Constitutive models and variational formulations in finite and classical elasto-plasticity

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Dedicated to Professor Nicolaie D. Cristescu on the occasion of his 85-th birthday

Abstract - We unify the methods for solving the initial and boundary value problems, by considering appropriate variational inequalities coupled with update algorithms associated with the rate type constitutive models, for large and small elasto-plastic deformation framework. To do that, we reformulate the definition of the plastic factor as a point-wise inequality on one hand and on the other hand the incremental equilibrium equation is written in terms of the rate of nominal stress which, in its weak form together with the rate type constitutive equations, leads to the variational inequality.

Key words and phrases : variational inequality, update algorithm, elasto-plastic deformation, plastic factor, rate-tupe model, boundary value problems.

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1. Introduction

The paper deals with initial and boundary value problems formulated in finite and classical elasto-plasticity. Here we unify the methods for solving the initial and boundary value problems, by considering appropriate variational inequalities coupled with update algorithms associated with the rate type constitutive models, for large and small elasto-plastic deformation framework. The variational inequalities have been formulated based on the incremental representation of the equilibrium equations for various rate-type constitutive framework, see Cleja-Ţigoiu in [2], [1], Cleja-Ţigoiu and Matei [6], Cleja-Ţigoiu and Stoicuţa [10].

The efficiency of variational formulations in solving the initial and boundary value problems which rise from elasto-plasticity or viscoplasticity has been proved, see for instance in Glowinski et al. [12], Kinderlehrer and Stampacchia [17], Ionescu and Sofonea [16] and Sofonea and Matei [25]. In [25], Sofonea and Matei presented various classes of variational inequalities for which they prove existence results and, for some of them, they prove uniqueness, regularity, and convergence results. Cleja-Tigoiu and Matei in

[6] described an appropriate variational inequalities related to the rate quasistatic boundary value problem and associated with a generic stage of the process in our approach to finite elasto-plastic materials in Eulerian and Lagrangian descriptions.

The rate type constitutive equations are associated with certain yield conditions and are dependent on the plastic factor, which is generally accepted to be defined via the Kuhn-Tucker and consistency conditions within the rate independent plasticity. On one hand the definition of the plastic factor is reformulated as a point-wise inequality, by the procedure proposed by Nguyen [22]. On the other hand the incremental equilibrium equation is written in terms of the rate of nominal stress and, in its weak form together with the rate type constitutive equations, leads to the variational inequality, see [2], [6]. To derive the variational inequality the reformulated definition of the plastic factor is a key point of the problem. The variational inequality has to be solved for the velocity and plastic factor. The coefficients of the variational inequalities are dependent on the current state of the elasto-plastic materials. Just the update algorithms allow us to compute the state of stress, strain and internal variables at every time-step. The update algorithms associated with rate type constitutive models have been formulated and numerically analyzed by Cleja-Ţigoiu and Matei [6] for rate-independent large deformation elasto-plastic model, Cleja-Ţigoiu and Paşcan [8] for elasto-viscoplastic models with dislocations, Cleja-Ţigoiu and Stoicuţa [11] for elasto-plastic mixed hardening models in the case of small deformation formalism .

The finite element methods (FEM) is applied to derive the discretized variational problems, which are coupled with the appropriate update algorithms. The numerical methods developed in [15], [12], [17] have been efficiently applied for solving initial and boundary value problems, see for instance Cleja-Ţigoiu and Paşcan [8], Cleja-Ţigoiu and Stoicuţa [10], [11].

First we introduce the variational inequalities for finite elasto-plasticity in the Eulerian setting, in Section 3 and secondary we derive the variational inequalities which correspond to small strains from the previously formulated inequalities. In Section 4 we derive directely the variational inequality for elasto-plastic mixed hardening model in the small deformation framework.

We mention certain methods for solving initial and boundary value problems for classical rate-independent elasto-plasticity, which could be connected with the point of view adopted here, in Section 4. We also make comments on various numerical methods for solving initial and boundary value problems.

2. Incremental equilibrium equation

We formulate the incremental form of the equilibrium equations at a certain moment of time, having in mind the rate-type constitutive models adopted for finite deformation elasto-plastic models, see Cleja-Ţigoiu [2], Cleja-Ţigoiu and Matei [6].

We denote by $\mathbf{T}(\mathbf{y}, \tau)$ the current value of the Cauchy stress at time τ in the material point localized at $\mathbf{y} = \chi(\mathbf{X}, \tau)$, where χ denotes the motion function. The Cauchy stress $\mathbf{T}(\mathbf{y}, \tau)$ satisfies the *equilibrium equation* at any time τ, which is written as

$$\operatorname{div} \mathbf{T}(\mathbf{y}, \tau) + \rho(\mathbf{y}, \tau)\mathbf{b}(\mathbf{y}, \tau) = 0, \quad \text{in} \quad \Omega_\tau \tag{2.1}$$

where $\mathbf{b}$ are the body forces.

The first Piola-Kirchhoff stress, $\mathbf{S}(\mathbf{X}, \tau)$, satisfies the balance equation (2.1)

$$\operatorname{div} \mathbf{S}(\mathbf{X}, \tau) + \mathbf{b}_0(\mathbf{X}, \tau) = 0,$$

$$\mathbf{S}(\mathbf{X}, \tau) = \det \mathbf{F}(\mathbf{X}, \tau)\mathbf{T}(\mathbf{y}, \tau)(\mathbf{F}(\mathbf{y}, \tau))^{-T}, \tag{2.2}$$

while the symmetry of Cauchy stress tensor leads to

$$\mathbf{S}(\mathbf{X}, \tau)\mathbf{F}^T(\mathbf{X}, \tau) = \mathbf{F}(\mathbf{X}, \tau)\mathbf{S}^T(\mathbf{X}, \tau). \tag{2.3}$$

Proposition 2.1. *The balance equation at time τ can be equivalently expressed, with respect to the configuration at time t, through*

$$div\, \mathbf{S}_t(\mathbf{x}, \tau) \quad + \rho(\mathbf{x}, t)\mathbf{b}_t(\mathbf{x}, \tau) = 0\,,$$

$$with \quad \mathbf{b}_t(\mathbf{x}, \tau) = \mathbf{b}(\chi_t(\mathbf{x}, \tau), \tau) \tag{2.4}$$

$$\mathbf{S}_t(\mathbf{x}, \tau)\mathbf{F}_t^T(\mathbf{x}, \tau) = \mathbf{F}_t(\mathbf{x}, \tau)\mathbf{S}_t^T(\mathbf{x}, \tau).$$

Here $\mathbf{F}_t(\mathbf{x}, \tau)$ denotes the relative deformation gradient at time τ, for $\mathbf{x} = \chi(\mathbf{X}, t)$, with respect to the actual configuration at time t,

$$\mathbf{F}_t(\mathbf{x}, \tau) = \mathbf{F}(\mathbf{X}, \tau)\mathbf{F}(\mathbf{X}, t). \tag{2.5}$$

Here the **nominal stress** or the *non-symmetric relative* Piola- Kirchhoff with respect to the actual configuration at time t, is defined by

$$\mathbf{S}_t(\mathbf{x}, \tau) \quad = (\det\mathbf{F}_t(\mathbf{x}, \tau))\mathbf{T}(\mathbf{y}, \tau)(\mathbf{F}_t(\mathbf{x}, \tau))^{-T},$$

$$\det\mathbf{F}_t(\mathbf{x}, \tau) = \frac{\rho(\mathbf{x}, t)}{\rho(\mathbf{y}, \tau)}, \tag{2.6}$$

in terms of the relative deformation gradient.

