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On socle and radical of modular lattices

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Dedicated to Professor Nicolaie D. Cristescu in honour of his 85th birthday

Abstract - In this paper we continue the investigation of preradicals in linear modular lattices started in our previous paper [T. Albu, M. Iosif, *Lattice preradicals,* submitted 2014]. Specifically, we show that the socle and the radical define preradicals in complete linear modular lattices. Then, we present equivalent forms of these preradicals for compactly generated modular lattices.

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Introduction

We introduced and investigated in [4] the general concept of a *lattice preradical.* The aim of this paper is to study two particular cases of lattice preradicals, namely the *socle* and the *radical* of complete linear modular lattices.

In Section 0 we list some notation and definitions about lattices, especially from [1] and [7]. Section 1 presents the concepts of linear morphism of lattices and lattice preradicals introduced and investigated in [2] and [4], respectively.

In Section 2 we show that the socle and the radical of a complete modular lattice L are preserved under a linear morphism of lattices. In case the complete modular lattice L is additionally compactly generated, we express in Section 3 the socle (respectively, the radical) of L as the meet of all its essential (respectively, proper maximal) elements. Usually, they are defined for any complete lattice L as the join of all atoms, respectively, small elements of L.

0. Preliminaries

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. Throughout this paper L will always denote such a lattice. By $\mathcal{M}$ we shall denote the class of all (bounded) modular lattices.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\ x \in L \mid a \leqslant x \leqslant b\ \}.$$

An *initial interval* of b/a is any interval c/a for some $c \in b/a$. We denote by L^o the *opposite* or *dual* lattice of L.

An element $a \in L$ is said to be an *atom* of L if $a \neq 0$ and $a/0 = \{0, a\}$. We denote by $A(L)$ the set, possibly empty, of all atoms of L. The *socle* $\mathrm{Soc}(L)$ of a complete lattice L is the join of all atoms of L, in other words, $\mathrm{Soc}\,(L) := \bigvee A(L)$; if $A(L) = \varnothing$ we put $\mathrm{Soc}\,(L) = 0$.

A *coatom* of L is an element $b \in L$ which is a maximal element of $L \setminus \{1\}$. We denote by $M(L)$ the set, possibly empty, of all coatoms of L, so $M(L) = A(L^o)$.

An element $e \in L$ is said to be *essential* (in L) if $e \wedge x \neq 0$ for every $x \neq 0$ in L. One denotes by $E(L)$ the set of all essential elements of L. An element $s \in L$ is called *small* or *superfluous* (in L) provided $s \in E(L^o)$. Thus, a small element s of L is characterized by the fact that $1 \neq s \vee a$ for every $a \in L$ with $a \neq 1$. We shall denote by $S(L)$ the set of all small elements of L, so that $S(L) = E(L^o)$. The *radical* $\mathrm{Rad}(L)$ of a complete lattice L is the join of all small elements of L, so, $\mathrm{Rad}(L) := \bigvee S(L)$.

Notice that a different concept of radical, much closer to its module-theoretical correspondent, has been considered in [6]: for a complete lattice L the *radical* $r_L := \bigwedge_{m \in M(L)} m$ of L is the meet of all coatoms of L, putting $r_L = 1$ if L has no coatoms. In order to avoid any confusion, we shall denote this r_L by $\mathrm{Jac}\,(L)$, and call it the *Jacobson radical* of L.

For all other undefined notation and terminology on lattices, the reader is referred to [1], [5], and [7].

1. Lattice preradicals

The property of a linear mapping $\varphi : M \longrightarrow N$ between two right modules M and N over a unital ring R to have a kernel $\mathrm{Ker}\,\varphi$ and to verify the Fundamental Theorem of Isomorphism $M/\mathrm{Ker}\,\varphi \simeq \mathrm{Im}\,\varphi$ has been taken in [2] as definition for the following concept.

A mapping $f : L \longrightarrow L'$ between a lattice L with least element 0 and greatest element 1 and a lattice L' with least element $0'$ and greatest element $1'$ is said to be a *linear morphism* if there exist $k \in L$, called a *kernel* of f, and $a' \in L'$ such that the following two conditions are satisfied.

