
&\langle \dot{\boldsymbol{u}}(T), \boldsymbol{v}(T) - \boldsymbol{u}(T)\rangle_{-\iota,\,\iota} - (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0) - \int_0^T (\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}} - \dot{\boldsymbol{u}})\, dt \\
&\quad + \int_0^T \left\{ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\dot{\boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}) - (\lambda, v_N - u_N)_{L^2(\Xi)} \right\} dt \qquad (3.2)\\
&+ \int_0^T \{ J(\lambda, \dot{\boldsymbol{u}}, \boldsymbol{v} + k\dot{\boldsymbol{u}} - \boldsymbol{u}) - J(\lambda, \dot{\boldsymbol{u}}, k\dot{\boldsymbol{u}}) \}\, dt \geq \int_0^T \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle\, dt \\
&\quad \forall\, \boldsymbol{v} \in L^\infty(0,T;\boldsymbol{V}) \cap W^{1,2}(0,T;\boldsymbol{H}), \quad \text{where} \ \ 1 > \iota > \frac{1}{2}, \ k > 0.
\end{aligned}$$

The formal equivalence between the variational problem P_v^1 and the classical problem (2.3)–(2.8) can be easily proved by using Green's formula and an integration by parts, where the Lagrange multiplier λ satisfies the relation $\lambda = \sigma_N$.

Let $\phi : (L^2_+(\Xi))^2 \times (\boldsymbol{V})^2 \to \mathbb{R}$ be defined by

$$\begin{aligned}
&\phi(\delta_1, \delta_2, \boldsymbol{v}, \boldsymbol{w}) = -(\delta_1 - \delta_2, w_N)_{L^2(\Xi)} + \int_\Xi \mu(\boldsymbol{v}_T)\,(\delta_1 + \delta_2)\,|\boldsymbol{w}_T|\, d\xi \qquad (3.3)\\
&\forall\, (\delta_1, \delta_2) \in (L^2_+(\Xi))^2, \ \forall\, \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2), \ \boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{V}.
\end{aligned}$$

Since $\eta \in \Lambda(\zeta)$ if and only if $(\eta_+, \eta_-) \in \Lambda_+(\zeta) \times \Lambda_-(\zeta)$, it follows that the variational problem P_v^1 is equivalent with the following problem.

Problem $\boldsymbol{P_v^2}$: Find $\boldsymbol{u} \in C^1([0,T];\boldsymbol{H}^{-\iota}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\lambda \in L^2(0,T;L^2(\Xi))$

such that $\boldsymbol{u}(0)=\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0)=\boldsymbol{u}_1$, $(\lambda_+,\lambda_-)\in\Lambda_+([u_N])\times\Lambda_-([u_N])$ and

$$\begin{aligned}&\langle\dot{\boldsymbol{u}}(T),\boldsymbol{v}(T)-\boldsymbol{u}(T)\rangle_{-\iota,\,\iota}-(\boldsymbol{u}_1,\boldsymbol{v}(0)-\boldsymbol{u}_0)\\&+\int_0^T\{-(\dot{\boldsymbol{u}},\dot{\boldsymbol{v}}-\dot{\boldsymbol{u}})+a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})+b(\dot{\boldsymbol{u}},\boldsymbol{v}-\boldsymbol{u})\}\,dt\\&+\int_0^T\{\phi(\lambda_+,\lambda_-,\dot{\boldsymbol{u}},\boldsymbol{v}+k\dot{\boldsymbol{u}}-\boldsymbol{u})-\phi(\lambda_+,\lambda_-,\dot{\boldsymbol{u}},k\dot{\boldsymbol{u}})\}\,dt\\&\geq\int_0^T\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}\rangle\,dt\quad\forall\,\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{V})\cap W^{1,2}(0,T;\boldsymbol{H}).\end{aligned}\tag{3.4}$$

The existence of variational solutions of the problem P_c will follow from some general existence results that will be proved in the next sections.

4. Existence results for some variational inequalities

Let U_0, $(V_0,\|.\|,\langle\,.\,,.\rangle)$, $(U,\|.\|_U)$ and $(H_0,|.|,(\,.\,,.))$ be four Hilbert spaces such that U_0 is a closed linear subspace of V_0 dense in H_0, $V_0\subset U\subseteq H_0$ with continuous embeddings and the embedding from V_0 into U is compact.

Let $B_r(\Xi)$, $B_r(\Xi_T)$ denote the closed balls with center 0 and radius r in $L^\infty(\Xi)$, $L^\infty(\Xi_T)$, respectively, where $\Xi_T:=\Xi\times(0,T)$ and $r>0$.

Let $a_0,\,b_0:V_0\times V_0\to\mathbb{R}$ be two bilinear and symmetric forms such that

$$\exists\,M_a,\,M_b>0\quad a_0(u,v)\leq M_a\,\|u\|\,\|v\|,\quad b_0(u,v)\leq M_b\,\|u\|\,\|v\|,\tag{4.1}$$

$$\exists\,m_a,\,m_b>0\quad a_0(v,v)\geq m_a\,\|v\|^2,\quad b_0(v,v)\geq m_b\,\|v\|^2\quad\forall\,u,v\in V_0.\tag{4.2}$$