Theorem 2.1. 1. *The contact (surface) force acting on the boundary of* $\mathcal{P}(\tau)$ *can be expressed in terms of the nominal stress* $\mathbf{S}_t(\mathbf{x}, \tau)$

$$
\begin{aligned}
\mathbf{f}_c(\mathcal{P}, \tau) \;&= \int_{\partial \mathcal{P}(\tau)} \mathbf{T}(\mathbf{y}, \tau)\mathbf{n}(\mathbf{y}, \tau)d\,A = \\
&= \int_{\partial \mathcal{P}(t)} \mathbf{S}_t(\mathbf{x}, \tau)\mathbf{n}(\mathbf{x}, t)d\,A,
\end{aligned}
\tag{2.7}
$$

with $\mathbf{n}(\mathbf{x})$ *the unit outward normal.*

2. *At time t, the nominal stress satisfies the following relations*

$$
\mathbf{S}_t(\mathbf{x}, t) = \mathbf{T}(\mathbf{x}, t) \quad and
$$

$$
\dot{\mathbf{S}}_t(\mathbf{x}, t) \equiv \frac{\partial}{\partial \tau}\mathbf{S}_t(\mathbf{x}, \tau)\mid_{\tau=t} = \rho(\mathbf{x}, t)\frac{\partial}{\partial \tau}(\frac{\mathbf{T}(\mathbf{y}, \tau)}{\rho(\mathbf{y}, \tau)})\mid_{\tau=t} -\mathbf{T}(\mathbf{x}, t)\mathbf{L}^T(\mathbf{x}, t).
\tag{2.8}
$$

Proposition 2.2. *The incremental quasi-static boundary value problem at time t*

$$
\begin{aligned}
&div\,\dot{\mathbf{S}}_t(\mathbf{x}, t) + \rho(\mathbf{x}, t)\dot{\mathbf{b}}_t(\mathbf{x}, t) = 0\,, \\
&\dot{\mathbf{S}}_t(\mathbf{x}, t)\mathbf{n}(t)\mid_{\Gamma_{1t}} = \dot{\mathbf{\hat{S}}}_\mathbf{t}(\mathbf{x}, \mathbf{t}), \quad \mathbf{v}(\mathbf{x}, \mathbf{t})\mid_{\Gamma_{2t}} = \dot{\mathbf{\hat{U}}}_\mathbf{t}(\mathbf{x}, \mathbf{t})
\end{aligned}
\tag{2.9}
$$

using the notation $\dot{\mathbf{b}}_t(\mathbf{x}, t)$ *for* $\dfrac{\partial}{\partial \tau}\mathbf{b}_t(\mathbf{x}, \tau\mid_{\tau=t}$.

The rate of the nominal stress, at time t is calculated in terms of the rate for Cauchy stress.

Theorem 2.2. *The weak formulation of the rate boundary value problem leads to*

$$
\int_{\Omega_t} \dot{\mathbf{S}}_t \cdot \nabla \mathbf{w}dx = \int_{\partial \Omega_t} \dot{\mathbf{S}}_t \mathbf{n} \cdot \mathbf{w}da + \int_{\Omega_t} \rho \dot{\mathbf{b}}_t \cdot \mathbf{w}dx,
\tag{2.10}
$$

$\forall \mathbf{w} \in \mathcal{V}_{ad}(t)$, *where*

$$
\dot{\mathbf{S}}_t = \rho\frac{d}{d\,t}(\frac{\mathbf{T}}{\rho}) - \mathbf{T}\mathbf{L}^T
\tag{2.11}
$$

Here $\mathcal{V}_{ad}(t)$ denotes the admissible velocities set at time t.

Remark. The key point in writhing the weak form of the equilibrium equation or the theorem on the virtual power is the relationship between the rate of nominal stress and Cauchy stress written in (2.11).

3. Rate type constitutive models in finite elasto-plasticity

The behaviour of elasto-plastic materials undergoing large deformations is described within the constitutive framework proposed and developed by Cleja-Ţigoiu [3], [4], Cleja-Ţigoiu and Soós [7].

The constitutive framework of finite elasto-plasticity adopted here is based on the multiplicative decomposition of the deformation gradient $\mathbf{F}$ into its elastic and plastic components, i.e. the so-called elastic and plastic distortions,

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p. \tag{3.1}$$

The physical motivation of the multiplicative decomposition (3.1) is based on the dislocation theory and available experimental data. The plastic distortion can be only locally introduced, see Teodosiu [26], Mandel [21]. At least, in principle it is assumed that a small material neighborhood of an arbitrary considered particle may be cut off from the deformed body and relax it, maintaining the content of dislocation. Local deformation from the reference configuration to the local relaxed configuration, K_t, is denoted by $\mathbf{F}^p$. The reversible elastic distortion represents the deformation of the crystalline lattice, which remains unchanged by the dislocation motion.

The indetermination in choosing the local relaxed configuration is determined by assuming that the corresponding crystalline directions are parallel to each other. The configurations obtained by this procedure are called the local relaxed *isoclinic* configuration. The non-uniqueness in choosing the local relaxed isoclinic configuration is related to the material symmetry of elasto-plastic materials, formalized by Cleja-Ţigoiu and Soós [7].

3.1. Elasto-plastic models with relaxed configurations and internal variables

• A larger class of elasto-plastic materials with relaxed configurations can be described in **strain formulation**, with respect to time dependent K_t- relaxed isoclinic configurations. The elastic type constitutive equation delivers the current value Piola-Kirchhoff stress tensor

$$\frac{\mathbf{\Pi}}{\tilde{\rho}} = \mathbf{h}(\mathbf{G}, \boldsymbol{\alpha}, \kappa) \,, \text{with} \quad \mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e, \tag{3.2}$$

and satisfies the *relaxation property*

$$\mathbf{h}(\mathbf{S}, \boldsymbol{\alpha}, \kappa) = 0 \quad \text{for} \quad \mathbf{S} \in Sym^+ \quad \Longleftrightarrow \quad \mathbf{S} = \mathbf{I}. \tag{3.3}$$

The evolution equations for the plastic part of deformation $\mathbf{F}^p$ and internal variables $\boldsymbol{\alpha}, \kappa$, with respect to the current relaxed configuration K_t are

written under the form

$$\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \lambda \, \widetilde{\mathcal{B}}(\mathbf{C}^e, \boldsymbol{\alpha}, \kappa),$$

$$\dot{\boldsymbol{\alpha}} = \lambda \, \widetilde{\mathbf{l}}(\mathbf{C}^e, \boldsymbol{\alpha}, \kappa), \ \dot{\kappa} = \lambda \, \widetilde{\mathbf{m}}(\mathbf{C}^e, \boldsymbol{\alpha}, \kappa).$$

(3.4)

The evolution equations are associated with the yield condition, defined in terms of the yield function $\widetilde{\mathcal{F}}$, and with the plastic factor λ expressed by Kuhn-Tucker and consistency conditions

$$\lambda \geq 0, \quad \lambda \widetilde{\mathcal{F}} = 0, \quad \widetilde{\mathcal{F}} \leq 0, \quad \lambda \dot{\widetilde{\mathcal{F}}} = 0. \tag{3.5}$$

The yield function $\widetilde{\mathcal{F}}$ is dependent on the elastic strain tensor $\mathbf{C}^e$ and internal variables $\boldsymbol{\alpha}$.

