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Seepage flow analysis in gravity and in variable acceleration fields

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In the occasion of his eighty-fifth birthday we would like to dedicate this paper to our old friend Nick Cristescu from whom we learned so much in Applied Mechanics

Abstract - Some finite element approaches are discussed for the analysis of seepage flows in a gravity field (referred to as quasi static conditions) and in the presence of an acceleration field variable in time and space (i.e. during an earthquake). The equations governing the flow of a liquid within a porous skeleton in quasi static conditions and their finite element formulation are recalled first. In this case the problem can be solved considering the hydraulic head as the only nodal variable. Then the dynamic case is considered. The governing equations are combined in two differential equations that, reduced in their weak form, lead to three alternative finite element formulations. They involve different sets of nodal variables, namely: the components of the discharge velocity only; both velocity and pore pressure; the pore pressure only. The advantages and shortcomings of these solution procedures are discussed and, to validate them, a test examples is solved and compared with results presented in the literature.

Key words and phrases : finite element, seepage flow, dynamic analysis, earthquake effects.

Mathematics Subject Classification (2010) **:** 76S05, 76M10, 86A17.

1. Introduction

The literature provides exhaustive theoretical bases and broadly accepted methods for the numerical analysis of seepage and of the coupled effective stress-flow problem under a gravity field, e.g. [3, 4, 6]. In dynamic conditions however, e.g. during earthquakes, the analysis of seepage becomes less straightforward [1, 2] since recourse cannot be made anymore to the usual concept of hydraulic head. This led to various numerical approaches for the dynamic coupled problem that involve different assumptions, different governing equations and different free variables [7, 8]. Note that a correct analysis of dynamic seepage is particularly relevant when dealing with granular deposits. In fact their relatively high hydraulic conductivity rules out the assumption of undrained conditions sometime adopted in engineering practice for fine grained soils. The relatively complex mathematical structure

of the dynamic problem and the associated computational burden makes the choice of the most appropriate numerical approach somewhat controversial. Here, the finite element formulation of quasi static seepage analysis will be recalled first. Then the fluid dynamic equations will be considered. Their weak form is derived and a first finite element formulation, having as free variable the discharge velocity, is presented. Some observations on the computational limits of this approach suggest considering alternative sets of free variables. Two alternative formulations are then derived. The first one involves as nodal variables both discharge velocity and pore pressure, while the second one adopts the pore pressure only. Finally, the solution of a bench mark problem is compared with Westergaard results (see [5]). In the following the discussion is limited to the seepage flow of a Newtonian liquid within a saturated and rigid porous skeleton in isothermal conditions. All variables are in general functions of time t. Upper case and lowercase underlined letters denote, respectively, matrices and column vectors. Superscript T means transpose.

2. Seepage flow in a gravity field

Let denote with $\underline{\hat{v}}$ and $\underline{v}$ the vectors of the velocity components of the liquid particles and the vector of the discharge velocity in Darcy sense, which pertains to the liquid phase. The two vectors are related to each other through the matrix $\underline{N}_S$ of the area, or surface, porosities the entries of which $(n_{S_x}, n_{S_y}, n_{S_z})$ are the ratios between the area of pores and the total area of the section through which the flow takes place.

$$\underline{v} = \underline{N}_S \underline{\hat{v}}. \tag{2.1}$$

Due to the difficulties in determining the area porosities, the volume porosity n in usually adopted, which represents the ratio between the volume of voids and the total volume of a soil element. The volume porosity can be seen as the average value of the area porosities (see [1]). Consequently, the following approximated relationship can be adopted.

$$\underline{v} = n\underline{\hat{v}}. \tag{2.2}$$

The flow continuity equation, expressing the conservation of mass of the liquid contained within an infinitesimal porous element, is customarily derived under the following assumptions: fully saturated porous medium; constant density ρ of the liquid; incompressible pore liquid and solid particles; rigid soil skeleton; absence of internal flow sources. Under these assumptions the continuity equation reads (cf. Figure 1),

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \underline{\nabla}^T \underline{v} = 0, \tag{2.3}$$

where v_x, v_y and v_z are the components of Darcy velocity collected in vector $\underline{v}$.

