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Generalized set-valued nonlinear variational-like inequalities

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Abstract - In this paper, we study a new class of generalized set-valued nonlinear variational-like inequalities (GSNVLI) in Hilbert spaces. By applying the auxiliary principle technique, we establish existence and uniqueness theorem for a solution of (GSNVLI) problem. We further suggest an algorithm to obtain a solution of (GSNVLI) problem and establish a convergence result in which sequence generated by the proposed algorithm converges to a solution of the (GSNVLI) problem. Our results represent refinement and improvement of the previously known results in variational inequalities.

Key words and phrases : variational-like inequalities, auxiliary principle, strongly monotone mapping, iterative algorithm.

Mathematics Subject Classification (2010) **:** 47J20, 65K10, 65K15, 90C33.

1. Introduction

Variational inequality theory provides techniques to solve a variety of problems arising in mathematical physics, economics, optimization, nonlinear programming, transportation and engineering; see [9, 10, 14]. In recent years, various generalizations and extensions of variational inequalities are suggested and analyzed in different directions using novel and innovative techniques; for details, we refer to [24].

One of the most important problems in variational inequality theory is the development of efficient and implementable iterative algorithms for solving various classes of variational inequalities and variational inclusions. The most effective numerical technique to solve variational inequalities is the projection method and its variant forms. It is worth mentioning that the projection type technique cannot be used to suggest iterative algorithms for variational-like inequalities, since it is not possible to find projection of the solution. Glowinski et al. (see [10]) suggested another technique, which does not depend on the projection. This technique is called the auxiliary principle technique. Noor (see [21, 22, 23]) modified and extended the auxiliary principle technique to study the existence of a solution of some special cases of variational-like inequalities and to develop some iterative algorithms. Chidume et al. (see [2]), Huang and Deng (see [11]) and Zeng et al. (see [29]) extended this technique to suggest and analyze a number of algorithms for solving various classes of variational inequalities. Variational-like inequality was introduced by Parida and Sen (see [25]) in 1987. It is an important and useful generalization of variational inequality. In recent years a number of papers appeared to deal with variational-like inequalities see, for example, [12, 13, 15, 16].

Motivated and inspired by the research going in this direction, in this paper we introduce and study a generalized set-valued nonlinear variational-like inequality, which includes several kinds of variational-like inequalities as special cases. An existence result of solution for this problem is established. Using auxiliary principle technique, we construct an iterative algorithm for finding the approximate solution of the generalized set-valued nonlinear variational-like inequality and obtain the convergence of the algorithm under certain conditions. The results proved in this paper represent a significant improvement and refinement of previously known results in this field.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $CB(\mathcal{H})$ be the family of all nonempty closed bounded subsets of $\mathcal{H}$. Let $A, B, C : \mathcal{H} \to CB(\mathcal{H})$ be set valued operators. Given operators $N : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, we consider the problem of finding $x \in \mathcal{H}$, $w \in A(x)$, $v \in B(x)$, $u \in C(x)$ such that for a given $f \in \mathcal{H}$,

$$\langle N(w, v, u) - f,\, \eta(y, x)\rangle + a(x, y - x) + b(x, y) - b(x, x) \geq 0\,, \quad \forall\, y \in \mathcal{H}, \tag{2.1}$$

where the forms $a(\cdot, \cdot), b(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to (-\infty, +\infty)$ have the properties :

(A1) a is a continuous function which is linear in both arguments;

(A2) $\exists\, \nu > 0$ such that $a(x, x) \geq \nu \|x\|^2\,, \ \forall\, x \in \mathcal{H}$;

(A3) $\exists\, \omega > 0$ such that $a(x, y) \leq \omega \|x\| \cdot \|y\|\,, \ \forall\, x, y \in \mathcal{H}$,

it follows from (A2) and (A3) that $\nu \leq \omega$ and

(A4) $|a(x, y)| \leq \omega \|x\| \cdot \|y\|\,, \ \forall\, x, y \in \mathcal{H}$.

The form $b(\cdot, \cdot)$ satisfies the following conditions:

(B1) $b(x, y)$ is linear in the first argument;

(B2) $b(x,y)$ is bounded, that is, there exists a constant $\varrho > 0$ such that

$$|b(x,y)| \leq \varrho \, \|x\| \cdot \|y\| \;, \quad \forall \, x,y \in \mathcal{H} \, .$$

(B3) $b(x,y) - b(x,z) \leq b(x, y-z) \,, \quad \forall \, x,y,z \in \mathcal{H}$.