We add the initial conditions, say

$$\mathbf{F}^p(t_0) = \mathbf{I}, \quad \boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}_0, \kappa(t_0) = \kappa_0. \tag{3.6}$$

• The constitutive and evolution equations for $\boldsymbol{\pi}-$ models can be introduced, if all the evolution functions are considered to be dependent on the stress measure $\boldsymbol{\pi}$ and the set of internal variables. The elastic type constitutive equation is defined by (3.2), and the irreversible behaviour is characterized by the rate-independent evolution equation of the form

$$\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \lambda \, \mathcal{B}(\frac{\boldsymbol{\pi}}{\widetilde{\rho}}, \boldsymbol{\alpha}, \kappa),$$

$$\dot{\boldsymbol{\alpha}} = \lambda \, \mathbf{l}(\frac{\boldsymbol{\pi}}{\widetilde{\rho}}, \boldsymbol{\alpha}, \kappa), \ \dot{\kappa} = \lambda \, m(\frac{\boldsymbol{\pi}}{\widetilde{\rho}}, \boldsymbol{\alpha}, \kappa).$$

(3.7)

Thus *a strain type model* can be associated by the procedure which is described below

$$\widetilde{\mathcal{F}}(\mathbf{C}^e, \boldsymbol{\alpha}) = \mathcal{F}(h(\mathbf{C}^e, \boldsymbol{\alpha}, \kappa), \boldsymbol{\alpha}, \kappa),$$

$$\widetilde{\mathcal{B}}(\mathbf{C}^e, \boldsymbol{\alpha}, \kappa) = \mathcal{B}(h(\mathbf{C}^e, \boldsymbol{\alpha}, \kappa), \boldsymbol{\alpha}, \kappa),$$

(3.8)

and similar definitions for $\widetilde{\mathbf{l}}, \widetilde{m}$.

• When we pass to the actual configuration, the Piola-Kirchoff stress tensor with respect to the local relaxed configuration will be transformed in the Cauchy stress tensor $\mathbf{T}$. The (symmetric) tensorial internal variable $\boldsymbol{\alpha}$ is replaced by $\mathbf{a}$, as a consequence of the following relationships

$$\frac{\mathbf{T}}{\rho} = \mathbf{F}^e \, \frac{\boldsymbol{\pi}}{\widetilde{\rho}} (\mathbf{F}^e)^T, \quad \mathbf{a} = \mathbf{F}^e \boldsymbol{\alpha} (\mathbf{F}^e)^T, \tag{3.9}$$

while the scalar internal variable remains unchanged.

The new plastic function is introduced in the deformed configuration by

$$
\begin{aligned}
\overline{\mathcal{F}}(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \kappa, \mathbf{F}^e) \;&= \mathcal{F}(\mathbf{F}^e)^{-1}\frac{\mathbf{T}}{\rho}(\mathbf{F}^e)^{-T}, (\mathbf{F}^e)^{-1}\mathbf{a}(\mathbf{F}^e)^{-T}, \kappa) \\
&\equiv \mathcal{F}(\frac{\boldsymbol{\pi}}{\tilde{\rho}}, \boldsymbol{\alpha}, \kappa).
\end{aligned}
\tag{3.10}
$$

Similarly the new evolution functions could be defined in the deformed configuration by

$$
\begin{aligned}
\overline{\mathcal{B}}(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{F}^e) = \mathbf{F}^e \mathcal{B}(\frac{\boldsymbol{\pi}}{\tilde{\rho}}, \boldsymbol{\alpha})((\mathbf{F}^e)^{-1}, \\
\bar{\mathbf{l}}(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{F}^e) = \mathbf{F}^e \mathbf{l}(\frac{\boldsymbol{\pi}}{\tilde{\rho}}, \boldsymbol{\alpha}) \, (\mathbf{F}^e)^T.
\end{aligned}
\tag{3.11}
$$

The elastic coefficients with respect to the actual configuration, which are characterized by $\overline{\mathcal{E}}$ with respect to the local relaxed configuration, are given through the relationships

$$
\overline{\mathcal{E}}[\mathbf{A}] = \mathbf{F}^e \, (\mathcal{E}[(\mathbf{F}^e)^T \mathbf{A} \mathbf{F}^e])(\mathbf{F}^e)^T \quad \forall \quad \mathbf{A} \in Sym.
\tag{3.12}
$$

Theorem 3.1. 1. *The rate type constitutive equations for the anisotropic models, pushed forward to the actual configuration, are described under the form of the differential system for the unknowns* $(\dfrac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E})$

$$
\begin{aligned}
\frac{d}{dt}(\frac{\mathbf{T}}{\rho}) \qquad &= \mathbf{L}\frac{\mathbf{T}}{\rho} + \frac{\mathbf{T}}{\rho}\mathbf{L}^T + \overline{\mathcal{E}}[\mathbf{D}] - \frac{\beta}{\bar{h}_c}\,(\overline{\mathcal{E}}[\overline{\mathcal{B}}^s] + 2\,\{\overline{\mathcal{B}}\frac{\mathbf{T}}{\rho}\}^s), \\[2mm]
(\frac{d}{dt}\mathbf{F}^e)(\mathbf{F}^e)^{-1} &= \mathbf{L} - \frac{\beta}{\bar{h}_c}\,\overline{\mathcal{B}}, \\[2mm]
\frac{d}{dt}\mathbf{a} \qquad &= \mathbf{La} + \mathbf{aL}^T = \frac{\beta}{\bar{h}_c}\,(\bar{\mathbf{l}} - \overline{\mathcal{B}}\,\mathbf{a} - \mathbf{a}\overline{\mathcal{B}}^T), \\[2mm]
\frac{d}{dt}\kappa \qquad &= \frac{\beta}{\bar{h}_c}\,m
\end{aligned}
\tag{3.13}
$$

with the plastic multiplier β *and hardening parameter* $\bar{h}_c$ *defined by*

$$
\begin{aligned}
\beta &= \partial_{\frac{\mathbf{T}}{\rho}}\,\overline{\mathcal{F}} \cdot \overline{\mathcal{E}}[\mathbf{D}] \cdot \partial_{\frac{\mathbf{T}}{\rho}}\overline{\mathcal{F}}, \\[2mm]
\bar{h}_c &= \overline{\mathcal{E}}\,[\{\overline{\mathcal{B}}\}^s] - \partial_{\frac{\mathbf{T}}{\rho}}\overline{\mathcal{F}} \cdot \bar{\mathbf{l}} - \partial_\kappa\overline{\mathcal{F}} \cdot \mathbf{m},
\end{aligned}
\tag{3.14}
$$

$\bar{h}_c$ is assumed to be positive.

     2. The plastic factor $\beta \geq 0$ can be express by the inequality

$$(\delta - \beta)(-\beta + \partial_{\frac{\mathbf{T}}{\rho}} \overline{\mathcal{F}} \cdot \overline{\mathcal{E}}[\mathbf{D}]) \leq 0, \quad \beta \overline{\mathcal{F}} = 0. \tag{3.15}$$

where the hardening parameter $\bar{h}_c$ is defined in $(3.14)_2$.