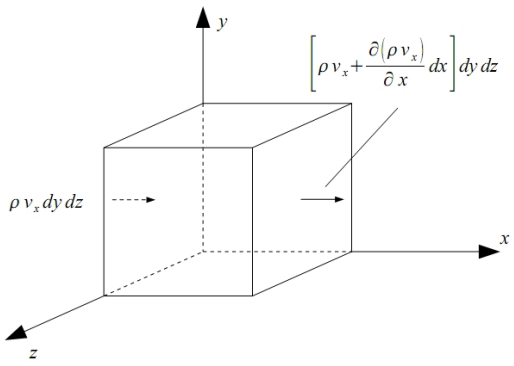

Figure 1. Relevant quantities for the equation of mass continuity in the x direction.

In a gravity field the velocity vector $\underline{v}$ can be related to the gradient $\underline{i}$ of the hydraulic head h,

$$h = \frac{p}{\rho g} + y + \frac{|\underline{v}|^2}{2g}; \tag{2.4a}$$

$$v_x = -k_x \frac{\partial h}{\partial x}, \quad \text{etc.}; \tag{2.4b}$$

$$\underline{i}^T = \left\{ \frac{\partial h}{\partial x} \quad \frac{\partial h}{\partial y} \quad \frac{\partial h}{\partial z} \right\} = (\underline{\nabla} h)^T \tag{2.4c}$$

$$\underline{v} = \underline{P} \cdot \underline{i}, \tag{2.4d}$$

where p is the pore pressure (positive if tensile); ρ is the density of liquid; g is the acceleration of gravity; y is the vertical coordinate in the direction opposite to that of the acceleration of gravity; $|\underline{v}|$ is the modulus of the velocity vector; k_x is the coefficient of hydraulic conductivity in the x direction (note that the Cartesian axes are assumed as principal directions of permeability) and $\underline{P}$ is the matrix collecting these coefficients.

In most cases of interest in geomechanics the velocity of flow is relatively small and the kinetic term in equation (2.4a) can be neglected. This has the advantage of reducing the mathematical complexity of the problem, otherwise the velocity components would depend on the square modulus of the velocity vector.

The finite element form of equation (2.3) is easily reached by writing it in a weak form,

$$\int_\Omega h^* \cdot \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) dV = 0, \tag{2.5}$$

where the weight function h^* represents a virtual variation of hydraulic head and Ω is the arbitrary volume of the porous medium.

Integrating by parts of equation (2.5),

$$\int_\Omega \left(\frac{\partial h^* v_x}{\partial x} + \frac{\partial h^* v_y}{\partial y} + \frac{\partial h^* v_z}{\partial z} \right) d\Omega - \int_\Omega \left(\frac{\partial h^*}{\partial x} v_x + \frac{\partial h^*}{\partial y} v_y + \frac{\partial h^*}{\partial z} v_z \right) d\Omega = 0 \tag{2.6}$$

and applying Green-Gauss theorem to the first term of equation (2.6), one obtains

$$\int_\Gamma h^* \cdot (v_x \cdot \alpha_x + v_y \cdot \alpha_y + v_z \cdot \alpha_z)\, d\Gamma - \int_\Omega \left(\frac{\partial h^*}{\partial x} v_x + \frac{\partial h^*}{\partial y} v_y + \frac{\partial h^*}{\partial z} v_z \right) d\Omega = 0. \tag{2.7}$$

Here Γ is the surface bounding volume Ω and α_x, α_y and α_z are the direction cosines of the outward vector normal to Γ. Note that the quantity within parentheses in the first integral of equation (2.7) represents the modulus of the flow velocity normal to surface Γ.