Let $l:V_0\to L^2(\Xi)$ and $\phi_0:[0,T]\times(L^2_+(\Xi))^2\times(V_0)^2\to\mathbb{R}$ be two mappings satisfying the following conditions:

$$\begin{aligned}&\exists\,k_1>0\text{ such that }\forall\,v_1,v_2\in V_0,\\&\|l(v_1)-l(v_2)\|_{L^2(\Xi)}\leq k_1\|v_1-v_2\|_U,\end{aligned}\tag{4.3}$$

$$\forall\,t\in[0,T],\ \ \forall\,\gamma_1,\,\gamma_2\in L^2_+(\Xi),\ \forall\,v,\,v_1,\,v_2\in V_0,$$

$$\phi_0(t,\gamma_1,\gamma_2,v,v_1+v_2)\leq\phi_0(t,\gamma_1,\gamma_2,v,v_1)+\phi_0(t,\gamma_1,\gamma_2,v,v_2),\tag{4.4}$$

$$\phi_0(t,\gamma_1,\gamma_2,v,\theta v_1)=\theta\,\phi_0(t,\gamma_1,\gamma_2,v,v_1),\quad\forall\,\theta\geq0,\tag{4.5}$$

$$\phi_0(t,\gamma_1,\gamma_2,v,w)=0,\quad\forall\,w\in U_0,\tag{4.6}$$

$$\phi_0(0,0,0,0,v)=0,\tag{4.7}$$

$$
\begin{aligned}
&\forall r>0,\ \exists\, k_2(r)>0 \ \text{ such that } \ \forall t_1,t_2\in[0,T],\\
&\forall \gamma_1,\, \gamma_2,\, \delta_1,\, \delta_2 \in L^2_+(\Xi)\cap B_r(\Xi),\, \forall v_1,v_2,w_1,w_2\in V_0,\\
&|\phi_0(t_1,\gamma_1,\gamma_2,v_1,w_1)-\phi_0(t_1,\gamma_1,\gamma_2,v_1,w_2)\\
&+\phi_0(t_2,\delta_1,\delta_2,v_2,w_2)-\phi_0(t_2,\delta_1,\delta_2,v_2,w_1)|\\
&\le k_2(r)(|t_1-t_2|+\|\gamma_1-\delta_1\|_{L^2(\Xi)}+\|\gamma_2-\delta_2\|_{L^2(\Xi)}\\
&\qquad +\|v_1-v_2\|_U)\|w_1-w_2\|_U,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&\text{if } (\gamma_1^n,\gamma_2^n)\in (L^2_+(\Xi_T))^2 \ \text{ for all } n\in\mathbb{N}\\
&\text{and } \ (\gamma_1^n,\gamma_2^n)\rightharpoonup(\gamma_1,\gamma_2) \text{ in } (L^2(0,T;L^2(\Xi)))^2, \text{ then}\\
&\int_0^T \phi_0(s,\gamma_1^n,\gamma_2^n,v,w)\,ds \to \int_0^T \phi_0(s,\gamma_1,\gamma_2,v,w)\,ds \quad \forall\, v,w\in L^2(0,T;V_0).
\end{aligned}
\tag{4.9}
$$

Remark 4.1. i) Since by (4.5) $\phi_0(\cdot,\cdot,\cdot,\cdot,0)=0$, from (4.8), for $w_1=w$, $w_2=0$, we have

$$
\begin{aligned}
&\forall t_1,t_2\in[0,T], \forall \gamma_1,\, \gamma_2,\, \delta_1,\, \delta_2 \in L^2_+(\Xi)\cap B_r(\Xi),\, \forall v_1,v_2,w\in V_0,\\
&|\phi_0(t_1,\gamma_1,\gamma_2,v_1,w)-\phi_0(t_2,\delta_1,\delta_2,v_2,w)|\\
&\le k_2(r)(|t_1-t_2|+\|\gamma_1-\delta_1\|_{L^2(\Xi)}+\|\gamma_2-\delta_2\|_{L^2(\Xi)}+\|v_1-v_2\|_U)\|w\|_U.
\end{aligned}
\tag{4.10}
$$

ii) From (4.7) and (4.8), for $t_1=t$, $t_2=0$, $\delta_1=\delta_2=0$ and $v_1=v$, $v_2=0$ we derive

$$
\begin{aligned}
&\forall t\in[0,T], \forall \gamma_1,\, \gamma_2\, \in L^2_+(\Xi)\cap B_r(\Xi),\, \forall v,w_1,w_2\in V_0,\\
&|\phi_0(t,\gamma_1,\gamma_2,v,w_1)-\phi_0(t,\gamma_1,\gamma_2,v,w_2)|\\
&\le k_2(r)(t+\|\gamma_1\|_{L^2(\Xi)}+\|\gamma_2\|_{L^2(\Xi)}+\|v\|_U)\|w_1-w_2\|_U.
\end{aligned}
\tag{4.11}
$$

iii) If $v_n\to v$, $w_m\to w$ in $L^2(0,T;U)$, $(\gamma_1^n,\gamma_2^n)\in(L^2_+(\Xi_T)\cap B_r(\Xi_T))^2$, for all $n\in\mathbb{N}$, and $(\gamma_1^n,\gamma_2^n)\rightharpoonup(\gamma_1,\gamma_2)$ in $(L^2(0,T;L^2(\Xi)))^2$, then

$$
\lim_{n,m\to\infty}\int_0^T \phi_0(s,\gamma_1^n,\gamma_2^n,v_n,w_m)\,ds \to \int_0^T \phi_0(s,\gamma_1,\gamma_2,v,w)\,ds, \tag{4.12}
$$

which can be proved by taking into account (4.11) in the following relations:

$$
\begin{aligned}
&|\int_0^T \{\phi_0(s,\gamma_1^n,\gamma_2^n,v_n,w_m)-\phi_0(s,\gamma_1,\gamma_2,v,w)\}\,ds|\\
&\le \int_0^T |\phi_0(s,\gamma_1^n,\gamma_2^n,v_n,w_m)-\phi_0(s,\gamma_1^n,\gamma_2^n,v_n,w)|\,ds\\
&+\int_0^T |\{\phi_0(s,\gamma_1^n,\gamma_2^n,v_n,w)-\phi_0(s,\gamma_1^n,\gamma_2^n,v,w)\}|\,ds\\
&+|\int_0^T \{\phi_0(s,\gamma_1^n,\gamma_2^n,v,w)-\phi_0(s,\gamma_1,\gamma_2,v,w)\}\,ds|\\
&\le \int_0^T k_2(r)(\|\gamma_1^n\|_{L^2(\Xi)}+\|\gamma_2^n\|_{L^2(\Xi)}+\|v_n\|_U)\|w_m-w\|_U\,ds\\
&+\int_0^T k_2(r)(\|\gamma_1^n\|_{L^2(\Xi)}+\|\gamma_2^n\|_{L^2(\Xi)}+\|v_n-v\|_U)\|w\|_U\,ds\\
&+|\int_0^T \{\phi_0(s,\gamma_1^n,\gamma_2^n,v,w)-\phi_0(s,\gamma_1,\gamma_2,v,w)\}\,ds|,
\end{aligned}
$$

and passing to limits by using that $(\gamma_{1,2}^n)_n$ are bounded and (4.9).