Here $\mathbf{D} = \mathbf{L}^s$.

3.2. Variational inequality in Eulerian description

In the theorem of virtual power (2.10), we substitute the rate of the Cauchy stress at time t, namely of $\dfrac{\mathbf{T}}{\rho}$, calculated from (3.1). We replace $\mathbf{L}$ from the kinematic relationships that follows from the multiplicative decomposition of the deformation gradient (3.1)

$$\mathbf{L}^e = \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}, \quad \mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1},$$

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}, \quad \mathbf{L} = \nabla\mathbf{v} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{3.16}$$

Theorem 3.2. *(Eulerian setting) At every time t, the velocity field, $\mathbf{v}$, and the equivalent plastic factor β satisfy the following relationships*

$$\left. \begin{aligned} &\int_{\Omega_t} \rho\{\nabla\mathbf{v}\frac{\mathbf{T}}{\rho} + \overline{\mathcal{E}}[\{\nabla\mathbf{v}\}^s]\} \cdot (\{\nabla\mathbf{w}\}^s - \{\nabla\mathbf{v}\}^s)dx - \\ &- \int_{\Omega_t^p} \rho\frac{\beta}{h_c} \left(\overline{\mathcal{E}}[\{\overline{\mathcal{B}}\}^s] + \overline{\mathcal{B}}\frac{\mathbf{T}}{\rho} + \frac{\mathbf{T}}{\rho}\overline{\mathcal{B}}^T\right) \cdot (\{\nabla\mathbf{w}\}^s - \{\nabla\mathbf{v}\}^s)dx = \\ &\int_{\Omega_t} \rho\dot{\mathbf{b}}_t \cdot (\mathbf{w} - \mathbf{v})dx + \int_{\Gamma_{2t}} \dot{\hat{\mathbf{S}}}_\mathbf{t} \cdot (\mathbf{w} - \mathbf{v})\mathbf{da} \end{aligned} \right\} \tag{3.17}$$

and

$$\int_{\Omega_t^p} \frac{\rho}{h_c} \left(\overline{\mathcal{E}}[\partial_{\frac{\mathbf{T}}{\rho}}\mathcal{F}] \cdot \{\nabla\mathbf{v}\}^s - \beta)(\delta - \beta)dx \leq 0, \tag{3.18}$$

which hold for every admissible vector field $\mathbf{w} \in \mathcal{V}_{ad}(t)$, and for all $\delta \in \mathcal{M}(t)$.

     In the theorem of virtual power (2.10) we substitute the rate of the nominal stress taking into account the rate form of the constitutive equation $(3.1)_1$

$$\dot{\mathbf{S}}_t(\mathbf{x}, t) = \rho\{\mathbf{L}\frac{\mathbf{T}}{\rho} + \overline{\mathcal{E}}[\mathbf{D}] - \overline{\mu} \, (\overline{\mathcal{E}}[\overline{\mathcal{B}}] + 2\,\{\overline{\mathcal{B}}\frac{\mathbf{T}}{\rho}\}^s)\}. \tag{3.19}$$

Let us introduce four maps $K[\cdot,\cdot]$, $A[\cdot,\cdot]$, $B_1[\cdot,\cdot]$ and $B_2[\cdot,\cdot]$ as follows

$$K[\mathbf{v},\mathbf{w}] = \int_{\Omega_t} \rho\big(\nabla\mathbf{v}\frac{\mathbf{T}}{\rho}\cdot\nabla\mathbf{w} + \overline{\mathcal{E}}[\{\nabla\mathbf{v}\}^s]\cdot\{\nabla\mathbf{w}\}^s\big)dx$$
$$\forall\,\mathbf{v},\mathbf{w}\in\mathcal{V}_{ad}(t),$$

$$A[\beta,\delta] = \int_{\Omega_t^p}\frac{\rho}{h_c}\,\beta\,\delta dx$$
$$\forall\,\delta,\,\beta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0},$$

$$B_1[\beta,\mathbf{w}] = -\int_{\Omega_t^p}\rho\frac{\overline{\beta}}{h_c}\big(\overline{\mathcal{E}}[\{\overline{\mathcal{B}}\}^s] + \overline{\mathcal{B}}\frac{\mathbf{T}}{\rho} + \frac{\mathbf{T}}{\rho}\overline{\mathcal{B}}^T\big)\cdot\{\nabla\mathbf{w}\}^s dx$$
$$\forall\,\beta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0},\ \forall\,\mathbf{w}\in\mathcal{V}_{ad}(t),$$

$$B_2[\mathbf{v},\delta] = -\int_{\Omega_t^p}\frac{\rho}{h_c}\delta\,\overline{\mathcal{E}}[\partial_{\frac{\mathbf{T}}{\rho}}\overline{\mathcal{F}}]\cdot\{\nabla\mathbf{v}\}^s dx$$
$$\forall\,\mathbf{v}\in\mathcal{V}_{ad}(t),\quad \forall\,\delta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0}.$$

Using these four maps, we define a **bilinear and non-symmetric form** $a[\cdot,\cdot]$,

$$a[\mathbf{V},\mathbf{W}] = K[\mathbf{v},\mathbf{w}] + B_1[\beta,\mathbf{w}] + B_2[\mathbf{v},\delta] + A[\beta,\delta] \qquad (3.20)$$
$$\forall\,\mathbf{V}=(\mathbf{v},\beta),\ \mathbf{W}=(\mathbf{w},\delta),\ \mathbf{v},\mathbf{w}\in\mathcal{V}_{ad}(t),\ \beta,\delta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0}.$$

Finally, we define a map $f[\cdot]$,

$$f[\mathbf{W}] = \int_{\Gamma_{1t}} \dot{\hat{\mathbf{S}}}_{\mathbf{t}}\cdot\mathbf{w}\mathbf{da} + \int_{\mathbf{\Omega_t}} \rho\dot{\mathbf{b}}_{\mathbf{t}}\cdot\mathbf{w}\mathbf{dx}, \qquad (3.21)$$
$$\forall\,\mathbf{W}=(\mathbf{w},\delta),\ \mathbf{w}\in\mathcal{V}_{ad}(t),\ \delta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0};\ \Gamma_{1t}\subset\partial\Omega_t.$$

Therefore, in *Eulerian setting*, the π-elasto-plastic model has the following formulation.