In view of (B2) and (B3), we can see for all $x,y,z \in \mathcal{H}$

$$\begin{aligned} b(x,y) - b(x,z) &\leq b(x,y-z) \leq \varrho \, \|x\| \, \|y-z\| \\ b(x,z) - b(x,y) &\leq b(x,z-y) \leq \varrho \, \|x\| \, \|z-y\| \;, \end{aligned}$$

that is

(B4) $|b(x,y) - b(y,z)| \leq \varrho \, \|x\| \, \|y-z\|$, for all $x,y,z \in \mathcal{H}$.

The problem (2.1) is called the generalized set-valued nonlinear variational-like inequality (GSNVLI) problem. We now consider some special cases of the inequality (2.1).

Special Cases :

1. If we consider $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $a \equiv 0$, then we get following variational-like inequality from the inequality (2.1): find $x \in \mathcal{H}$ such that $w \in A(x)$, $v \in B(x)$ such that, for a given $f \in \mathcal{H}$,

$$\langle N(w,v) - f, \, \eta(y,x) \rangle + b(x,y) - b(x,x) \geq 0 \,, \ \ \forall \, y \in \mathcal{H} \,,$$

   which was studied by Ding et al. (see [8]) in Banach spaces.

2. If we consider $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $a \equiv 0$ and $f = 0$, then we get following variational-like inequality from the inequality (2.1): find $x \in \mathcal{H}$, $w \in A(x)$, $v \in B(x)$ such that

$$\langle N(w,v), \, \eta(y,x) \rangle + b(x,y) - b(x,x) \geq 0 \,, \ \ \forall \, y \in \mathcal{H} \,,$$

   which was studied by Ding (see [6]) and Noor (see [22]).

3. If we consider $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $a \equiv 0$, $b \equiv 0$ and $f = 0$, then we get following variational-like inequality from the inequality (2.1): find $x \in \mathcal{H}$, $w \in A(x)$, $v \in B(x)$ such that

$$\langle N(w,v), \, \eta(y,x) \rangle \geq 0 \,, \ \ \forall \, y \in \mathcal{H} \,,$$

   which was studied by Noor (see [23]).

4. If we consider $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $\eta(x,y) = x - y$, $a \equiv 0$ and $f = 0$, then we get following variational-like inequality from the inequality (2.1): find $x \in \mathcal{H}$, $w \in A(x)$, $v \in B(x)$ such that

$$\langle N(w,v), \, y - x \rangle + b(x,y) - b(x,x) \geq 0 \,, \ \ \forall \, y \in \mathcal{H} \,,$$

   which was studied by Noor (see [22]).

5. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, then the generalized set valued nonlinear variational-like inequality (2.1) is equivalent to find $x \in \mathcal{H}$ such that, for a given $f \in \mathcal{H}$,

$$\langle N(Ax, Bx, Cx) - f,\, \eta(y,x)\rangle + a(x, y-x) + b(x,y) - b(x,x) \geq 0\,, \forall\, y \in \mathcal{H}$$

which was studied by Liu et al. (see [15]).

6. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $f = 0$, then we get following inequality from (2.1): find $x \in \mathcal{H}$ such that

$$\langle N(Ax, Bx),\, \eta(y,x)\rangle + a(x, y-x) + b(x,y) - b(x,x) \geq 0\,, \quad \forall\, y \in \mathcal{H}$$

which was studied by Liu et al. (see [17]) in Banach spaces.

7. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x + y$, $a \equiv 0$ and $f = 0$. Then from the inequality (2.1), we get following: find $x \in \mathcal{H}$ such that

$$\langle Ax + Bx,\, \eta(y,z)\rangle + b(x,y) - b(x,x) \geq 0\,, \quad \forall\, y \in \mathcal{H}$$

which was studied by Noor (see [20]).

8. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x - y$, $a \equiv 0$ and $f = 0$. Then from the inequality (2.1), we get following: find $x \in \mathcal{H}$ such that

$$\langle Ax - Bx,\, \eta(y,z)\rangle + b(x,y) - b(x,x) \geq 0\,, \quad \forall\, y \in \mathcal{H}$$

which was studied by Ding (see [5]) in Banach spaces.

9. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x$, $\eta(s,t) = s - t$, $a \equiv 0$ and $f = 0$. Then from the inequality (2.1), we get following: find $x \in \mathcal{H}$ such that

$$\langle Ax,\, y - x\rangle + b(x,y) - b(x,x) \geq 0\,, \quad \forall\, y \in \mathcal{H}$$

which was studied by Bose (see [1]), Duvaut and Lions (see [9]).

10. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x$, $\eta(s,t) = g(s) - g(t)$, where $g : \mathcal{H} \to \mathcal{H}$ is a given mapping and $f = 0$, then from the inequality (2.1), we get following: find $x \in \mathcal{H}$ such that

$$\langle Ax,\, g(y) - g(x)\rangle + a(x, y-x) + b(x,y) - b(x,x) \geq 0\,, \quad \forall\, y \in \mathcal{H}$$

which was studied by Ding and Tarafdar (see [7]).

11. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x - y$, $f = 0$, $a \equiv 0$ and $b(x,y) = g(y)$ for all $x, y \in \mathcal{H}$, where $g : \mathcal{H} \to \mathcal{H}$ be a given mapping, then we get following variational-like inequality from the inequality (2.1): find $x \in \mathcal{H}$ such that

$$\langle A(x) - B(x),\, \eta(y,x)\rangle + g(y) - g(x) \geq 0\,, \quad \forall\, y \in \mathcal{H}\,,$$

which was studied by Zeng (see [28]).

12. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x - y$ and $a \equiv 0$, $b \equiv 0$ and $f = 0$, then we get following from (2.1): find $x \in \mathcal{H}$ such that

$$\langle Ax - Bx,\, \eta(y,x)\rangle \geq 0\,,\ \ \forall\, y \in \mathcal{H}$$

inequality of type above is studied by Noor (see [19]).

13. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x$, $\eta(s,t) = s - t$, $a \equiv 0$, $f = 0$ and $b(s,t) = \phi(t)$ where $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a given mapping. Then from the inequality (2.1), we get following: find $x \in \mathcal{H}$ such that

$$\langle Ax,\, y - x\rangle + \phi(y) - \phi(x) \geq 0\,,\ \ \forall\, y \in \mathcal{H}$$

which was studied by Cohen (see [3]).

14. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x$, $\eta(s,t) = g(s) - g(t)$ where $g : \mathcal{H} \to \mathcal{H}$ is a given mapping, $a \equiv 0$, $b \equiv 0$ and $f = 0$. Then from the inequality (2.1), we get following: find $x \in \mathcal{H}$ such that

$$\langle Ax,\, g(y) - g(x)\rangle \geq 0\,,\ \ \forall\, y \in \mathcal{H}$$

which was studied by Yao (see [26]).

15. If we take $A, B, C : \mathcal{H} \to \mathcal{H}$, $N(x,y,z) = x$, $f = 0$, $a \equiv 0$, $\eta(s,t) = s - t$ and $b(x,y) = g(y)$, where $g : \mathcal{H} \to \mathcal{H}$ be a mapping, for all $x, y \in \mathcal{H}$ then, we get following variational-like inequality from the inequality (2.1): find $x \in \mathcal{H}$ such that

$$\langle A(x),\, y - x\rangle + g(y) - g(x) \geq 0\,,\ \ \forall\, y \in \mathcal{H}\,,$$

which was studied by Zeng (see [27]).

In brief, for appropriate and suitable choice of the mappings N, η, A, B, C, the functions a, b and choice of $f \in \mathcal{H}$, one can obtain a number of the known classes of variational inequalities and variational-like inequalities as special cases from the problem (2.1).

We also need the following concepts:

Definition 2.1. *For all $x_1, x_2 \in \mathcal{H}$, the operator $N(\cdot,\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be*

(i) strongly monotone in the first argument with constant ζ, if there exists a constant $\zeta > 0$ such that

$$\langle N(w_1,\cdot,\cdot) - N(w_2,\cdot,\cdot),\ x_1 - x_1\rangle \geq \zeta\,\|x_1 - x_2\|^2\ ;$$

$\forall\, w_1 \in A(x_1), w_2 \in A(x_2)$*;*

(ii) Lipschitz continuous in the first argument with constant α, if there exists a constant $\alpha > 0$ such that

$$\|N(x_1,\cdot,\cdot) - N(x_2,\cdot,\cdot)\| \leq \alpha \|x_1 - x_2\| .$$

In a similar way, we can define Lipschitz continuity of the $N(\cdot,\cdot,\cdot)$ in the second and third arguments.