With reference to the first integral in equation (2.7), consider that surface S is subdivided into its pervious and impervious parts. The hydraulic head is known on the pervious boundary, consequently the variation h^* vanishes on it, whilst the normal flow velocity vanishes on the impervious portion of Γ. Consequently, the entire surface integral vanishes. Hence, the weak form of equation (2.3) reduces to

$$\int_\Omega \left(\frac{\partial h^*}{\partial x} v_x + \frac{\partial h^*}{\partial y} v_y + \frac{\partial h^*}{\partial z} v_z \right) d\Omega = 0 \tag{2.8}$$

Now, consider a finite element within which the distribution of hydraulic head $h(x, y, z)$ and, hence, of pore pressure is governed by interpolation functions, grouped in vector $\underline{b}(x, y, z)$, and by the nodal hydraulic heads $\underline{h}^e$

$$h(x, y, z) = \underline{b}^T(x, y, z) \cdot \underline{h}^e = (\underline{h}^e)^T \cdot \underline{b}(x, y, z). \tag{2.9}$$

The velocity field within the element can be expressed by substituting equation (2.9) into equations (2.4b)-(2.4d),

$$\underline{v}(x, y, z) = \underline{P} \cdot \underline{B}(x, y, z) \cdot \underline{h}^e, \tag{2.10}$$

where $\underline{B}$ is the matrix of the space derivatives of the interpolation functions $\underline{b}$. Introducing equations (2.9), (2.10) into equation (2.8), denoting with V^e the volume of the finite element and eliminating the virtual vector of nodal hydraulic heads, the following system of linear equations is arrived at,

$$\underline{M} \cdot \underline{h}^e = \underline{0}, \tag{2.11}$$

where $\underline{M}$ is the element flow matrix

$$\underline{M} = \int_{V^e} \underline{B}^T\, \underline{P}\, \underline{B}\, dV. \tag{2.12}$$

3. Governing equations in the case of variable acceleration field

The relative velocity $\underline{w}$ of the fluid phase with respect to that of the solid one, $\underline{\dot{u}}$, is expressed as (cf. (2.2)).

$$\underline{w} = n\underline{\hat{v}} - \underline{\dot{u}}. \tag{3.1}$$

Since a rigid porous skeleton is assumed here, the velocity $\underline{\dot{u}}$ is simply the time integral of the imposed acceleration (due to gravity and to the earthquake) and, hence, is known. As a consequence, the problem can be tackled subjecting the fluid phase to a known acceleration field varying with time. This allows using $\underline{v}$ instead of $\underline{w}$ as a free variable.

3.1. Equation of compatibility

This equation expresses the relationship between the strain rates of the liquid, collected in vector $\underline{\dot{\varepsilon}}$, and the velocity $\underline{\hat{v}}$ through a differential operator $\underline{C}$ analogous to that governing the strain-displacement relationship for solids.

$$\underline{\dot{\varepsilon}} = \begin{Bmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \cdots \\ \dot{\gamma}_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \cdot \begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{Bmatrix} = \underline{C}(\underline{\hat{v}}). \tag{3.2}$$

As usual, the first three entries of vector $\underline{\dot{\varepsilon}}$ correspond to normal strain rates and the remaining ones to shear strain rates. The above equation, rewritten in terms of the discharge velocity v, becomes

$$\underline{\dot{\varepsilon}} = \frac{1}{n}\underline{C}(\underline{v}). \tag{3.3}$$

3.2. Shear stress-shear strain rate relationship

Assuming that the fluid phase behaves as a Newtonian liquid, a linear relationship can be established between the deviatoric strain rates $\underline{\dot{e}}$ and the deviatoric stresses $\underline{\tau}$. To this purpose the following quantities are introduced for convenience,

$$p = \frac{1}{3}\underline{m}^T\underline{\sigma}, \tag{3.4a}$$

$$\underline{\tau} = \underline{\sigma} - \underline{m}p; \tag{3.4b}$$

$$\dot{\varepsilon}_{vol} = \underline{m}^T\underline{\dot{\varepsilon}}, \tag{3.5a}$$

$$\underline{\dot{e}} = \underline{\dot{\varepsilon}} - \frac{1}{3}\underline{m}\dot{\varepsilon}_{vol}. \tag{3.5b}$$

Here $\underline{\dot{\varepsilon}}$ and $\underline{\sigma}$ are the strain rate and the stress vectors; $\dot{\varepsilon}_{vol}$ is the volumetric strain rate and $\underline{m}$ is a vector the entries of which are equal to 1 if they correspond to normal strains, otherwise they vanish.