Problem 3.1. *Find* $\mathbf{V}=(\mathbf{v},\beta)\in\widetilde{\mathcal{K}}$ *such that*

$$a[\mathbf{V},\mathbf{W}-\mathbf{V}]\geq f[\mathbf{W}-\mathbf{V}]\quad \forall\,\mathbf{W}\in\widetilde{\mathcal{K}}. \qquad (3.22)$$

Here $\widetilde{\mathcal{K}}$ is the convex set defined in

$$\widetilde{\mathcal{K}} = \{(\mathbf{w},\delta)\mid\mathbf{w}\in\mathcal{V}_{ad}(t),\quad \delta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0}\}. \qquad (3.23)$$

3.3. Variational inequality for small distortions

Let us introduce the **hypothesis** of small deformations

$$\mathbf{F}=\mathbf{I}+\mathbf{H},\quad \mathbf{H}=\nabla\mathbf{u},\quad \sup_{X\in\mathcal{B},t\in R}\mid\mathbf{H}(\mathbf{X},t)\mid<<1. \qquad (3.24)$$

Here $\mathbf{F}$ is the deformation gradient and $\mathbf{H}$ is the gradient of the displacement vector $\mathbf{u}$, and moreover

$$\mathbf{F}^e = \mathbf{I} + \mathbf{H}^e, \quad \mathbf{F}^p = \mathbf{I} + \mathbf{H}^p, \quad \mid \mathbf{H}^e \mid << 1, \quad \mid \mathbf{H}^p \mid << 1,$$

$$\mathbf{H} = \mathbf{H}^e + \mathbf{H}^p, \quad \varepsilon = \varepsilon^e + \varepsilon^p, \varepsilon = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}), \tag{3.25}$$

$$\varepsilon^e = \frac{1}{2}(\mathbf{H}^e + (\mathbf{H}^e)^T), \quad \varepsilon^p = \frac{1}{2}(\mathbf{H}^p + (\mathbf{H}^p)^T).$$

Consequently, the small deformations counter-part of finite deformations, previously defined, can be emphasized. The basic kinematic relationships (3.16) become

$$\begin{aligned} \mathbf{L}^e &= \dot{\mathbf{H}}^e, \quad \mathbf{L}^p = \dot{\mathbf{H}}^p, \\ \dot{\mathbf{H}} &= \dot{\mathbf{H}}^e + \dot{\mathbf{H}}^p, \end{aligned} \tag{3.26}$$

while the stress measures, namely Piola-Kirchoff $\boldsymbol{\pi}$, nominal stress $\mathbf{S}$ and Cauchy stresses, coincide

$$\boldsymbol{\pi} \simeq \mathbf{S} \simeq \mathbf{T} \equiv \boldsymbol{\sigma}. \tag{3.27}$$

Proposition 3.1. *Let us restrict to linear elastic type constitutive equation, namely* $\dfrac{\boldsymbol{\pi}}{\tilde{\rho}} = \dfrac{1}{2}\mathcal{E}(\mathbf{C}^e - \mathbf{I})$, *written in plastically deformed configuration. Under the hypothesis of small elastic strains, which means*

$$\boldsymbol{\Delta} = \frac{1}{2}(\mathbf{C}^e - \mathbf{I}) \simeq \boldsymbol{\epsilon}^e, \quad \mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e, \quad where \tag{3.28}$$

$$\mathbf{U}^e = \mathbf{I} + \boldsymbol{\epsilon}^e, \quad \mathbf{C}^e = \mathbf{I} + 2\boldsymbol{\epsilon}^e \quad with \quad \mid \boldsymbol{\epsilon}^e \mid \leq 1,$$

$\mathbf{R}^e \in Ort$, $\mathbf{U}^e \in Psym$, $\mathbf{R}^e$ *is the elastic rotation, the following estimations hold:*

$$\mid \nabla\mathbf{w}\frac{\mathbf{T}}{\rho} \cdot \nabla\mathbf{w} \mid \leq \mid \nabla\mathbf{w} \mid^2 \mid \mathcal{E} \mid_4 \mid \boldsymbol{\epsilon}^e \mid \tag{3.29}$$

$$\mid \overline{\mathcal{E}}[\{\nabla\mathbf{w}\}^s] \cdot \{\nabla\mathbf{w}\}^s \mid \leq \ Const \mid \nabla\mathbf{w} \mid^2 \mid \mathcal{E} \mid_4 .$$

The hypothesis of small elastic strains with large elastic rotations and large plastic distortions has been introduced by Mandel [21] written in 3.28 are in good agreement with experimental data put into evidence in the crystalline materials, like metals.

As a straightforward consequence of the hypothesis (3.28) the kinematical relationships between the velocity gradient $\mathbf{L}$ and the rates of elastic distortion $\mathbf{L}^e$ is derived

$$\dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1} = \mathbf{R}^e\dot{\boldsymbol{\varepsilon}}^e(\mathbf{R}^e)^T. \tag{3.30}$$

In the case of small elastic strains and large elastic rotations, when we separate the symmetric and skew-symmetric parts from the kinematic relationship we obtain

$$\mathbf{D} = \mathbf{R}^e \dot{\boldsymbol{\epsilon}}^e (\mathbf{R}^e)^T + \mathbf{R}^e \mathbf{D}^p (\mathbf{R}^e)^T, \quad \mathbf{W} = \dot{\mathbf{R}}^e (\mathbf{R}^e)^{\mathbf{T}} + \mathbf{R}^e \mathbf{W}^p (\mathbf{R}^e)^T. \quad (3.31)$$

$\mathbf{D}$ is the rate of strain and $\mathbf{W}$ is the plastic spin.

Conclusions: 1. In the case of small elastic strains, the first term in the bilinear form $K[\cdot, \cdot]$ can be neglected in the presence of the second one. Moreover, the bilinear form $B_1[\beta, \mathbf{w}]$ is reduced to the first term. Let us remark that $B_1[\delta, \mathbf{w}]$ becomes equal to $B_2[\mathbf{w}, \delta]$ if an associated flow rule is considered, which means that $\overline{\mathcal{B}} = \partial_{\underset{\rho}{\mathbf{T}}} \overline{\mathcal{F}}$.

2. If the behavior of the body with small elastic strain only is elastic, i.e. $\beta = 0$, the bilinear form $a[\mathbf{V}, \mathbf{V}]$ for $\mathbf{V} = (\mathbf{v}, 0)$ is symmetric and positive definite.

Thus we derive the variational inequalities which correspond to small strains from the previously formulated inequalities (3.17).

$$\left. \begin{aligned} & \int_{\Omega_t} \rho \mathcal{E}[\{\nabla \mathbf{v}\}^s] \cdot (\{\nabla \mathbf{w}\}^s - \{\nabla \mathbf{v}\}^s) dx - \\ & - \int_{\Omega_t^p} \rho \frac{\beta}{h_c} \mathcal{E}[\{\overline{\mathcal{B}}\}^s] \cdot (\{\nabla \mathbf{w}\}^s - \{\nabla \mathbf{v}\}^s) dx = \\ & \int_{\Omega_t} \rho \dot{\mathbf{b}}_t \cdot (\mathbf{w} - \mathbf{v}) dx + \int_{\Gamma_{2t}} \dot{\overline{\mathbf{S}}}_{\mathbf{t}} \cdot (\mathbf{w} - \mathbf{v}) \mathbf{da} \end{aligned} \right\} \quad (3.32)$$

and

$$\int_{\Omega_t^p} \frac{\rho}{h_c} (\mathcal{E}[\partial_{\underset{\rho}{\mathbf{T}}} \mathcal{F}] \cdot \{\nabla \mathbf{v}\}^s - \beta)(\delta - \beta) dx \le 0, \quad (3.33)$$

which hold for every admissible vector field $\mathbf{w} \in \mathcal{V}_{ad}(t)$, and for all $\delta \in \mathcal{M}(t)$.