(1) $f(x) = f(x \vee k), \forall\, x \in L$.

(2) f induces an isomorphism of lattices

$$\bar{f} : 1/k \xrightarrow{\sim} a'/0',\ \bar{f}(x) = f(x), \forall\, x \in 1/k.$$

By [2, Proposition 2.2], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, and called the category of *linear modular lattices,* with morphisms between two such lattices as linear morphisms; moreover, a subobject K of an $L \in \mathcal{LM}$ is exactly an initial interval $a/0$ of $L = 1/0$, and we denote this by $K \leqslant L$.

The latticial counterpart of the concept of preradical for modules has been introduced in [4] as follows. A *lattice preradical* is any functor

$$r : \mathcal{LM} \longrightarrow \mathcal{LM}$$

satisfying the following two conditions:

(1) $r(L) \leqslant L$ for any $L \in \mathcal{LM}$.

(2) For any morphism $f : L \longrightarrow L'$ in $\mathcal{LM}$, $r(f) : r(L) \longrightarrow r(L')$ is the restriction and corestriction of f to $r(L)$ and $r(L')$, respectively.

In other words, a lattice preradical is exactly a subfunctor of the identity functor $1_{\mathcal{LM}}$ of the category $\mathcal{LM}$.

Let $r : \mathcal{LM} \longrightarrow \mathcal{LM}$ be a lattice preradical. For any $L \in \mathcal{LM}$ and $a \in L$, the subobject $r(a/0)$ of L in $\mathcal{LM}$ is necessarily an initial interval of $a/0$. We denote

$$r(a/0) := a^r/0.$$

If $a \leqslant b$ in L, the inclusion mapping $\iota : a/0 \hookrightarrow b/0$ is clearly a linear morphism, so, applying r we obtain $r(\iota) : a^r/0 \longrightarrow b^r/0$ as a restriction of ι, and so $a^r \leqslant b^r$.

With notation above, $r(L) = r(1/0) = 1^r/0$. As in [4], we say that a preradical r is a *radical* if $r(1/1^r) = 1^r/1^r$ for all $L = 1/0 \in \mathcal{LM}$. Further, r is said to be an *idempotent* (respectively, a *left exact* or *hereditary*) preradical if for all $L \in \mathcal{LM}$, $r(r(L)) = r(L)$ (respectively, $a^r = a \wedge 1^r$ for every $a \in L$).

2. The socle and radical of a complete modular lattice

In this section we show that the socle and the radical of a lattice define two preradicals on the full subcategory $\mathcal{LM}_c$ of $\mathcal{LM}$ consisting of all complete linear modular lattices.

Recall that if L is any (bounded) lattice, then $A(L)$ denotes the set of all atoms of L, $S(L)$ denotes the set of all small elements of L, and if L is a complete lattice then

$$\operatorname{Soc}(L) := \bigvee_{a \in A(L)} a = \bigvee A(L),$$

$$\operatorname{Rad}(L) := \bigvee_{s \in S(L)} s = \bigvee S(L).$$

Notice that $\mathrm{Soc}\,(L) = 0 \iff A(L) = \varnothing$.

For two lattices L and L' we shall denote by 0 (respectively, 1) the least (respectively, greatest) element of L, and by $0'$ (respectively, $1'$) the least (respectively, greatest) element of L'.

Lemma 2.1. *Let L and L' be complete modular lattices. The following statements hold for a linear morphism $f : L \longrightarrow L'$ and an element $x \in L$.*

(1) $x \in A(L) \Longrightarrow f(x) \in \{0'\} \cup A(L')$.

(2) $x \in S(L) \Longrightarrow f(x) \in S(L')$.