Assume that $f_0 \in W^{1,\infty}(0,T;V_0)$, u^0, $u^1 \in V_0$ are given, and that the following compatibility condition holds: $\overline{\kappa}(l(u^0))=0$ and $\exists\, p_0 \in H_0$ such that

$$
(p_0,w)+a_0(u^0,w)+b_0(u^1,w)=\langle f_0(0),w\rangle \quad \forall\, w\in V_0. \tag{4.13}
$$

We consider the following problem.
Problem Q_1: Find $u \in W_0$, $\lambda \in L^2(0,T;L^2(\Xi))$ such that $u(0)=u^0$, $\dot u(0)=u^1$, $(\lambda_+,\lambda_-)\in \Lambda_+(l(u))\times\Lambda_-(l(u))$ and

$$
\begin{aligned}
&\langle \dot u(T), v(T)-u(T)\rangle_{U'\times U}-(u_1,v(0)-u_0)\\
&+\int_0^T \{-(\dot u,\dot v-\dot u)+a_0(u,v-u)+b_0(\dot u,v-u)\}\,dt\\
&+\int_0^T \{\phi_0(t,\lambda_+,\lambda_-,\dot u,v+k\dot u-u)-\phi_0(t,\lambda_+,\lambda_-,\dot u,k\dot u)\}\,dt\\
&\ge \int_0^T \langle f_0, v-u\rangle\,dt \quad \forall\, v\in L^\infty(0,T;V_0)\cap W^{1,2}(0,T;H_0),
\end{aligned}
$$

where $W_0 := C^1([0,T];U')\cap W^{1,2}(0,T;V_0)$.

The sets $\Lambda_+(\zeta)$, $\Lambda_-(\zeta)$ and $\Lambda(\zeta)$ have the following useful properties, proved in [8], see also [27].

Lemma 4.1. *Let $\zeta\in L^2(0,T;L^2(\Xi))$ and $(\eta_1,\eta_2)\in\Lambda_+(\zeta)\times\Lambda_-(\zeta)$. Then $\eta_1\eta_2=0$ a.e. in Ξ_T and there exists $\eta\in\Lambda(\zeta)$ such that $\eta_+=\eta_1$, $\eta_-=\eta_2$ a.e. in Ξ_T.*

Based on the previous lemma, consider the following problem, which has the same solution u as the problem Q_1, and the solutions λ_1, λ_2 satisfy the relation $\lambda = \lambda_1 - \lambda_2$, where λ is a solution of Q_1 .

Problem Q_2 : Find $u \in W_0$, $\lambda_1, \lambda_2 \in L^2(0,T;L^2(\Xi))$ such that $u(0) = u^0$, $\dot{u}(0) = u^1$, $(\lambda_1, \lambda_2) \in \Lambda_+(l(u)) \times \Lambda_-(l(u))$ and

$$\begin{aligned} &\langle \dot{u}(T), v(T) - u(T)\rangle_{U' \times U} - (u_1, v(0) - u_0) \\ &+ \int_0^T \{-(\dot{u}, \dot{v} - \dot{u}) + a_0(u, v-u) + b_0(\dot{u}, v-u)\}\, dt \\ &+ \int_0^T \{\phi_0(t, \lambda_1, \lambda_2, \dot{u}, v + k\dot{u} - u) - \phi_0(t, \lambda_1, \lambda_2, \dot{u}, k\dot{u})\}\, dt \\ &\geq \int_0^T \langle f_0, v - u\rangle\, dt \quad \forall\, v \in L^\infty(0,T;V_0) \cap W^{1,2}(0,T;H_0). \end{aligned} \tag{4.14}$$

For the convenience of the reader, an existence and uniqueness result proved in [11] will be restated here under an adapted and more general form that will enable to study problem Q_2.

Let $\beta : V_0 \to \mathbb{R}$ and $\phi_1 : [0,T] \times V_0^3 \to \mathbb{R}$ be two sequentially weakly continuous mappings such that

$$\beta(0) = 0 \ \text{ and } \ \phi_1(t,z,v,w_1 + w_2) \leq \phi_1(t,z,v,w_1) + \phi_1(t,z,v,w_2), \tag{4.15}$$

$$\phi_1(t,z,v,\theta w) = \theta\, \phi_1(t,z,v,w), \tag{4.16}$$

$$\phi_1(0,0,0,w) = 0 \ \ \forall\, t \in [0,T],\ \forall\, z,\ v,\ w,\ w_{1,2} \in V_0,\ \forall\, \theta \geq 0, \tag{4.17}$$

$$\begin{aligned} &\exists\, k_3 > 0 \text{ such that } \ \forall\, t_{1,2} \in [0,T], \forall\, u_{1,2}, v_{1,2}, w \in V_0, \\ &|\phi_1(t_1,u_1,v_1,w) - \phi_1(t_2,u_2,v_2,w)| \\ &\quad \leq k_3(|t_1 - t_2| + |\beta(u_1 - u_2)| + \|v_1 - v_2\|_U)\, \|w\|, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\exists\, k_4 > 0 \text{ such that } \ \forall\, t_{1,2} \in [0,T],\ \forall\, u_{1,2}, v_{1,2}, w_{1,2} \in V_0, \\ &|\phi_1(t_1,u_1,v_1,w_1) - \phi_1(t_1,u_1,v_1,w_2) + \phi_1(t_2,u_2,v_2,w_2) \\ &\quad -\phi_1(t_2,u_2,v_2,w_1)| \leq k_4\,(\,|t_1 - t_2| + \|u_1 - u_2\| + \|v_1 - v_2\|_U)\, \|w_1 - w_2\|. \end{aligned} \tag{4.19}$$