In the theorem of virtual power (2.10) we substitute the rate of the nominal stress taking into account the rate form of the constitutive equation $(3.13)_1$

$$\dot{\mathbf{S}}_t(\mathbf{x}, t) = \rho \{ \mathcal{E}[\mathbf{D}] - \overline{\mu} \, \mathcal{E}[\overline{\mathcal{B}}] \}. \quad (3.34)$$

When the small deformation case is considered the four maps $K[\cdot, \cdot]$,

$A[\cdot,\cdot]$, $B_1[\cdot,\cdot]$ and $B_2[\cdot,\cdot]$ are defined as follows

$$
\begin{aligned}
K[\mathbf{v},\mathbf{w}] \quad &= \int_{\Omega_t} \rho \mathcal{E}[\{\nabla\mathbf{v}\}^s]\cdot\{\nabla\mathbf{w}\}^s)dx, \\
&\qquad\qquad \forall\, \mathbf{v},\mathbf{w}\in\mathcal{V}_{ad}(t),
\end{aligned}
$$

$$
\begin{aligned}
A[\beta,\delta] \quad &= \int_{\Omega_t^p} \frac{\rho}{h_c}\,\beta\,\delta dx, \\
&\qquad\qquad \forall\,\delta,\ \beta:\Omega_t \longrightarrow \mathbf{R}_{\geq 0},
\end{aligned}
$$

$$
\begin{aligned}
B_1[\beta,\mathbf{w}] \quad &= -\int_{\Omega_t^p} \rho\frac{\overline{\beta}}{h_c}\left(\overline{\mathcal{E}}[\{\overline{\mathcal{B}}\}^s]\right)\cdot\{\nabla\mathbf{w}\}^s dx, \\
&\qquad\qquad \forall\,\beta:\Omega_t \longrightarrow \mathbf{R}_{\geq 0},\ \forall\,\mathbf{w}\in\mathcal{V}_{ad}(t),
\end{aligned}
$$

$$
\begin{aligned}
B_2[\mathbf{v},\delta] \quad &= -\int_{\Omega_t^p} \frac{\rho}{h_c}\delta\,\mathcal{E}[\partial_{\frac{\mathbf{T}}{\rho}}\mathcal{F}]\cdot\{\nabla\mathbf{v}\}^s dx, \\
&\qquad\qquad \forall\,\mathbf{v}\in\mathcal{V}_{ad}(t),\quad \forall\,\delta:\Omega_t \longrightarrow \mathbf{R}_{\geq 0}.
\end{aligned}
$$

(3.35)

Using these four maps, we define a **bilinear and non-symmetric form** $a[\cdot,\cdot]$,

$$
a[\mathbf{V},\mathbf{W}] = K[\mathbf{v},\mathbf{w}] + B_1[\beta,\mathbf{w}] + B_2[\mathbf{v},\delta] + A[\beta,\delta] \tag{3.36}
$$
$$
\forall\,\mathbf{V}=(\mathbf{v},\beta),\ \mathbf{W}=(\mathbf{w},\delta),\ \mathbf{v},\mathbf{w}\in\mathcal{V}_{ad}(t),\ \beta,\delta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0}.
$$

Finally, we define a map $f[\cdot]$,

$$
f[\mathbf{W}] = \int_{\Gamma_{1t}} \dot{\mathbf{S}}_{\mathbf{t}}\cdot\mathbf{w}\mathbf{da} + \int_{\mathbf{\Omega_t}} \rho\dot{\mathbf{b}}_{\mathbf{t}}\cdot\mathbf{w}\mathbf{dx}, \tag{3.37}
$$
$$
\forall\,\mathbf{W}=(\mathbf{w},\delta),\ \mathbf{w}\in\mathcal{V}_{ad}(t),\ \delta:\Omega_t\longrightarrow\mathbf{R}_{\geq 0};\ \Gamma_{1t}\subset\partial\Omega_t.
$$

Therefore, in the case of small deformation formalism the variational inequality has the same representation as those written in the **Problem 1**, were $\mathbf{v}=\dot{\mathbf{u}}$ and the bilinear mappings are defined by the relationships (3.35) and (3.36).

4. Constitutive framework of classical plasticity

The rate independent constitutive equations are described by Kachanov [18], Khan and Huang [19], Cleja-Ţigoiu and Cristescu [5], Paraschiv-Munteanu et al. [23], Simo and Hughes [24]. In Computational Inelasticity by Simo and Hughes [24], the authors analyzed the rate-independent constitutive equations of the classical elasto-plastic materials for small strains in chapter 2. The analysis is performed to motivate the return-mapping algorithms adopted by the authors.

4.1. Armstrong- Frederick and Chaboche type mixed hardening model

We consider an elasto-plastic body described by a mixed hardening material, within the small deformation framework as in [20]. An initial and boundary value problem associated with the Armstrong-Frederick kinematic hardening model coupled with an isotropic hardening law has been formulated and numerically solved by Claja-Ţigoiu and Stoicuţa [11], based on the variational inequality coupled with an update algorithm.

1. The rate of the strain tensor $\dot{\boldsymbol{\varepsilon}}$ is decomposed into the rate of elastic and plastic part, denoted by $\dot{\varepsilon}^e$ and $\dot{\varepsilon}^p$, respectively:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \tag{4.1}$$

2. The irreversible properties of the material are described in terms of the yield function $\mathcal{F} : \Omega \subset Sim \times \mathbb{R} \to \mathbb{R}_{\leq 0}$ which is depending on the stress and the hardening scalar variable k:

$$\mathcal{F}(\boldsymbol{\sigma}, k) = \|dev\boldsymbol{\sigma}\| - [Hk + \sigma_Y],$$
$$dev\boldsymbol{\sigma} = \boldsymbol{\sigma} - tr(\boldsymbol{\sigma})\mathbf{I}, \tag{4.2}$$

with $H > 0$ the hardening constant and σ_Y represents the initial yield constant, in terms of the deviatoric part of the stress tensor $dev\boldsymbol{\sigma} \equiv \mathbf{s}$.

3. The rate of the plastic strain tensor is described by the associated flow rule through:

   The rate of the plastic strain tensor is described by the associated flow rule through:

$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \frac{\partial \mathcal{F}(\boldsymbol{\sigma}, k)}{\partial \boldsymbol{\sigma}} \quad \text{or} \quad \dot{\boldsymbol{\varepsilon}}^p = \lambda \frac{\mathbf{s}}{\|\mathbf{s}\|}, \tag{4.3}$$

   or

$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \mathbf{n}, \quad \mathbf{n} = \frac{\mathbf{s}}{\|\mathbf{s}\|} \tag{4.4}$$

4. Here λ the so-called plastic factor is a function, which is define through the Kuhn-Tucker and consistency condition:

$$\lambda \geq 0, \quad \mathcal{F} \leq 0, \quad \lambda\mathcal{F} = 0, \quad \lambda\dot{\mathcal{F}} = 0. \tag{4.5}$$

5. The elastic type constitutive equation is expressed in terms of the Cauchy stress tensor, or by its associated rate form

$$\boldsymbol{\sigma} = \mathcal{E}(\varepsilon^e)$$
$$\dot{\boldsymbol{\sigma}} = \mathcal{E}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p) \quad \text{or} \quad \dot{\boldsymbol{\sigma}} = \mathcal{E}(\varepsilon(\dot{\mathbf{u}}) - \lambda\mathbf{n}) \tag{4.6}$$

where the fourth order tensor of elastic moduli $\mathcal{E}$ has the form:

$$\mathcal{E} = k_a \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I}_{dev}; \quad \text{where}$$

$$\mathbf{I}_{dev}\boldsymbol{\sigma} = dev\boldsymbol{\sigma}, \quad \Longleftrightarrow \quad \mathbf{I} \otimes \mathbf{I}(\boldsymbol{\sigma}) = tr(\boldsymbol{\sigma})\mathbf{I} \quad \forall \quad \boldsymbol{\sigma} \in Sim,$$

(4.7)

in the isotropic case, with λ_a, μ the Lame constants, and $k_a = \lambda_a + \frac{2}{3}\mu$ the bulk modulus.