Definition 2.2. *An operator $\eta(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be*

(i) strongly monotone with constant σ, if there exists a constant $\sigma > 0$ such that

$$\langle \eta(x,y),\ x-y\rangle \geq \sigma \|x-y\|^2 , \ \ \forall\, x,y \in \mathcal{H};$$

(ii) Lipschitz continuous with constant τ, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau \|x-y\| , \ \ \forall\, x,y \in \mathcal{H}.$$

Assumption 2.1. The operator $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ satisfies the following conditions:

(1) $\eta(x,y) = \eta(x,z) + \eta(z,y) , \quad \forall\, x,y,z \in \mathcal{H};$

(2) $\eta(x,y) = -\eta(y,x) , \quad \forall\, x,y \in \mathcal{H}.$

Clearly $\eta(x,x) = 0$ for all $x \in \mathcal{H}$.

Let $\widehat{H}(\cdot,\cdot)$ be the Hausdorff metric on $CB(\mathcal{H})$ defined by

$$\widehat{H}(A,B) = \max\left\{ \sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A) \right\}, \ \ \forall\, A,B \in CB(\mathcal{H}).$$

Definition 2.3. *An operator $A : \mathcal{H} \to CB(\mathcal{H})$ is said to be $\widehat{H}-$Lipschitz continuous with constant λ if there exists a constant $\lambda > 0$ such that*

$$\widehat{H}(Ax, Ay) \leq \lambda \|x-y\| , \quad \forall\, x,y \in \mathcal{H} .$$

Lemma 2.1. (see [18]) *Let (X,d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping. Then for any $\varepsilon > 0$ and $x,y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \leq (1+\varepsilon)\, \widehat{H}(T(x), T(y)) .$$

Lemma 2.2. (see [18]) *Let (X,d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping satisfying*

$$\widehat{H}(T(x), T(y)) \leq k\, d(x,y) \quad \forall x,y \in X ,$$

where $0 \leq k < 1$ is a constant. Then the mapping T has a fixed point in X.

3. Auxiliary problem and existence of solution

Theorem 3.1. *Let the operator $N(\cdot,\cdot,\cdot)$ be strongly monotone in the first argument with constant $\zeta > 0$ and Lipschitz continuous in the first, second and third arguments with constants $\alpha, \beta, \gamma > 0$ respectively. Let $A, B, C : \mathcal{H} \to CB(\mathcal{H})$ be set valued $\widehat{H}-$Lipschitz continuous mappings with constants λ, μ, δ respectively. Let $\eta(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\tau > 0$ and let the forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ satisfy the conditions* (A1)-(A3) *and* (B1)-(B3), *respectively. If Assumption* 2.1 *holds and*

$$0 < \rho < \frac{2\,(\zeta - (\kappa + \vartheta)}{\beta^2\lambda^2 + (\kappa + \vartheta)^2}\,, \quad \rho(\kappa + \vartheta) < 1, \;\; (\kappa + \vartheta) < \zeta\,, \tag{3.1}$$

where

$$\begin{aligned} \kappa &= \alpha\lambda\sqrt{1 - 2\sigma + \tau^2} \\ \vartheta &= \tau\,(\beta\mu + \gamma\delta) + (\omega + \varrho)\;, \end{aligned} \tag{3.2}$$

then the generalized set-valued variational-like inequality problem (2.1) *has a unique solution.*

Proof. (a) Uniqueness.

Let $x_1, x_2 \in H$, $x_1 \neq x_2$ be two solutions of (2.1), that is

$$\langle N(w_1, v_1, u_1) - f, \eta(y, x_1)\rangle + a(x_1, y - x_1) + b(x_1, y) - b(x_1, x_1) \geq 0, \;\forall y \in K, \tag{3.3}$$

and

$$\langle N(w_2, v_2, u_2) - f,\, \eta(y, x_2)\rangle + a(x_2, y - x_2) + b(x_2, y) - b(x_2, x_2) \geq 0, \;\forall y \in K. \tag{3.4}$$

Taking $y = x_2$ in (3.3) and $y = x_1$ in (3.4), we get

$$\langle N(w_1, v_1, u_1) - f, \eta(x_2, x_1)\rangle + a(x_1, x_2 - x_1) + b(x_1, x_2) - b(x_1, x_1) \geq 0, \;\forall y \in K, \tag{3.5}$$

and

$$\langle N(w_2, v_2, u_2) - f, \eta(x_1, x_2)\rangle + a(x_2, x_1 - x_2) + b(x_2, x_1) - b(x_2, x_2) \geq 0, \;\forall y \in K. \tag{3.6}$$