Let $L \in W^{1,\infty}(0,T;V_0)$ and assume the following compatibility condition on the initial data: $\exists\, p_1 \in H_0$ such that

$$(p_1, w) + a_0(u^0, w) + b_0(u^1, w) + \phi_1(0, u^0, u^1, w) = \langle L(0), w\rangle \ \forall\, w \in V_0. \tag{4.20}$$

Consider the following problem.

Problem Q_3 : Find $u \in W^{2,2}(0,T;H_0) \cap W^{1,2}(0,T;V_0)$ such that $u(0) = u^0$, $\dot{u}(0) = u^1$, and for almost all $t \in (0,T)$

$$\begin{aligned} &(\ddot{u}, v - \dot{u}) + a_0(u, v - \dot{u}) + b_0(\dot{u}, v - \dot{u}) \\ &\quad + \phi_1(t,u,\dot{u},v) - \phi_1(t,u,\dot{u},\dot{u}) \geq \langle L, v - \dot{u}\rangle \quad \forall\, v \in V_0. \end{aligned} \tag{4.21}$$

We have the following existence and uniqueness result.

Theorem 4.1. *Under the assumptions* (4.1), (4.2), (4.15)-(4.20)*, there exists a unique solution to the problem* Q_3.

The proof, which will be presented in a forthcoming paper, is based on a similar method to that used to prove the Theorem 3.2 established in [11] and on a useful estimate, see [22] or [32], which, when applied to the spaces $V_0 \subset U \subseteq H_0$, implies the following result: for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|u\|_U \leq \epsilon \,\|u\| + C_\epsilon \,|u| \quad \forall\, u \,\in\, V_0. \tag{4.22}$$

Lemma 4.2. *Assume that* (4.1), (4.2), (4.4), (4.5), (4.7), (4.8) *and* (4.13) *hold. If* $r > 0$ *then, for each* $(\gamma_1, \gamma_2) \in (W^{1,\infty}(0,T; L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2 \cap (B_r(\Xi_T))^2$ *with* $\gamma_1(0) = \gamma_2(0) = 0$*, there exists a unique solution* $u = u_{(\gamma_1,\gamma_2)}$ *of the following evolution variational inequality: find* $u \in W^{2,2}(0,T; H_0) \cap W^{1,2}(0,T; V_0)$ *such that* $u(0) = u^0$*,* $\dot{u}(0) = u^1$*, and for almost all* $t \in (0,T)$

$$\begin{aligned} &(\ddot{u}, v - \dot{u}) + a_0(u, v - \dot{u}) + b_0(\dot{u}, v - \dot{u}) \\ &+\phi_0(t, \gamma_1, \gamma_2, \dot{u}, v) - \phi_0(t, \gamma_1, \gamma_2, \dot{u}, \dot{u}) \geq \langle f_0, v - \dot{u}\rangle \quad \forall\, v \,\in V_0. \end{aligned} \tag{4.23}$$

Proof. We apply Theorem 4.1 to $\beta = 0$, $L = f_0$ and

$$\phi_1(t, z, v, w) = \phi_0(t, \gamma_1(t), \gamma_2(t), v, w) \quad \forall\, t \,\in\, [0,T],\ \forall\, z,\ v,\ w \,\in V_0.$$

Since ϕ_0 satisfies (4.4), (4.5) and (4.7), one can easily verify the properties (4.15)-(4.17). Also, (4.13) and (4.17) imply the condition (4.20).

Using (4.8), for some $k_2 = k_2(r) > 0$ we have

$$\begin{aligned} &\forall\, t_{1,2} \,\in [0,T],\ \forall\, u_{1,2},\ v_{1,2},\ w_{1,2} \in V_0, \\ &|\phi_1(t_1, u_1, v_1, w_1) - \phi_1(t_1, u_1, v_1, w_2) + \phi_1(t_2, u_2, v_2, w_2) - \phi_1(t_2, u_2, v_2, w_1)| \\ &= |\phi_0(t_1, \gamma_1(t_1), \gamma_2(t_1), v_1, w_1) - \phi_0(t_1, \gamma_1(t_1), \gamma_2(t_1), v_1, w_2)| \\ &\quad +\phi_0(t_2, \gamma_1(t_2), \gamma_2(t_2), v_2, w_2) - \phi_0(t_2, \gamma_1(t_2), \gamma_2(t_2), v_2, w_1)| \\ &\leq k_2(|t_1 - t_2| + \|\gamma_1(t_1) - \gamma_1(t_2)\|_{L^2(\Xi)} + \|\gamma_2(t_1) - \gamma_2(t_2)\|_{L^2(\Xi)} \\ &\qquad\qquad\qquad +\|v_1 - v_2\|_U)\|w_1 - w_2\|_U \\ &\leq k_2((1 + C_{\gamma_1} + C_{\gamma_2})|t_1 - t_2| + \|v_1 - v_2\|_U)\|w_1 - w_2\|_U, \\ &\leq k_5(|t_1 - t_2| + \|v_1 - v_2\|_U)\|w_1 - w_2\|_U, \end{aligned}$$

where C_{γ_1}, C_{γ_2} denote the Lipschitz constants of γ_1, γ_2, respectively, and $k_5 = k_2\,(1 + C_{\gamma_1} + C_{\gamma_2})$.