6. The rate of the isotropic hardening variable k is given by the relation:

$$\dot{k} = \sqrt{\dot{\varepsilon}^p \cdot \dot{\varepsilon}^p} \quad \text{or} \quad \dot{k} = \lambda \tag{4.8}$$

7. We add the *initial condition*

$$\boldsymbol{\sigma}(0) = 0, \ \boldsymbol{\varepsilon}(0) = 0, \ \boldsymbol{\varepsilon}^p(0) = 0, \ k(0) = 0 \tag{4.9}$$

Proposition 4.1. *The plastic factor as a function of the point in the domain, namely* $\lambda : \Omega \times I \to \mathbb{R}$, *is calculated on the yield surface* $\mathcal{F}(\boldsymbol{\sigma}, k) = 0$ *by:*

$$\lambda = \frac{\langle \beta \rangle}{h}\mathcal{H}(\mathcal{F}) \text{ with } \beta = \mathbf{n} \cdot \mathcal{E}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \quad \text{and} \tag{4.10}$$

$$h = \mathbf{n} \cdot \mathcal{E}\mathbf{n} + H \tag{4.11}$$

where the hardening parameter h *is supposed to be strictly positive, and* $\mathcal{H}(\mathcal{F})$ *denotes the Heaviside function:*

$$\mathcal{H}(\mathcal{F}) = \begin{cases} 0, & \text{if } \mathcal{F} < 0 \\ 1, & \text{if } \mathcal{F} \geq 0 \end{cases} \tag{4.12}$$

Problem P2. Find functions $\sigma, \varepsilon^p, \alpha, k$ defined in $\Omega \times [0, T)$ and satisfying the following differential-type constitutive equations

$$\begin{aligned}
\dot{\varepsilon}^p &= \frac{\beta}{h_c}\partial_\sigma \mathcal{F}\mathcal{H}(\mathcal{F}) \\
\dot{\sigma} &= \mathcal{E}\varepsilon(\dot{u}) - \frac{\beta}{h_c}\mathcal{E}\partial_\sigma \mathcal{F}\mathcal{H}(\mathcal{F}) \\
\dot{\alpha} &= \frac{\beta}{h_c}\left(C\partial_\sigma \mathcal{F} - \gamma\alpha\right)\mathcal{H}(\mathcal{F}) \\
\dot{k} &= \frac{\beta}{h_c}\mathcal{H}(\mathcal{F})
\end{aligned} \tag{4.13}$$

with β, with $\beta \geq 0$, defined by the inequality

$$(\delta - \beta)(\partial_\sigma \mathcal{F} \cdot \mathcal{E}\varepsilon(\dot{u}) - \beta) \leq 0, \quad \forall \, \delta \geq 0, \quad \text{and} \quad \beta\mathcal{F} = 0, \tag{4.14}$$

the positive hardening parameter h_c is positive.

Problem P3. Let the body force **b** and boundary condition be given on $\partial\Omega$. Find the the displacement **u**, stress $\boldsymbol{\sigma}$, internal variables $(\boldsymbol{\alpha}, \kappa)$ and plastic strain $\boldsymbol{\varepsilon}^p$ as functions defined on $\Omega \times I$, which satisfies the equilibrium equation of the body, which occupies the domain Ω and is given by

$$div\sigma + b = 0 \quad \text{in} \quad \Omega \times I, \tag{4.15}$$

and the elasto-plastic model with Armstrong-Frederick type mixed hardening is described by (4.19).

The boundary conditions are given by the following relations

$$\begin{cases} \boldsymbol{\sigma}\mathbf{n} = f & \text{in} \quad \Gamma_\sigma \times I \\ \mathbf{u} = g & \text{in} \quad \Gamma_u \times I \end{cases} \tag{4.16}$$

4.2. Variational inequality in small strain elasto-plasticity

The following variational inequality has been formulated by Cleja-Ţigoiu and Stoicuţa in [11].

At every time $t \in I$, the velocity field $\dot{u}$ and the complementary plastic factor β, i.e. $(\dot{u}, \beta)$ satisfy the following relations

$$\begin{aligned}
&\int_\Omega \mathcal{E}\epsilon(\dot{u}) \cdot \epsilon(w - \dot{u})dx - \int_{\Omega^p} \frac{\beta}{h_P}\mathcal{E}\partial_s\tilde{\mathcal{F}} \cdot \varepsilon(w - \dot{u})dx = \\
&= \int_\Omega \dot{b} \cdot (w - \dot{u})dx + \int_{\Gamma_\sigma} \dot{f} \cdot (w - \dot{u})d\Gamma
\end{aligned} \tag{4.17}$$

$$-\int_{\Omega^p} \frac{1}{h_P}(\delta - \beta)\,\partial_s\tilde{\mathcal{F}}\mathcal{E} \cdot \varepsilon(\dot{u})dx + \int_{\Omega^p} \frac{1}{h_P}\beta(\delta - \beta)\,dx \geq 0 \tag{4.18}$$

for every admissible vector field $w \in \mathcal{V}_{ad}(t)$ and for all $\delta \in \mathcal{M}_{ad}(t)$.

To the differential type constitutive model (4.13) an update algorithm is associated by the procedure listed below. The generalized midpoint rule is described in the book ([24]).

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function, and consider the Cauchy problem in $[0, T]$

$$\begin{aligned}
\dot{\mathbf{x}} &= f(x(t)) \\
\mathbf{x} &= \mathbf{x}_n
\end{aligned} \tag{4.19}$$

The following integration algorithm, called the generalized midpoint, consists of

$$\begin{aligned}
\mathbf{x}_{n+1} &= \mathbf{x}_n + \Delta t\, f(x_{n+\theta}) \\
x_{n+\theta} &= \theta\mathbf{x}_{n+1} + (1 - \theta)\mathbf{x}_n, \quad \theta \in [0, 1].
\end{aligned} \tag{4.20}$$

Here $\mathbf{x}_{n+1} \simeq \mathbf{x}(t_{n+1})$ denotes the algorithmic approximation to the exact value at time $t_n + \Delta t$. We note that in particular

$$
\begin{array}{ll}
\theta = 0 & \text{forward (explicit) Euler} \\[2pt]
\theta = \dfrac{1}{2} & \text{midpoint rule} \\[2pt]
\theta = 1 & \text{backward (implicit) Euler.}
\end{array}
\tag{4.21}
$$

When we build the update algorithm concerning the rate type constitutive equations (4.13), β is replaced by the numerical solution of the variational inequality computed at any time t_n.

Lubarda and Benson [20] described the Return Mapping Algorithm for the elasto-plastic model with Armstrong-Frederick type mixed hardening, which holds for the loading process only, but the rate quasi-static boundary value problem associated with the mixed hardening model is solved for homogeneous case only.