Adding inequalities (3.5) and (3.6), using assumptions on $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we obtain

$$\begin{aligned} &\langle N(w_1, v_1, u_1) - N(w_2, v_2, u_2), \eta(x_1, x_2)\rangle \\ &\qquad\qquad \leq b(x_1 - x_2,\, x_2 - x_1) + a(x_1 - x_2,\, x_2 - x_1). \end{aligned} \tag{3.7}$$

Inequality (3.7) can be written as

$$\begin{aligned}\langle N(w_1,v_1,u_1)-N(w_2,v_1,u_1),x_1-x_2\rangle \\ &\le \langle N(w_1,v_1,u_1)-N(w_2,v_1,u_1),x_1-x_2-\eta(x_1,x_2)\rangle \\ &\quad+\langle N(w_2,v_2,u_2)-N(w_2,v_1,u_1),\eta(x_1,x_2)\rangle \\ &\quad+b(x_1-x_2,\,x_2-x_1)+a(x_1-x_2,\,x_2-x_1) \\ &= \langle N(w_1,v_1,u_1)-N(w_2,v_1,u_1),x_1-x_2-\eta(x_1,x_2)\rangle \\ &\quad+\langle N(w_2,v_2,u_2)-N(w_2,v_2,u_1),\eta(x_1,x_2)\rangle \\ &\quad+\langle N(w_2,v_2,u_1)-N(w_2,v_1,u_1),\eta(x_1,x_2)\rangle \\ &\quad+b(x_1-x_2,\,x_2-x_1)+a(x_1-x_2,\,x_2-x_1). \end{aligned} \tag{3.8}$$

Using the strong monotonicity of $N(\cdot,\cdot,\cdot)$ in the first argument with constant ζ, the Cauchy-Schwartz inequality and (3.8), we obtain

$$\begin{aligned}\zeta\|x_1-x_2\|^2 &\le \|N(w_1,v_1,u_1)-N(w_2,v_1,u_1)\|\,\|x_1-x_2-\eta(x_1,x_2)\| \\ &\quad+\|N(w_2,v_2,u_2)-N(w_2,v_2,u_1)\|\,\|\eta(x_1,x_2)\| \\ &\quad+\|N(w_2,v_2,u_1)-N(w_2,v_1,u_1)\|\,\|\eta(x_1,x_2)\| \\ &\quad+\varrho\|x_1-x_2\|^2+\omega\|x_1-x_2\|^2. \end{aligned} \tag{3.9}$$

Since η is strongly monotone with constant σ and Lipschitz continuous with constant τ, so we have

$$\begin{aligned}\|x_1-x_2-\eta(x_1,x_2)\|^2 &= \|x_1-x_2\|^2-2\langle\eta(x_1,x_2),\,x_1-x_2\rangle+\left\|\eta(x_1,x_2)^2\right\| \\ &\le (1-2\sigma+\tau^2)\|x_1-x_2\|^2. \end{aligned} \tag{3.10}$$

Using the Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the first argument with constant α and $\widehat{H}-$Lipschitz continuity of A with constant λ, we have

$$\begin{aligned}\|N(w_1,v_1,u_1)-N(w_2,v_1,u_1)\| &\le \alpha\|v_1-v_2\| \\ &\le \alpha(1+\varepsilon)\widehat{H}(Ax_1,Ax_2) \\ &\le \alpha\lambda(1+\varepsilon)\|x_1-x_2\|. \end{aligned} \tag{3.11}$$

Similarly, using the Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the second argument with constant β and $\widehat{H}-$Lipschitz continuity of B with constant μ, we get

$$\|N(w_2,v_2,u_1)-N(w_2,v_1,u_1)\| \le \beta\mu(1+\varepsilon)\|x_1-x_2\|. \tag{3.12}$$

Also, the Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the third argument with constant γ and $\widehat{H}-$Lipschitz continuity of C with constant δ, gives

$$\|N(w_2,v_2,u_1)-N(w_2,v_2,u_2)\| \le \gamma\delta(1+\varepsilon)\|x_1-x_2\|. \tag{3.13}$$