Thus,

$$
\begin{aligned}
&|\phi_1(t_1,u_1,v_1,w_1)-\phi_1(t_1,u_1,v_1,w_2)+\phi_1(t_2,u_2,v_2,w_2)\\
&-\phi_1(t_2,u_2,v_2,w_1)|\le k_5(|t_1-t_2|+\|v_1-v_2\|_U)\|w_1-w_2\|_U \qquad (4.24)\\
&\qquad\forall\, t_{1,2}\in[0,T],\ \forall\, u_{1,2},\, v_{1,2},\, w_{1,2}\in V_0,
\end{aligned}
$$

and, since by the continuous embedding $V_0\subset U$ there exists $C_U>0$ such that $\|w\|_U\le C_U\|w\|\ \ \forall\, w\in V_0$, it follows that ϕ_1 satisfies (4.19) with $k_4=k_5C_U$.

Taking in (4.24) $w_1=w$, $w_2=0$, by (4.16) with $\theta=0$, we obtain

$$
\begin{aligned}
&|\phi_1(t_1,u_1,v_1,w)-\phi_1(t_2,u_2,v_2,w)|\le k_5(|t_1-t_2|+\|v_1-v_2\|_U)\|w\|_U\\
&\qquad\forall\, t_{1,2}\in[0,T],\ \forall\, u_{1,2},\, v_{1,2},\, w\in V_0,
\end{aligned}
\qquad (4.25)
$$

and using the continuous embedding $V_0\subset U$, it follows that ϕ_1 satisfies (4.18) with $k_3=k_5C_U$.

Now, taking in (4.24) $t_1=t$, $t_2=0$, $u_1=z$, $v_1=v$, $u_2=v_2=0$, by (4.17) we have

$$
\begin{aligned}
&|\phi_1(t,z,v,w_1)-\phi_1(t,z,v,w_2)|\le k_5(t+\|v\|_U)\|w_1-w_2\|_U\\
&\qquad\forall\, t\in[0,T],\ \forall\, z,\, v,\, w_{1,2}\in V_0.
\end{aligned}
\qquad (4.26)
$$

As the embedding from V_0 into U is compact, from (4.25) and (4.26) it follows that ϕ_1 is weakly sequentially continuous.

By Theorem 4.1 there exists a unique solution $u=u_{(\gamma_1,\gamma_2)}$ of the variational inequality (4.23). □

Also, we have the following result, which is similar to Lemma 3.3 in [9].

Lemma 4.3. *Consider $r>0$, (γ_1,γ_2), $(\delta_1,\delta_2)\in(W^{1,\infty}(0,T;L^2(\Xi)))^2\cap(L^2_+(\Xi_T))^2\cap(B_r(\Xi_T))^2$ such that $\gamma_1(0)=\gamma_2(0)=\delta_1(0)=\delta_2(0)=0$ and let $u_{(\gamma_1,\gamma_2)}, u_{(\delta_1,\delta_2)}$ be the corresponding solutions of* (4.23)*. Then there exists a constant $C_0>0$, independent of (γ_1,γ_2), (δ_1,δ_2), such that for all $t\in[0,T]$*

$$
\begin{aligned}
&|\dot u_{(\gamma_1,\gamma_2)}(t)-\dot u_{(\delta_1,\delta_2)}(t)|^2+\|u_{(\gamma_1,\gamma_2)}(t)-u_{(\delta_1,\delta_2)}(t)\|^2\\
&\qquad+\int_0^t\|\dot u_{(\gamma_1,\gamma_2)}-\dot u_{(\delta_1,\delta_2)}\|^2\,ds\\
&\le C_0\int_0^t\{\phi_0(s,\gamma_1,\gamma_2,\dot u_{(\gamma_1,\gamma_2)},\dot u_{(\delta_1,\delta_2)})-\phi_0(s,\gamma_1,\gamma_2,\dot u_{(\gamma_1,\gamma_2)},\dot u_{(\gamma_1,\gamma_2)}) \qquad (4.27)\\
&\quad+\phi_0(s,\delta_1,\delta_2,\dot u_{(\delta_1,\delta_2)},\dot u_{(\gamma_1,\gamma_2)})-\phi_0(s,\delta_1,\delta_2,\dot u_{(\delta_1,\delta_2)},\dot u_{(\delta_1,\delta_2)})\}\,ds.
\end{aligned}
$$

5. An equivalent fixed point problem

Since $\mathcal{D}(0,T;L^2(\Xi))$ is dense in $L^2(0,T;L^2(\Xi))$, which is classically proved by using the convolution product with suitable mollifiers, it follows that for every $\gamma \in L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T)$, there exist $r > 0$ and a sequence $(\gamma^n)_n$ in $W^{1,\infty}(0,T;L^2(\Xi)) \cap L^2_+(\Xi_T) \cap B_r(\Xi_T)$ such that $\gamma^n(0) = 0$, for all $n \in \mathbb{N}$, and $\gamma^n \to \gamma$ in $L^2(0,T;L^2(\Xi))$.