Proof. Denote by k the kernel of f. By definition, there exists $a' \in L'$ such that f induces a lattice isomorphism

$$\bar{f} : 1/k \stackrel{\sim}{\longrightarrow} a'/0',\ \bar{f}(x) = f(x),\ \forall\, x \in 1/k.$$

(1) Consider $x \in A(L)$. If $x \leqslant k$, then

$$0' \ \leqslant\ f(x)\ \leqslant\ f(k)\ =\ 0',$$

so $f(x) = 0'$. Suppose that $x \nleqslant k$. Then $x \wedge k < x$ and since x is an atom, it follows that $x \wedge k = 0$. By modularity, we have

$$x/0\ =\ x/(x \wedge k)\ \simeq\ (x \vee k)/k,$$

so $x \vee k \in A(1/k)$. Since $\bar{f}$ is an isomorphism, we deduce that

$$f(x)\ =\ f(x \vee k)\ =\ \bar{f}(x \vee k)\ \in\ A(a'/0')\ \subseteq\ A(L').$$

(2) Let $x \in S(L)$. Then $x \vee k \in 1/k$. Let $y \in 1/k$ with $(x \vee k) \vee y = 1$. Then $x \vee y = 1$, and, since x is small in L, it follows that $y = 1$. Thus $x \vee k \in S(1/k)$. Since $\bar{f}$ is a lattice isomorphism, we have

$$f(x)\ =\ f(x \vee k)\ =\ \bar{f}(x \vee k)\ \in\ S(a'/0').$$

Now we are going to show that $S(a'/0') \subseteq S(L')$. To see this, pick $x' \in S(a'/0')$, and let $y' \in L'$ be such that $1' = x' \vee y'$. By modularity, we have $a' = (x' \vee y') \wedge a' = x' \vee (y' \wedge a')$. Since $y' \wedge a' \in a'/0'$ and $x' \in S(a'/0')$, we deduce that $y' \wedge a' = a'$, so $x' \leqslant a' \leqslant y'$. Thus $1' = x' \vee y' = y'$. Hence $x' \in S(L')$. Therefore $f(x) \in S(L')$, as desired. □

Proposition 2.2. *Let L and L' be complete modular lattices. For any linear morphism $f : L \longrightarrow L'$ in $\mathcal{LM}$ we have*

$$f(\mathrm{Soc}\,(L)) \leqslant \mathrm{Soc}\,(L') \quad \textit{and} \quad f(\mathrm{Rad}\,(L)) \leqslant \mathrm{Rad}\,(L').$$

Proof. Set $s := \mathrm{Soc}\,(L)$. By the proof of Lemma 2.1, we have

$$\begin{aligned} s \vee k &= \Big(\bigvee_{x \in A(L)} x \Big) \vee k = \bigvee_{x \in A(L)} (x \vee k) \\ &= \Big(\bigvee_{x \in A(L),\ x \leqslant k} (x \vee k) \Big) \vee \Big(\bigvee_{x \in A(L),\ x \nleqslant k} (x \vee k) \Big) \\ &= k \vee \Big(\bigvee_{x \in A(L),\ x \nleqslant k} (x \vee k) \Big) = \bigvee_{x \in A(L),\ x \nleqslant k} (x \vee k) \leqslant \mathrm{Soc}\,(1/k). \end{aligned}$$

Now, using the fact that $\bar{f}$ is a lattice isomorphism and the fact that any lattice isomorphism between complete lattices commutes with arbitrary joins, we obtain

$$f(s) = f(s \vee k) \leqslant f(\mathrm{Soc}\,(1/k)) = \bar{f}(\mathrm{Soc}\,(1/k)) = \mathrm{Soc}\,(a'/0') \leqslant \mathrm{Soc}\,(L').$$

For the second statement, set $q := \mathrm{Rad}\,(L)$. Using again the proof of Lemma 2.1, we have

$$q \vee k = \Big(\bigvee_{x \in S(L)} x \Big) \vee k = \bigvee_{x \in S(L)} (x \vee k) \leqslant \bigvee_{z \in S(1/k)} z = \mathrm{Rad}\,(1/k).$$

Since $\bar{f}$ is a lattice isomorphism, we obtain

$$f(q) = f(q \vee k) \leqslant f(\mathrm{Rad}\,(1/k)) = \bar{f}(\mathrm{Rad}\,(1/k)) = \mathrm{Rad}\,(a'/0').$$