On these bases, the following relationship holds for a Newtonian fluid,

$$\underline{\tau} = \mu \underline{I}_0 \underline{\dot{e}}, \tag{3.6}$$

where μ is its deviatoric viscosity and $\underline{I}_0$ is a diagonal matrix the entries of which are equal to 2 if they correspond to normal stresses, otherwise they are equal to 1.

Substitution of equations (3.5), (3.3) into equation (3.6) leads to the strain rate vs. shear stress relationship,

$$\underline{\tau} = \mu \underline{I}_1 \underline{\dot{\varepsilon}} = \frac{\mu}{n} \underline{I}_1 \underline{C}(\underline{v}), \tag{3.7}$$

where

$$\underline{I}_1 = \underline{I}_0 - \frac{1}{3} \underline{I}_0 \underline{m} m^T = \underline{I}_0 - \frac{2}{3} \underline{m} m^T. \tag{3.8}$$

3.3. Equation of motion

Adopting an Eulerian approach, the equation of motion expresses the momentum balance of the liquid contained within an infinitesimal volume of the porous medium. This implies that the rate of increase of momentum be equal to the difference between the inward and outward momentum rates plus the contribution of the external forces acting on the liquid.

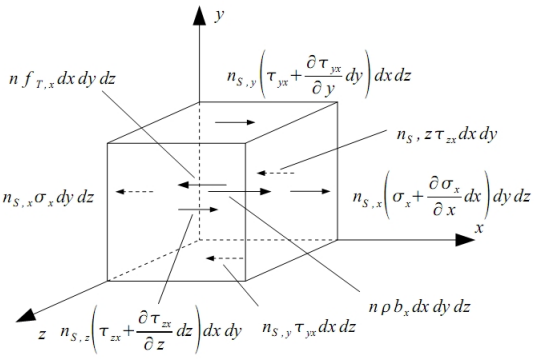

Figure 2. Relevant quantities for the equation of motion in the x direction.

The equation of motion for a one dimensional flow in the x direction is

(cf. Fig. 1),

$$\frac{\partial(\rho v_x)}{\partial t} = -\left[\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z}\right] + \left(n_{S_x}\frac{\partial \sigma_x}{\partial x} + n_{S_y}\frac{\partial \tau_{yx}}{\partial y} + n_{S_z}\frac{\partial \tau_{zx}}{\partial z}\right) + n\rho\bar{b}_x - f_{D_x}, \tag{3.9}$$

where $\bar{b}_x$ is the component of the imposed acceleration field in the x direction and f_{D_x} is the drag force due to the interaction between the flowing liquid and the porous skeleton. Introducing the simplifying assumption expressed by equation (2.2), the matrix form of the equation of motion for a three dimensional flow becomes

$$\frac{\partial(\rho \underline{v})}{\partial t} = -\left[\underline{m}^T \underline{C}\left(\rho \underline{v}\underline{v}^T\right)\right]^T + n\underline{C}^T(\underline{\sigma}) + n\rho\underline{\bar{b}} - \underline{f}_D. \tag{3.10}$$

In equation (3.9) vectors $\underline{f}_D$ and $\underline{\bar{b}}$ collect, respectively, the drag forces and the given acceleration components in the Cartesian directions. Confining our attention to laminar flows, the following relationship holds between the drag forces and the discharge velocity, where $\underline{K}'$ is the intrinsic permeability matrix of the skeleton,

$$\underline{f}_D = \mu(\underline{K}')^{-1}\underline{v}. \tag{3.11}$$

3.4. Mass continuity equation

If internal flow sources are neglected, the continuity of mass requires that the mass of fluid cumulated within an infinitesimal volume of the porous medium in a unit time coincides with the difference between the rate of masses entering and leaving it.