Combining (3.9), (3.10), (3.11), (3.12), (3.13) and Lipschitz continuity of η with constant τ, we have

$$\zeta\|x_1-x_2\|^2\le\left[(1+\varepsilon)\left\{\alpha\lambda\sqrt{1-2\sigma+\tau^2}+\tau(\gamma\delta+\beta\mu)\right\}+(\varrho+\omega)\right]\|x_1-x_2\|^2.$$

Letting $\varepsilon \to 0$, we get

$$\begin{aligned}\zeta \|x_1 - x_2\|^2 &\le \left[\alpha\lambda\sqrt{1-2\sigma+\tau^2} + \tau(\gamma\delta+\beta\mu) + (\varrho+\omega)\right] \|x_1 - x_2\|^2 \\ &= (\kappa+\vartheta)\|x_1 - x_2\|^2 , \quad \text{using (3.2)}\end{aligned}$$

which implies that

$$(\zeta - (\kappa+\vartheta)\|x_1 - x_2\|^2 \le 0 ,$$

since $(\kappa+\vartheta) < \zeta$, it follows that $x_1 = x_2$, the uniqueness of the solution.

(b) Existence.

We use the auxiliary principle technique of Glowinski et al. (see [10]) as developed by Noor (see [21, 22, 23]) to prove the existence of a solution of the generalized set-valued variational-like inequality (2.1). For a given $x \in \mathcal{H}$, we consider the problem of finding $z \in \mathcal{H}$, $w \in A(x), v \in B(x), u \in C(x)$ satisfying the variational-like inequality

$$\begin{aligned}\langle z, y - z\rangle \ge \langle x, y-z\rangle &- \rho\,\langle N(w,v,u) - f, \eta(y,z)\rangle \\ &- \rho a(x, y-z) - \rho b(x,y) + \rho b(x,z), \qquad (3.14)\end{aligned}$$

for all $y \in \mathcal{H}$, where $\rho > 0$ is a constant.

The relation (3.14) defines a mapping $x \mapsto z$. In order to prove the existence of a solution of (2.1), it is sufficient to show that the mapping $x \mapsto z$ defined by (3.14) has a fixed point belongs to $\mathcal{H}$ satisfying (2.1). We denote the mapping $x \mapsto z$ by F.

For any $x_1 \in \mathcal{H}$, there exist $z_1 \in F(x_1)$, $w_1 \in A(x_1)$, $v_1 \in B(x_1)$ and $u_1 \in C(x_1)$ such that

$$\begin{aligned}\langle z_1, y - z_1\rangle \ge \langle x_1, y-z_1\rangle &- \rho\,\langle N(w_1,v_1,u_1) - f, \eta(y,z_1)\rangle \\ &- \rho a(x_1, y-z_1) - \rho b(x_1,y) + \rho b(x_1,z_1) , \quad \forall\, y \in \mathcal{H} . \quad (3.15)\end{aligned}$$

For any $x_2 \in \mathcal{H}$, there exist $z_2 \in F(x_2)$, $w_2 \in A(x_2)$, $v_2 \in B(x_2)$ and $u_2 \in C(x_2)$ such that

$$\begin{aligned}\langle z_2, y - z_2\rangle \ge \langle x_2, y-z_2\rangle &- \rho\,\langle N(w_2,v_2,u_2) - f, \eta(y,z_2)\rangle \\ &- \rho a(x_2, y-z_2) - \rho b(x_2,y) + \rho b(x_2,z_2) , \quad \forall\, y \in \mathcal{H} . \quad (3.16)\end{aligned}$$

Since (3.15) and (3.16) hold for all $y \in \mathcal{H}$, replacing y in (3.15) and (3.16) by z_2 and z_1 respectively, we get

$$\begin{aligned}\langle z_1, z_2 - z_1\rangle \ge \langle x_1, z_2-z_1\rangle &- \rho\,\langle N(w_1,v_1,u_1) - f, \eta(z_2,z_1)\rangle \\ &- \rho a(x_1, z_2-z_1) - \rho b(x_1,z_2) + \rho b(x_1,z_1) . \qquad (3.17)\end{aligned}$$

$$\begin{aligned}\langle z_2, z_1 - z_2\rangle \ge \langle x_2, z_1-z_2\rangle &- \rho\,\langle N(w_2,v_2,u_2) - f, \eta(z_1,z_2)\rangle \\ &- \rho a(x_2, z_1-z_2) - \rho b(x_2,z_1) + \rho b(x_2,z_2) . \qquad (3.18)\end{aligned}$$