But $S(a'/0') \subseteq S(L')$, so $\mathrm{Rad}\,(a'/0') \leqslant \mathrm{Rad}\,(L')$, and hence $f(q) \leqslant \mathrm{Rad}\,(L')$, as desired. □

Let $\mathcal{LM}_c$ be the full subcategory of $\mathcal{LM}$ consisting of all complete modular lattices. Clearly we may define a preradical on $\mathcal{LM}_c$ as being a subfunctor of the identity functor $1_{\mathcal{LM}_c}$ of $\mathcal{LM}_c$.

Proposition 2.2 can now be reformulated as follows: the "socle" and "radical" define the preradicals

$$\sigma : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c\,,\ \sigma(L) = \mathrm{Soc}\,(L)/0,$$

and

$$\varrho : \mathcal{LM}_c \longrightarrow \mathcal{LM}_c\,,\ \varrho(L) = \mathrm{Rad}\,(L)/0,$$

on $\mathcal{LM}_c$, respectively.

Remark 2.3. The fact that the functor σ defined above is a lattice preradical on $\mathcal{LM}_c$ follows from a more general result of [4] involving the trace $\mathrm{Tr}\,(\mathcal{X}, L)$ of a cohereditary class of lattices $\mathcal{X}$ in a complete modular lattice L. However, we preferred in this paper to provide a direct proof of it. □

3. The socle, radical, and Jacobson radical of compactly generated modular lattices

In this section we present equivalent forms of the socle and radical of a complete modular lattice L in case additionally L is compactly generated. In particular, we show that for such an L we have $\mathrm{Rad}\,(L) = \mathrm{Jac}\,(L)$.

Recall that an element c of a lattice L is called *compact* in L if whenever $c \leqslant \bigvee_{x\in A} x$ for a subset A of L, there exists a finite subset F of A such that $c \leqslant \bigvee_{x\in F} x$. One denotes by $K(L)$ the set of all compact elements of L. The lattice L is said to be *compact* if 1 is a compact element in L, and *compactly generated* if it is complete and every element of L is a join of compact elements.

Also, recall that for a lattice L we have denoted by $M(L)$ the set, possibly empty, of all maximal elements of $L \setminus \{1\}$ and by $\mathrm{Jac}\,(L)$ the *Jacobson radical* of L, i.e.,

$$\mathrm{Jac}\,(L) := \bigwedge M(L).$$

Proposition 3.1. *The following assertions hold for a complete lattice L.*

(1) $\mathrm{Soc}\,(L) \leqslant \bigwedge E(L)$ *and* $\mathrm{Rad}\,(L) \leqslant \mathrm{Jac}\,(L)$.

(2) *If additionally L is a compactly generated modular lattice, then*

$$\mathrm{Soc}\,(L) = \bigwedge E(L) \quad \textit{and} \quad \mathrm{Rad}\,(L) = \mathrm{Jac}\,(L).$$

Proof. (1) If $a \in A(L)$ and $e \in E(L)$, then $a \wedge e \neq 0$, and since a is an atom, it follows that $a \leqslant e$. Thus, because L is complete, we have

$$\mathrm{Soc}\,(L) = \bigvee_{a\in A(L)} a \leqslant \bigwedge_{e\in E(L)} e = \bigwedge E(L).$$

Now, let $s \in S(L)$ and $m \in M(L)$. If we assume that $s \nleqslant m$, then we have $m < s \vee m$, so $s \vee m = 1$. Because s is small in L, it follows that $m = 1$, which is a contradiction. Consequently, $s \leqslant m$. Because L is complete, we have

$$\mathrm{Rad}\,(L) = \bigvee_{s\in S(L)} s \leqslant \bigwedge_{m\in M(L)} m = \bigwedge M(L).$$

(2) Assume that L is a compactly generated modular lattice. Then, the equality $\mathrm{Soc}\,(L) = \bigwedge E(L)$ is exactly [7, Chapter III, Proposition 6.7].