As to the rate of mass accumulation, $\dot{M}$, two contributions exist. The first one is the change in mass due to the volumetric strain rate of the fluid phase that, neglecting the possible volumetric viscosity of the liquid, can be related to the pore pressure rate through the bulk modulus B of the liquid and the volume porosity n of the skeleton.

$$\dot{M}_1 = -n\frac{\dot{p}}{B}\rho. \tag{3.12}$$

The second contribution depends on the change in density of the liquid.

$$\dot{M}_2 = n\dot{\rho}. \tag{3.13}$$

On these bases, the following equation expresses the mass continuity for a three dimensional flow,

$$n\dot{\rho} - \frac{n\rho}{B}\dot{p} = \left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right], \tag{3.14}$$

which in matrix form becomes

$$n\dot{\rho} - \frac{n\rho}{B}\dot{p} = \underline{m}^T\underline{C}(\rho\underline{v}). \tag{3.15}$$

3.5. Equation of state for the liquid phase

An additional scalar relationship is necessary to reach the balance between the numbers of unknowns and equations. This is represented by the equation of state that expresses the variation of the density of liquid ρ with temperature and pore pressure.

As previously mentioned, it is reasonable to treat the dynamic seepage flow under isothermal conditions, thus neglecting the variation of density with temperature. In addition, considering the high bulk modulus of water, also the change of density with the pore pressure is marginal. Consequently, the density of liquid ρ was assumed as constant in equation (3.15).

4. Final system of matrix equations

Introducing the assumption of constant density, the governing equations (3.7), (3.4b) and (3.10) can be re-written in the following form.

$$\underline{\sigma} = \frac{\mu}{n}\underline{I}_1\underline{C}(\underline{v}) + \underline{m}p \tag{4.1}$$

$$\rho\dot{\underline{v}} = -\rho\left[\underline{m}^T\underline{C}\left(\underline{v}\underline{v}^T\right)\right]^T + n\underline{C}^T(\underline{\sigma}) + n\rho\underline{\bar{b}} - \mu(\underline{K}')^{-1}\underline{v}. \tag{4.2}$$

The final system of governing differential equations consists of equations (3.15) and of the combination of equations (4.1) and (4.2)

$$\underline{m}^T\underline{C}(\underline{v}) - \frac{n}{B}\dot{p} = 0 \tag{4.3}$$

$$\frac{\rho}{n}\dot{\underline{v}} = -\frac{\rho}{n}\left[\underline{m}^T\underline{C}\left(\underline{v}\underline{v}^T\right)\right]^T + \frac{\mu}{n}\underline{C}^T\left[\underline{I}_1\underline{C}(\underline{v})\right] + \underline{C}^T(\underline{m}p) + \rho\underline{\bar{b}} - \frac{\mu}{n}(\underline{K}')^{-1}\underline{v}, \tag{4.4}$$

which correspond to four scalar differential equations involving as free variables the discharge velocity components and the pore pressure.

As to the boundary conditions, consider a porous domain having surface Γ and volume Ω. The surface can be subdivided into its impervious part, Γ_V, where the velocity normal to it vanishes and its pervious part, Γ_P, where the pore pressure $\overline{p}$ is known, i.e.

$$\underline{\alpha}^T\underline{v} = 0 \quad \text{on } \Gamma_V; \tag{4.5a}$$

$$p = \overline{p} \quad \text{on } \Gamma_P. \tag{4.5b}$$

Here $\underline{\alpha}$ collects the direction cosines of the outward vector normal to Γ_V.

5. Velocity approach

A first finite element formulation is derived here in which the discharge velocity represents the only nodal variable. To this purpose equation (4.4)

and the boundary condition equation (4.5b) are written in weak form introducing a virtual variation $\underline{v}^*$ of the discharge velocity that fulfils equation (4.5a). Note that the quadratic term on the right hand side of equation (4.4), equivalent to the kinetic part of equation (2.4), is disregarded.

$$\int_\Omega (\underline{v}^*)^T \left\{ \frac{\rho}{n}\dot{\underline{v}} - \frac{\mu}{n}\underline{C}^T [\underline{I}_1 \underline{C}(\underline{v})] - \underline{C}^T(\underline{m}p) - \rho\underline{\bar{b}} \right\} d\Omega + \\ + \int_\Omega (\underline{v}^*)^T \left\{ \frac{\mu}{n}(\underline{K}')^{-1}\underline{v} \right\} d\Omega + \int_{\Gamma_P} (\underline{v}^*)^T \underline{\alpha}(p-\bar{p}) d\Gamma = 0. \tag{5.1}$$