4.3. History of problems in classical elasto-plasticity

We make some references to certain procedures proposed to solve problems in classical plasticity, which are closed to the proposed here constitutive framework.

The elastic type constitutive equation together with the evolution equation for plastic strain

$$
\boldsymbol{\sigma} = \mathcal{E}[\boldsymbol{\epsilon}^e], \quad \dot{\boldsymbol{\epsilon}}^p = \lambda \mathbf{r}, \quad \text{with}
$$
$$
\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p,
\tag{4.22}
$$

are rewritten as

$$
\dot{\boldsymbol{\sigma}} = \mathcal{E}[\dot{\boldsymbol{\epsilon}} - \lambda \mathbf{r}].
\tag{4.23}
$$

The plastic factor or plastic multiplier, λ, is defined by Kuhn- Tucker and consistency conditions (4.5).

The formula (2.2.22) from [24] introduces the so-called *tensor of tangent elasto-plastic moduli*, denoted by $\mathbf{C}^{ep}$, which is defined by the following relationships

$$
\mathcal{E}^{ep} = \begin{cases}
\mathcal{E} & \text{if} \quad \lambda = 0 \\[6pt]
\mathcal{E} - \dfrac{\mathcal{E}\mathbf{r} \otimes \mathcal{E}\partial_{\boldsymbol{\sigma}}\mathcal{F}}{h_c} & \text{if} \quad \lambda > 0
\end{cases}
\tag{4.24}
$$

under the hypothesis that the hardening parameter is positive, $h_c > 0$. The constitutive function $\mathbf{r}$ is defined by $\mathbf{r} = \partial_{\boldsymbol{\sigma}}\mathcal{F}$, if the associated flow rule is considered. Such kind of the formula becomes useful if and only if the

plastic factor is replaced by its expression, dependent on the deformation process. Simo and Hughes [24] proposed a Returne Mapping Algorithm which numerically avoided such type of *tangent elasto-plastic moduli*, as whose denoted by $\mathbf{C}^{ep}$.

In [14], Han and Reddy formulated variational inequalities and performed the mathematical analysis of the abstract boundary value problem for elasto-plastic models within the constitutive framework of standard generalized materials, see Halphen and Nguyen [13]. Han and Reddy considered the elasto-plastic materials with linear kinematic hardening, when the Cauchy stress $\boldsymbol{\sigma}$ and the kinematic hardening variable, the so-called back stress, $\boldsymbol{\chi}$, are defined by the following constitutive equations:

$$\boldsymbol{\sigma} = \mathcal{E}(\boldsymbol{\varepsilon}(u) - \varepsilon^p), \quad \boldsymbol{\chi} = -k_1\varepsilon^p, \tag{4.25}$$

and the strain-like hardening variable, $\boldsymbol{\xi}$, is conjugate variable associated with $\boldsymbol{\chi}$, namely $\boldsymbol{\xi} = H^{-1}\chi$. Here k_1 and H are constant parameters. Note that the finite form of the constitutive equations is adopted at every time t.

The flow rule is described by Han and Reddy [14] in terms of the dissipation function, which is non-negative, convex, positively homogeneous and l.s.c. (local inferior semi-continuous) defined on domain K_p, i.e. $(\dot{\varepsilon}^p, \dot{\boldsymbol{\xi}}) \in K_p$ are given in a such way to have

$$\begin{aligned} D(\mathbf{q}, \boldsymbol{\eta}) \quad &\geq D(\dot{\varepsilon}^p, \dot{\boldsymbol{\xi}}) + \boldsymbol{\sigma} \cdot (\mathbf{q} - \dot{\varepsilon}^p) + \boldsymbol{\chi} \cdot (\boldsymbol{\eta} - \dot{\boldsymbol{\xi}}), \\ &\forall \quad (\mathbf{q}, \boldsymbol{\eta}) \in K_p. \end{aligned} \tag{4.26}$$

Here (q, η) are the virtual rates that correspond to $(\dot{\varepsilon}^p, \dot{\boldsymbol{\xi}})$ We remark that, consequently these fields can not be strains, they are rate of strains.

The variational formulation for the problem follows from the dissipation inequality integrated over the full domain Ω, occupied by the elasto-plastic body (not only by the plastically deformed subdomain), together with the weak formulation of the equilibrium equation. In [14], the dissipation function is considered to be given in the space of strain by: $D(q) = c_0 \|q\|$, for q. Here $D(q)$ corresponds to Mises yield condition described in strain space. Here q ought to be an admissible rate of strain, and not a strain.

In the aforementioned book the following bilinear form $a : Z \times Z \to R$ is introduced by:

$$a(w, z) = \int_\Omega [\mathcal{E}(\boldsymbol{\varepsilon}(u) - \varepsilon^p) \cdot (\varepsilon(v) - q) + k_1\varepsilon^p \cdot q]dx, \tag{4.27}$$

where Z is an appropriate Hilbert space.

The linear functional $l(t)$ is associated with the body force, and the functional $j(\cdot)$ is related with the dissipation function, $D(q)$, above introduced.

The variational formulation of elasto-plastic problem, called by Han and Reddy **the primal problem** [14], is written formally as

Problem PRIM. Given the functional l, find $w = (u, \varepsilon^p) : [0, T] \to Z$, $w(0) = 0$, such that for almost all $t \in [0, T]$ the inequality

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq \langle l(t), z - \dot{w}(t) \rangle, \qquad (4.28)$$

holds for $\forall z = (\nu, q) \in Z$. Here $a(\cdot, \cdot)$ is symmetric and $j(\cdot)$ is a convex, positively homogeneous, nonnegative and Lipsschitz continuous on Z.

The time-discrete problem is rewritten in a form such that the increment of w is the primary unknown. The last variational inequality is reformulated as a minimization problem. In [14], a two steps method (predictor-corrector) for solving the variational inequality is described and a convergence analysis is also performed.

The variational inequality (4.28) and the proposed here variational inequality in Section 4 are completely different.

We make references to the variational characterization of the plastic response performed by Simo and Hughes [24], in Section 4.2. The assumption concerning the global free energy functional is formulated in terms of the density of elastic stored energy, $W(\varepsilon^e)$ and the quadratic contribution of hardening, i.e. $\mathbf{q} \cdot D^{-1} \mathbf{q}$. In addition to the free energy, the Lagrangian functional is associated with the plastic dissipation over the entire body, i.e. $\mathcal{D}^p \equiv \dot{\varepsilon}^p \cdot \boldsymbol{\sigma} - \dot{\mathbf{q}} \cdot D^{-1} \mathbf{q} \geq 0$. The constraint that the variables $(\boldsymbol{\sigma}, \mathbf{q})$ are in the closure of the elastic domain is removed through the method of Lagrange multiplier, which is interpreted as plastic factor. The authors proceed to the discretized functionals and the variational form of the governing equations have been derived. The integration algorithms for the nonlinear initial and boundary value problems are developed in the chapter 3 of [24].

Cleja-Ţigoiu et al. [9] analysed various numerical algorithms for solving the elasto-plastic problems with mixed hardening and the role played by Return Mapping Algorithms, with the references to the book written by Simo and Hughes [24]

Acknowledgments

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