Theorem 5.1. *Assume that* (4.1), (4.2), (4.4)-(4.9) *and* (4.13) *hold. For each* $(\gamma_1,\gamma_2) \in (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2$, *let* $(\gamma_1^n,\gamma_2^n)_n$ *be a sequence included in* $(W^{1,\infty}\ (0,T;L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2 \cap (B_r(\Xi_T))^2$, *for some* $r>0$, *such that* $(\gamma_1^n,\gamma_2^n) \rightharpoonup (\gamma_1,\gamma_2)$ *in* $(L^2(0,T;L^2(\Xi)))^2$, $\gamma_1^n(0)=\gamma_2^n(0)=0$, *and let* $u_{(\gamma_1^n,\gamma_2^n)}$ *be the solution of* (4.23) *corresponding to* (γ_1^n,γ_2^n), *for every* $n \in \mathbb{N}$. *Then* $(u_{(\gamma_1^n,\gamma_2^n)})_n$ *is strongly convergent in* W_0, *its limit, denoted by* $u = u_{(\gamma_1,\gamma_2)}$, *is independent of the chosen sequence converging to* (γ_1,γ_2) *with the same properties as* $(\gamma_1^n,\gamma_2^n)_n$ *and is a solution of the following evolution variational inequality:* $u(0)=u^0$, $\dot u(0)=u^1$,

$$
\begin{aligned}
&\langle \dot u(T), v(T)-u(T)\rangle_{U'\times U} - (u^1, v(0)-u^0) \\
&+\int_0^T \{-(\dot u, \dot v-\dot u) + a_0(u,v-u) + b_0(\dot u, v-u)\}\,dt \\
&+\int_0^T \{\phi_0(t,\gamma_1,\gamma_2,\dot u, v-u+k\dot u) - \phi_0(t,\gamma_1,\gamma_2,\dot u,k\dot u)\}\,dt \\
&\geq \int_0^T \langle f_0, v-u\rangle\,dt \quad \forall\, v \in L^\infty(0,T;V_0)\cap W^{1,2}(0,T;H_0).
\end{aligned}
\tag{5.1}
$$

The proof, based on Lemmas 4.2, 4.3 and on some compactness results established in [32], is similar to the proof of Theorem 3.2 in [9], so that will be not presented here.

Now, let $\Phi : (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2 \to 2^{(L^2_+(\Xi_T)\cap B_{r_0}(\Xi_T))^2} \setminus \{\emptyset\}$ be the set-valued mapping defined by

$$
\begin{aligned}
&\Phi(\gamma_1,\gamma_2) = \Lambda_+(l(u_{(\gamma_1,\gamma_2)})) \times \Lambda_-(l(u_{(\gamma_1,\gamma_2)})) \\
&\quad \text{for all } (\gamma_1,\gamma_2) \in (L^2_+(\Xi_T)\cap B_{r_0}(\Xi_T))^2,
\end{aligned}
\tag{5.2}
$$

where $u_{(\gamma_1,\gamma_2)}$ is the solution of the variational inequality (5.1) which corresponds to (γ_1,γ_2) by the procedure described in Theorem 5.1.

It is clear that if (λ_1,λ_2) is a fixed point of Φ, i.e. $(\lambda_1,\lambda_2) \in \Phi(\lambda_1,\lambda_2)$, then $(u_{(\lambda_1,\lambda_2)},\lambda_1,\lambda_2)$ is a solution of the problem Q_2.

Consider a new problem, which consists in finding a fixed point of the set-valued mapping Φ, called also multivalued function or multifunction, which will provide a solution of problem Q_1.

The existence of a fixed point of the multifunction Φ will be obtained by using a corollary of the Ky Fan's fixed point theorem [15], proved in [27] in the particular case of a reflexive Banach space.

Definition 5.1. *Let Y be a reflexive Banach space, D a weakly closed set in Y, and $F : D \to 2^Y \setminus \{\emptyset\}$ be a multivalued function. F is called sequentially weakly upper semicontinuous if $z_n \rightharpoonup z$, $y_n \in F(z_n)$ and $y_n \rightharpoonup y$ imply $y \in F(z)$.*

Proposition 5.1. (see [27]) *Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y, and $F : D \to 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $F(z)$ is convex for every $z \in D$. Then F has a fixed point.*

Note that since Y is a reflexive Banach space and D is convex, closed and bounded, we don't need to assume that Y is separable, see [4, 27].

Theorem 5.2. *Assume that* (2.1), (2.2), (4.1)-(4.9) *and* (4.13) *hold. Then there exists $(\lambda_1, \lambda_2) \in (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2$ such that $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$. For each fixed point (λ_1, λ_2) of the multifunction Φ, $(u_{(\lambda_1,\lambda_2)}, \lambda)$ with $\lambda = \lambda_1 - \lambda_2$ is a solution of the problem Q_1.*

Proof. The proof is similar to the proof of Theorem 3.3 in [9] but, for the convenience of the reader, we shall present it. By Lemma 4.1, if $(\lambda_1, \lambda_2) \in \Phi(\lambda_1, \lambda_2)$, then $(u_{(\lambda_1,\lambda_2)}, \lambda)$ is clearly a solution to the problem Q_1.

We apply Proposition 5.1 to $Y = (L^2(0, T; L^2(\Xi)))^2$, $D = (L^2_+(\Xi_T) \cap B_{r_0}(\Xi_T))^2$ and $F = \Phi$.

The set $D \subset (L^2(0, T; L^2(\Xi)))^2$ is clearly convex, closed and bounded. Since for each $\zeta \in L^2(0, T; L^2(\Xi))$ the sets $\Lambda_+(\zeta)$ and $\Lambda_-(\zeta)$ are nonempty, convex, closed, and bounded by r_0, it follows that $\Phi(\gamma_1, \gamma_2)$ is a nonempty, convex and closed subset of D for every $(\gamma_1, \gamma_2) \in D$.