Integrating by parts the second term within brackets, and applying Green-Gauss theorem, introducing the interpolation or shape function matrix $\underline{S}_v^e$, expressing the velocity distribution within the e-th element as a function of the nodal velocities $\underline{v}^e$, after some manipulations one obtains

$$\underline{M}^e \dot{\underline{v}}^e + (\underline{X}^e + \underline{Z}^e)\underline{v}^e = \underline{f}^e + \underline{f}_P^e. \tag{5.2}$$

The following expressions hold for matrices and vectors in equations (5.2), where V^e is the volume of the element; Γ_P^e represents its sides where a known pore pressure $\bar{p}^e$ is imposed; $\underline{\bar{b}}$ is the vector of the imposed accelerations and p^e is the unknown pore pressure distribution within the e-th element,

$$\underline{M}^e = \frac{\rho}{n}\int_{V^e} \underline{S}_v^{eT}\underline{S}_v^e dV \tag{5.3a}$$

$$\underline{X}^e = \frac{\mu}{n}\int_{V^e} [\underline{C}(\underline{S}_v^e)]^T \underline{I}_1 \underline{C}(\underline{S}_v^e) dV \tag{5.3b}$$

$$\underline{Z}^e = \frac{\mu}{n}\int_{V^e} (\underline{S}_v^e)^T (\underline{K}')^{-1} \underline{S}_v^e dV \tag{5.3c}$$

$$\underline{f}^e = \rho\int_{V^e} (\underline{S}_v^e)^T \underline{\bar{b}}\, dV + \int_{\Gamma_P^e} (\underline{S}_v^e)^T \underline{m}\bar{p}^e d\Gamma \tag{5.3d}$$

$$\underline{f}_P^e = -\int_{V^e} [\underline{C}(\underline{S}_v^e)]^T \underline{m}p^e dV. \tag{5.3e}$$

The integration of equation (5.2) is carried through a series of time increments Δt_i, so that $t_i = t_{i-1} + \Delta t_i$, and assuming a linear variation of the nodal velocities within each increment. This leads to the following expressions for $\underline{v}^e$ and $\dot{\underline{v}}^e$ within the time interval,

$$\underline{v}^e(t_{i-1} + \beta\Delta t_i) = \beta\underline{v}^e(t_{i-1} + \Delta t_i) + (1-\beta)\underline{v}^e(t_{i-1}) \tag{5.4a}$$

$$\dot{\underline{v}}^e(t_{i-1} + \beta\Delta t_i) = \frac{1}{\Delta t_i}[\underline{v}^e(t_{i-1} + \Delta t_i) - \underline{v}^e(t_{i-1})], \tag{5.4b}$$

where $0 \leq \beta \leq 1$. Assuming $\beta = \frac{1}{2}$, substitution of equations (5.4) into equation (5.2) leads to

$$\left[\underline{M}^e + \frac{\Delta t_i}{2}(\underline{X}^e + \underline{Z}^e)\right] \underline{v}^e(t_i) = \left[\underline{M}^e - \frac{\Delta t_i}{2}(\underline{X}^e + \underline{Z}^e)\right] \underline{v}^e(t_{i-1}) + \\ + \frac{\Delta t_i}{2}\left[\underline{f}^e(t_i) + \underline{f}^e(t_{i-1})\right] + \frac{\Delta t_i}{2}\left[\underline{f}^e_P(t_i) + \underline{f}^e_P(t_{i-1})\right]. \quad (5.5)$$

Considering equation (5.3e), the pore pressure $p^e(t_i)$ at the end of the time interval that appears in vector $\underline{f}^e_P(t_i)$ can be expressed as

$$p^e(t_i) = p^e(t_{i-1}) + \int_{t_{i-1}}^{t_i} \dot{p}^e(t)dt. \quad (5.6)$$

Since the pore pressure rate and the velocity vary linearly within the time step, cf. equation (4.3), equation (5.6), the following incremental relationship holds,

$$p^e(t_i) = p^e(t_{i-1}) + \frac{\Delta t_i}{2}\frac{B}{n}\underline{m}^T \cdot \left\{\underline{C}[\underline{v}^e(t_{i-1})] + \underline{C}[\underline{v}^e(t_i)]\right\}, \quad (5.7)$$

where the pore pressure p^e is defined at the integration point of the element.