Adding (3.17) and (3.18) and rearranging the terms we get

$$\begin{aligned}
&\langle z_1 - z_2\,,\, z_1 - z_2\rangle \\
&\quad \le \langle x_1 - x_2\,,\, z_1 - z_2\rangle - \rho\,\langle N(w_1, v_1, u_1) - N(w_2, v_2, u_2), \eta(z_1, z_2)\rangle \\
&\qquad - \rho a(x_1 - x_2, z_1 - z_2) + \rho b(x_1 - x_2, z_2 - z_1) \\
&\quad \le \langle x_1 - x_2\,,\, z_1 - z_2\rangle - \rho\,\langle N(w_1, v_1, u_1) - N(w_2, v_1, u_1), \eta(z_1, z_2)\rangle \\
&\qquad - \rho\,\langle N(w_2, v_1, u_1) - N(w_2, v_2, u_2), \eta(z_1, z_2)\rangle \\
&\qquad - \rho a(x_1 - x_2, z_1 - z_2) + \rho b(x_1 - x_2, z_2 - z_1) \\
&\quad \le \langle x_1 - x_2\,,\, z_1 - z_2\rangle - \rho\,\langle N(w_1, v_1, u_1) - N(w_2, v_1, u_1),\, z_1 - z_2\rangle \\
&\qquad + \rho\,\langle N(w_1, v_1, u_1) - N(w_2, v_1, u_1),\, z_1 - z_2 - \eta(z_1, z_2)\rangle \\
&\qquad - \rho\,\langle N(w_2, v_1, u_1) - N(w_2, v_2, u_2), \eta(z_1, z_2)\rangle \\
&\qquad - \rho a(x_1 - x_2, z_1 - z_2) + \rho b(x_1 - x_2, z_2 - z_1) \\
&\quad \le \langle x_1 - x_2 - \rho\,\{N(w_1, v_1, u_1) - N(w_2, v_1, u_1)\}\,,\, z_1 - z_2\rangle \\
&\qquad + \rho\,\langle N(w_1, v_1, u_1) - N(w_2, v_1, u_1),\, z_1 - z_2 - \eta(z_1, z_2)\rangle \\
&\qquad - \rho\,\langle N(w_2, v_1, u_1) - N(w_2, v_2, u_2), \eta(z_1, z_2)\rangle \\
&\qquad - \rho a(x_1 - x_2, z_1 - z_2) + \rho b(x_1 - x_2, z_2 - z_1)\,. \qquad (3.19)
\end{aligned}$$

It follows from (3.19) that

$$\begin{aligned}
\|z_1 - z_2\|^2 \le\ & \|x_1 - x_2 - \rho\,\{N(w_1, v_1, u_1) - N(w_2, v_2, u_1)\}\|\,\|z_1 - z_2\| \\
& + \rho\,\|N(w_1, v_1, u_1) - N(w_2, v_1, u_1)\|\,\|z_1 - z_2 - \eta(z_1, z_2)\| \\
& + \rho\,\|N(w_2, v_1, u_1) - N(w_2, v_2, u_2)\|\,\|\eta(z_1, z_2)\| \\
& + \rho\omega\,\|x_1 - x_2\|\,\|z_1 - z_2\| + \rho\varrho\,\|x_1 - x_2\|\,\|z_2 - z_1\|\,. \qquad (3.20)
\end{aligned}$$

Using strong monotonicity and Lipschitz continuity of $N(\cdot, \cdot, \cdot)$ in the first argument with constants ζ and α respectively, and $\widehat{H}-$Lipschitz continuity of A with constant λ, we have

$$\begin{aligned}
&\|x_1 - x_2 - \rho\,\{N(w_1, v_1, u_1) - N(w_2, v_1, u_1)\}\|^2 \\
&\qquad = \|x_1 - x_2\|^2 - 2\rho\,\langle N(w_1, v_1, u_1) - N(w_2, v_1, u_1),\, x_1 - x_2\rangle \\
&\qquad\quad + \rho^2\,\|N(w_1, v_1, u_1) - N(w_2, v_1, u_1)\|^2 \\
&\qquad \le (1 - 2\rho\zeta)\,\|x_1 - x_2\|^2 + \rho^2\alpha^2\,\|w_1 - w_2\|^2 \\
&\qquad \le (1 - 2\rho\zeta)\,\|x_1 - x_2\|^2 + \rho^2\alpha^2\left\{(1 + \varepsilon)\widehat{H}(Ax_1, Ax_2)\right\}^2 \\
&\qquad \le \left(1 - 2\rho\zeta + \rho^2\alpha^2\lambda^2(1 + \varepsilon)^2\right)\|x_1 - x_2\|^2\,. \qquad (3.21)
\end{aligned}$$