In order to prove that the multifunction Φ is sequentially weakly upper semicontinuous, let $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$, $(\gamma_1^n, \gamma_2^n) \in D$, $(\eta_1^n, \eta_2^n) \in \Phi(\gamma_1^n, \gamma_2^n)$ $\forall\, n \in \mathbb{N}$, $(\eta_1^n, \eta_2^n) \rightharpoonup (\eta_1, \eta_2)$ and let us verify that $(\eta_1, \eta_2) \in \Phi(\gamma_1, \gamma_2)$. By Theorem 5.1, there exists a sequence $(\hat\gamma_1^n, \hat\gamma_2^n)_n$ in $(W^{1,\infty}(0, T; L^2(\Xi)))^2 \cap (L^2_+(\Xi_T))^2 \cap (B_r(\Xi_T))^2$, for some $r > 0$, such that $(\gamma_1^n, \gamma_2^n) - (\hat\gamma_1^n, \hat\gamma_2^n) \rightharpoonup 0$, $\hat\gamma_1^n(0) = \hat\gamma_2^n(0) = 0$ and

$$\|u_{(\hat\gamma_1^n,\hat\gamma_2^n)} - u_{(\gamma_1^n,\gamma_2^n)}\|_{W_0} \le \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \tag{5.3}$$

where $u_{(\hat\gamma_1^n,\hat\gamma_2^n)}$ is the solution of (4.23) corresponding to $(\hat\gamma_1^n, \hat\gamma_2^n)$, $u_{(\gamma_1^n,\gamma_2^n)}$ is the solution of (5.1) corresponding to (γ_1^n, γ_2^n) and to the procedure which enables to define $\Phi(\gamma_1^n, \gamma_2^n)$.

As $(\gamma_1^n, \gamma_2^n) \rightharpoonup (\gamma_1, \gamma_2)$ in $(L^2(0, T; L^2(\Xi)))^2$, Theorem 5.1 implies also that $u_{(\hat\gamma_1^n,\hat\gamma_2^n)} \to u_{(\gamma_1,\gamma_2)}$ in W_0, and by (5.3) and the triangle inequality, we obtain

$$u_{(\gamma_1^n,\gamma_2^n)} \to u_{(\gamma_1,\gamma_2)} \quad \text{in } W_0. \tag{5.4}$$

By Lemma 4.1, the relation $(\eta_1^n, \eta_2^n) \in \Phi(\gamma_1^n, \gamma_2^n)$ is equivalent to

$$\eta_1^n - \eta_2^n \in \Lambda(l(u_{(\gamma_1^n,\gamma_2^n)})) \tag{5.5}$$

which may be rewritten as

$$\underline{\kappa} \circ l_n \leq \eta_1^n - \eta_2^n \leq \overline{\kappa} \circ l_n \quad \text{a.e. in } \Xi_T, \tag{5.6}$$

for all $n \in \mathbb{N}$, where $l_n := l(u_{(\gamma_1^n,\gamma_2^n)})$, and also under the following equivalent form:

$$\int_\omega \underline{\kappa} \circ l_n \leq \int_\omega (\eta_1^n - \eta_2^n) \leq \int_\omega \overline{\kappa} \circ l_n, \tag{5.7}$$

for every measurable subset $\omega \subset \Xi_T$ and for all $n \in \mathbb{N}$.

Using (5.4), (4.3), the semi-continuity of $\underline{\kappa}$ and $\overline{\kappa}$, the relation (2.2), the convergence property $\int_\omega (\eta_1^n - \eta_2^n) \to \int_\omega (\eta_1 - \eta_2)$, and passing to limits according to Fatou's lemma (see also [27]), we obtain

$$\int_\omega \underline{\kappa} \circ l(u_{(\gamma_1,\gamma_2)}) \leq \int_\omega (\eta_1 - \eta_2) \leq \int_\omega \overline{\kappa} \circ l(u_{(\gamma_1,\gamma_2)}), \tag{5.8}$$

for every measurable subset $\omega \subset \Xi_T$, which implies $(\eta_1, \eta_2) \in \Phi(\gamma_1, \gamma_2)$. □

6. Existence of a solution to the contact problem

Theorem 6.1. *Under the assumptions of Section 3 there exists a solution of the problem P_v^1.*

Proof. We shall prove that there exists at least a solution $(\boldsymbol{u}, \lambda_+, \lambda_-)$ of the problem P_v^2 which will provide a solution $(\boldsymbol{u}, \lambda)$ of the problem P_v^1 with $\lambda = \lambda_+ - \lambda_-$.

We apply Theorem 5.2 to $U_0 = \boldsymbol{H}_0^1 = H_0^1(\Omega^1;\mathbb{R}^d) \times H_0^1(\Omega^2;\mathbb{R}^d)$, $V_0 = \boldsymbol{V}$, $U = \boldsymbol{H}^\iota$, $H_0 = \boldsymbol{H}$, $a_0 = a$, $b_0 = b$, $u^0 = \boldsymbol{u}_0$, $u^1 = \boldsymbol{u}_1$, $\phi_0 = \phi$, $f_0 = \boldsymbol{f}$ and to the mapping $l : \boldsymbol{V} \to L^2(\Xi)$ defined by $l(\boldsymbol{v}) = [v_N] \ \ \forall \boldsymbol{v} \in \boldsymbol{V}$.

Since $\mathcal{A}_{ijkl}^\alpha,\ \mathcal{B}_{ijkl}^\alpha \in L^\infty(\Omega^\alpha)\ \forall i,\ j,\ k,\ l = 1, \ldots, d,\ \alpha = 1, 2$, we obtain (4.1).

The condition $\text{meas}(\Gamma_U^\alpha) > 0$, the ellipticity properties of the coefficients $\mathcal{A}_{ijkl}^\alpha$, $\mathcal{B}_{ijkl}^\alpha$ and the Korn's inequality imply that there exist $m_a^\alpha, m_b^\alpha > 0$ such that

$$a^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{v}^\alpha) \geq m_a^\alpha \|\boldsymbol{v}^\alpha\|_{\boldsymbol{V}^\alpha}^2,\ b^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{v}^\alpha) \geq m_b^\alpha \|\boldsymbol{v}^\alpha\|_{\boldsymbol{V}^\alpha}^2 \ \ \forall \boldsymbol{v}^\alpha \in \boldsymbol{V}^\alpha,\ \alpha = 1, 2,$$

and we obtain

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq m_a \|\boldsymbol{v}\|^2,\ \ b(\boldsymbol{v}, \boldsymbol{v}) \geq m_b \|\boldsymbol{v}\|^2 \ \ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{6.1}$$

where $m_a = \min(m_a^1, m_a^2),\ m_b = \min(m_b^1, m_b^2)$.