On these bases, and knowing the flow velocities and pore pressures at time t_{i-1}, the following iterative process can be adopted to evaluate the quantities at the end of the step:

- The pore pressure $p^e(t_i)$ is approximated through equation (5.7) assuming $v^e(t_i) = v^e(t_{i-1})$;
- The velocity $v^e(t_i)$ is calculated solving the system of linear equations (5.5);
- The pore pressure $p^e(t_i)$ is updated through equation (5.7);
- The iterations end when $v^e(t_i)$ and $p^e(t_i)$ stabilize.

6. Velocity-pore pressure approach

The above iterative approach, where the pore pressure does not represent a nodal variable and, hence, does not have a continuous distribution throughout the mesh, might show some stability problem unless very small time integration steps are adopted. To overcome this drawback an alternative approach can be easily formulated where both discharge velocity and pore pressure represent nodal variables.

To this purpose equation (4.3) is written in the following weak form where p^* is a virtual pore pressure variation.

$$\frac{n}{B}\int_\Omega \delta p\, \dot{p}\, d\Omega = \int_\Omega \delta p\, \underline{m}^T \underline{C}(\underline{v}) d\Omega. \quad (6.1)$$

Introducing the interpolation function vector for the pore pressure, $\underline{s}_p^e$, that relates the pore pressure within the element to its nodal values $\underline{p}^e$, and integrating over the volume V^e of the e-th element, equation (6.1) leads to

$$\underline{A}_1^e \dot{\underline{p}}^e + \underline{A}_2^e \underline{v}^e = \underline{0}, \tag{6.2}$$

where

$$\underline{A}_1^e = \frac{n}{B} \int_{V^e} \underline{s}_P^e \, \underline{s}_P^{eT} \, dV; \tag{6.3a}$$

$$\underline{A}_2^e = - \int_{V^e} \underline{s}_P^e \underline{m}^T \underline{C}(\underline{S}_v^{eT}) dV. \tag{6.3b}$$

Now, expressing the pore pressure p^e in equation (5.3e) in terms of the nodal pore pressures $\underline{p}^e$ through the interpolation functions

$$\underline{f}_P^e = - \int_{V^e} [\underline{C}(\underline{S}_v^e)]^T \underline{m} \, \underline{s}_P^{eT} \, dV \cdot \underline{p}^e \tag{6.4}$$

and combining equations (5.2), (6.2) and (6.4), the following system is arrived at

$$\begin{bmatrix} \underline{M}^e & \underline{0} \\ \underline{0} & \underline{A}_1^e \end{bmatrix} \begin{Bmatrix} \dot{\underline{v}}^e \\ \dot{\underline{p}}^e \end{Bmatrix} + \begin{bmatrix} (\underline{X}^e + \underline{Z}^e) & (\underline{A}_2^e)^T \\ \underline{A}_2^e & \underline{0} \end{bmatrix} \begin{Bmatrix} \underline{v}^e \\ \underline{p}^e \end{Bmatrix} = \begin{Bmatrix} \underline{f}^e \\ \underline{0} \end{Bmatrix}. \tag{6.5}$$

The time integration of equation (6.5) is carried out assuming a linear variation of velocity during the time increment (cf. equations (5.4)). In addition, also a linear variation of the pore pressure should be assumed. This introduces an approximation since, as previously observed, the pore pressure rate should vary linearly with time.

Note that, to be consistent with equation (4.3), the interpolations functions of the pore pressure should coincide with the space derivatives of those of the flow velocity. This implies that higher order element should be used with respect to those adopted for the pore pressure. The consequent non negligible computational burden and the above approximation suggest disregarding this approach for practical applications.