We are now going to show that $\mathrm{Jac}\,(L) \leqslant \mathrm{Rad}\,(L)$. Because L is compactly generated, $\mathrm{Jac}\,(L)$ is a join of compact elements of L.

Let $c \in K(L)$ with $c \leqslant \mathrm{Jac}\,(L)$. We claim that $c \in S(L)$. Indeed, let $y \in L$ with $c \vee y = 1$, and show that $y = 1$. If not, we have $y \in L \setminus \{1\}$, and then, clearly $c \nleqslant y$.

Set $A_y := \{ z \in L \mid y \leqslant z,\, c \nleqslant z \}$. Clearly $A_y \neq \varnothing$ because $y \in A_y$. Let $\varnothing \neq T \subseteq A_y$ be a chain, and set $t := \bigvee T$. Then $t \in A_y$, for otherwise we would have $c \leqslant t$, and hence it would exist a finite subset F of T such that $c \leqslant \bigvee F$, i.e., $c \leqslant a$ for some $a \in T$ (because T is a chain), which is a contradiction. This shows that A_y is an inductive set. By Zorn's Lemma, A_y has a maximal element, say m.

Now, we are going to prove that $m \in M(L)$. To do that, let $u \in L$ with $m \leqslant u$. If $c \leqslant u$, then we deduce that $1 = c \vee y \leqslant u \vee m \leqslant u$, i.e., $u = 1$. If $c \nleqslant u$, then $u \in A_y$, so, because m is a maximal element of A_y, we deduce that $u = m$. This shows that $m \in M(L)$. Then $c \leqslant \mathrm{Jac}\,(L) \leqslant m$, which contradicts the fact that $m \in A_y$.

In conclusion, the assumption that $y \neq 1$ produces a contradiction, and therefore we have necessarily $y = 1$, which shows that $c \in S(L)$. This proves our claim. Consequently,

$$\mathrm{Jac}\,(L) = \bigvee \{ c \in K(L) \mid c \leqslant \mathrm{Jac}\,(L) \} \leqslant \bigvee S(L) = \mathrm{Rad}\,(L),$$

which finishes the proof. □

We denote by $\mathcal{LM}_u$ (respectively, $\mathcal{LM}_{cg}$) the full subcategory of $\mathcal{LM}$ consisting of all upper continuous (respectively, compactly generated) modular lattices.

Proposition 3.2. *The following statements hold for an upper continuous modular lattice L.*

(1) $\mathrm{Rad}\,(1/\mathrm{Rad}\,(L)) = \mathrm{Rad}\,(L)$ *if additionally L is assumed to be compactly generated, i.e., ϱ is a radical on $\mathcal{LM}_{cg}$.*

(2) $\mathrm{Soc}\,(\mathrm{Soc}\,(L)/0) = \mathrm{Soc}\,(L)$*, i.e., σ is an idempotent preradical on $\mathcal{LM}_c$.*

(3) $\mathrm{Soc}\,(a/0) = a \wedge \mathrm{Soc}\,(L)$ *for any $a \in L$, i.e., σ is a hereditary preradical on $\mathcal{LM}_u$.*

Proof. (1) First, observe that for any $a \in L$, the interval $1/a$ is compactly generated if L is so. By Lemma 3.1, $\mathrm{Rad}\,(L) = \bigwedge M(L)$, hence, if we denote $q := \mathrm{Rad}\,(L)$, then we have

$$\mathrm{Rad}\,(1/\mathrm{Rad}\,(L)) = \mathrm{Rad}\,(1/q) = \bigwedge M(1/q) = \bigwedge M(L) = \mathrm{Rad}\,(L)$$

because $M(1/q) = M(L)$.

(2) Clearly, $A(L) = A(\mathrm{Soc}\,(L)/0)$ entails the desired equality.

(3) Follows at once from [3, Proposition 1.1(1)]. □

Remark 3.3. The results in Propositions 3.1 and 3.2(1) are also true for classes of complete modular lattices other than the compactly generated ones (see [4]). □

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