Also, the properties (4.3)-(4.7), (4.9) and (4.13) can be easily verified.

Now, let C_{tr} be a positive constant such that $\|\boldsymbol{v}\|_{(L^2(\Xi))^d} \leq C_{tr}\|\boldsymbol{v}\|_{\boldsymbol{H}^\iota}$ for all $\boldsymbol{v} \in \boldsymbol{H}^\iota$. Using (3.3), the following estimates hold:

$\forall\, r > 0,\ \forall\, \gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2 \in L^2_+(\Xi) \cap B_r(\Xi),\ \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{V},$

$$
\begin{aligned}
&|\phi(\gamma_1,\gamma_2,\boldsymbol{v}_1,\boldsymbol{w}_1) - \phi(\gamma_1,\gamma_2,\boldsymbol{v}_1,\boldsymbol{w}_2) + \phi(\delta_1,\delta_2,\boldsymbol{v}_2,\boldsymbol{w}_2) - \phi(\delta_1,\delta_2,\boldsymbol{v}_2,\boldsymbol{w}_1)| \\
&\quad = |-(\gamma_1-\gamma_2, w_{1N})_{L^2(\Xi)} + \int_\Xi \mu(\boldsymbol{v}_{1T})\,(\gamma_1+\gamma_2)\,|\boldsymbol{w}_{1T}|\,d\xi \\
&\qquad +(\gamma_1-\gamma_2, w_{2N})_{L^2(\Xi)} - \int_\Xi \mu(\boldsymbol{v}_{1T})\,(\gamma_1+\gamma_2)\,|\boldsymbol{w}_{2T}|\,d\xi \\
&\qquad -(\delta_1-\delta_2, w_{2N})_{L^2(\Xi)} + \int_\Xi \mu(\boldsymbol{v}_{2T})\,(\delta_1+\delta_2)\,|\boldsymbol{w}_{2T}|\,d\xi \\
&\qquad +(\delta_1-\delta_2, w_{1N})_{L^2(\Xi)} - \int_\Xi \mu(\boldsymbol{v}_{2T})\,(\delta_1+\delta_2)\,|\boldsymbol{w}_{1T}|\,d\xi| \\
&\quad \leq |(\gamma_1-\gamma_2-\delta_1+\delta_2, w_{1N}-w_{2N})_{L^2(\Xi)}| \\
&\qquad +|\int_\Xi (\mu(\boldsymbol{v}_{1T})-\mu(\boldsymbol{v}_{2T}))\,(\gamma_1+\gamma_2)\,(|\boldsymbol{w}_{1T}|-|\boldsymbol{w}_{2T}|)\,d\xi| \\
&\qquad +|\int_\Xi \mu(\boldsymbol{v}_{2T})(\gamma_1+\gamma_2-\delta_1-\delta_2)\,(|\boldsymbol{w}_{1T}|-|\boldsymbol{w}_{2T}|)\,d\xi| \\
&\quad \leq (\|\gamma_1-\delta_1\|_{L^2(\Xi)} + \|\gamma_2-\delta_2\|_{L^2(\Xi)})\,\|w_{1N}-w_{2N}\|_{L^2(\Xi)} \\
&\qquad + \int_\Xi C_\mu\,|\boldsymbol{v}_{1T}-\boldsymbol{v}_{2T}|\,(|\gamma_1|+|\gamma_2|)\,|\boldsymbol{w}_{1T}-\boldsymbol{w}_{2T}|\,d\xi \\
&\qquad + \int_\Xi \mu(\boldsymbol{v}_{2T})\,(|\gamma_1-\delta_1|+|\gamma_2-\delta_2|)\,|\boldsymbol{w}_{1T}-\boldsymbol{w}_{2T}|\,d\xi \\
&\quad \leq (\|\gamma_1-\delta_1\|_{L^2(\Xi)} + \|\gamma_2-\delta_2\|_{L^2(\Xi)})\,\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{(L^2(\Xi))^d} \\
&\qquad +2rC_\mu \int_\Xi |\boldsymbol{v}_1-\boldsymbol{v}_2|\,|\boldsymbol{w}_1-\boldsymbol{w}_2|\,d\xi \\
&\qquad +M_\mu \int_\Xi (|\gamma_1-\delta_1|+|\gamma_2-\delta_2|)\,|\boldsymbol{w}_1-\boldsymbol{w}_2|\,d\xi \\
&\quad \leq (\|\gamma_1-\delta_1\|_{L^2(\Xi)} + \|\gamma_2-\delta_2\|_{L^2(\Xi)})\,\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{(L^2(\Xi))^d} \\
&\qquad +2rC_\mu\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{(L^2(\Xi))^d}\,\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{(L^2(\Xi))^d} \\
&\qquad +M_\mu(\|\gamma_1-\delta_1\|_{L^2(\Xi)} + |\gamma_2-\delta_2|_{L^2(\Xi)})\,\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{(L^2(\Xi))^d}
\end{aligned}
$$

$$\leq (1 + M_\mu C_{tr})(\|\gamma_1 - \delta_1\|_{L^2(\Xi)} + \|\gamma_2 - \delta_2\|_{L^2(\Xi)}) \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\boldsymbol{H}^\iota}$$

$$+2rC_\mu C_{tr}^2 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{H}^\iota} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\boldsymbol{H}^\iota},$$

and so (4.8) is satisfied with $k_2(r)=\max(1 + M_\mu C_{tr}, 2rC_\mu C_{tr}^2)$, where M_μ is an upper bound of μ. □

Note that the same method can provide a unified approach to study more complex dynamic surface interactions, for which the evolution of the intensity of adhesion is governed by a variational inequality or a differential equation, see e.g. [33], [31], [12], [8].

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