7. Pore pressure approach

Taking into account that in many cases of interest in geotechnical engineering the flow velocity is small, some terms of equation (4.4) can be disregarded since their contribution is likely to be marginal. These are the term of the left hand side of the equation (4.4), which depends on the acceleration, and the first term on the right hand side that depends on the square of velocity.

Consider now the terms linearly depending on velocity. The second one on the right hand side of equation (4.4), related to the curvature of the stream lines, plays a major role in standard flow problem. However, in the

case of seepage flows its contribution is in general smaller than that of the last term on the right hand side that represents the interaction between the flowing liquid and the solid particles. Hence, equation (4.4) reduces to

$$\underline{v} = \frac{n}{\mu}\underline{K}'[\underline{C}^T(\underline{m}p) + \rho\underline{\bar{b}}]. \tag{7.1}$$

Upon substitution of equation (7.1) into equation (4.3) one obtains the governing equation in terms of the pore pressure only (see [8]).

$$\underline{m}^T\underline{C}\left\{\frac{n}{\mu}\underline{K}'[\underline{C}^T(\underline{m}p) + \rho\underline{\bar{b}}]\right\} - \frac{n}{B}\dot{p} = 0. \tag{7.2}$$

Writing equation (7.2) in weak form, and following the same procedure previously outlined for the velocity approach, the pore pressure finite element formulation is arrived at.

$$\underline{A}_1^e\dot{\underline{p}}^e + \underline{A}_3^e\underline{p}^e = \underline{f}_2^e, \tag{7.3}$$

where

$$\underline{A}_3^e = \frac{n}{\mu}\int_{V^e}\left[\underline{C}^T(\underline{m}\,\underline{s}_P^{eT})\right]^T\underline{K}'\underline{C}^T(\underline{m}\,\underline{s}_P^{eT})\,dV; \tag{7.4a}$$

$$\underline{f}_2^e = \rho\int_{V^e}\left[\underline{C}^T(\underline{m}\,\underline{s}_P^{eT})\right]^T\underline{K}'\underline{\bar{b}}\,dV. \tag{7.4b}$$

8. Test example

The velocity and the pore pressure approaches were used for determining the water pressure distribution along a vertical rigid wall due to a dynamic excitation in the horizontal direction (see [5]).

The mesh consists of 200 four node quadrilateral elements and 231 nodes, 11 of which discretize the vertical wall.

The numerical results are reported, and compared with Westergaard solution, in Figures 3 and 4. They show, respectively, the maximum excess pressure distribution, with respect to the hydrostatic one, along the wall and the variation with time of the excess pressure at its base. In these figures H is the wall height and $p_{\max}$ is the maximum excess pressure at the wall base from Westergaard solution.

9. Conclusions

Three finite element approaches have been presented for dynamic seepage analysis, which involve different sets of free variables. Two of them, which seem more convenient from the numerical standpoint, have been used for

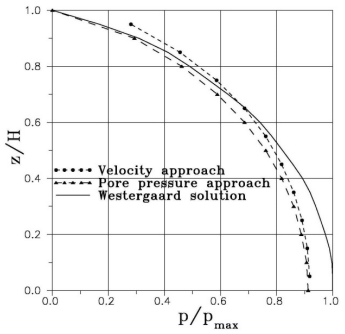

Figure 3. Excess pressure distribution along the vertical wall.

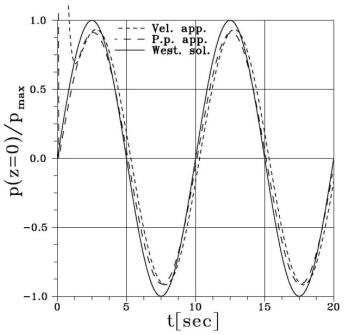

Figure 4. Variation with time of the excess pressure at the wall base.

solving a bench mark problem obtaining an acceptable agreement with Westergaard solution. The study will now proceed towards the finite element formulation of dynamic two phase problems in view of the analysis of the effects of earthquakes on structures embedded in saturated granular deposits.

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