Using the strong monotonicity and Lipschitz continuity of η with constants σ and τ respectively, we have

$$\begin{aligned}
\|z_1 - z_2 - \eta(z_1, z_2)\|^2 &\le \|z_1 - z_2\|^2 - 2\,\langle \eta(z_1, z_2),\, z_1 - z_2\rangle + \|\eta(z_1, z_2)\|^2 \\
&\le (1 - 2\sigma + \tau^2)\,\|z_1 - z_2\|^2\,. \qquad (3.22)
\end{aligned}$$

Using the Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the first argument with constant α and $\widehat{H}-$Lipschitz continuity of A, we get

$$\|N(w_1,v_1,u_1)-N(w_2,v_1,u_1)\| \leq \alpha \|w_1-w_2\| \leq \alpha\lambda(1+\varepsilon)\|x_1-x_2\|. \tag{3.23}$$

Also, by using the Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the second and third arguments with constants β and γ respectively, $\widehat{H}-$Lipschitz continuity of B with constant μ and $\widehat{H}-$Lipschitz continuity of C with constant δ, we have

$$\begin{aligned}\|N(w_2,v_1,u_1)-N(w_2,v_2,u_2)\| &\leq \|N(w_2,v_1,u_1)-N(w_2,v_2,u_1)\| \\ &\quad + \|N(w_2,v_2,u_1)-N(w_2,v_2,u_2)\| \\ &\leq \beta\|v_1-v_2\|+\gamma\|u_1-u_2\| \\ &\leq (\beta\mu+\gamma\delta)(1+\varepsilon)\|x_1-x_2\|. \end{aligned}\tag{3.24}$$

Substituting (3.21), (3.22), (3.23) and (3.24) into (3.20), we get

$$\begin{aligned}\|z_1-z_2\| &\leq \Big[\sqrt{1-2\rho\zeta+\rho^2\alpha^2\lambda^2(1+\varepsilon)^2} \\ &\quad +\rho(1+\varepsilon)\Big\{\alpha\lambda\sqrt{1-2\sigma+\tau^2}+\tau(\beta\mu+\gamma\delta)\Big\}+\rho(\omega+\varrho)\Big]\|x_1-x_2\| \\ &= (t(\rho_\varepsilon)+\rho(\kappa_\varepsilon+\vartheta_\varepsilon))\|x_1-x_2\| \\ &= \theta_\varepsilon\|x_1-x_2\|, \end{aligned}\tag{3.25}$$

where

$$\begin{aligned}\theta_\varepsilon &= t(\rho_\varepsilon)+\rho(\kappa_\varepsilon+\vartheta_\varepsilon), \\ t(\rho_\varepsilon) &= \sqrt{1-2\rho\zeta+\rho^2\alpha^2\lambda^2(1+\varepsilon)^2}, \\ \kappa_\varepsilon &= \alpha\lambda(1+\varepsilon)\sqrt{1-2\sigma+\tau^2}, \\ \vartheta_\varepsilon &= \tau(1+\varepsilon)(\beta\mu+\gamma\delta)+(\omega+\varrho). \end{aligned}$$

Using (3.25), we get

$$d(z_1,F(x_2)) = \inf_{z_2\in F(x_2)}\|z_1-z_2\| \leq \theta_\varepsilon\|x_1-x_2\|.$$

Since $z_1\in F(x_1)$ is arbitrary, we get

$$\sup_{z_1\in F(x_1)} d(z_1,F(x_2)) \leq \theta_\varepsilon\|x_1-x_2\|. \tag{3.26}$$

Similarly, we get that

$$\sup_{z_2\in F(x_2)} d(z_2,F(x_1)) \leq \theta_\varepsilon\|x_1-x_2\|. \tag{3.27}$$

From the definition of Hausdorff metric $\widehat{H}$, it follows from (3.26) and (3.27) that

$$\widehat{H}\left(F(x_1), F(x_2)\right) \leq \theta_\varepsilon \left\| x_1 - x_2 \right\|, \quad \forall\, x_1, x_2 \in \mathcal{H}.$$

Letting $\varepsilon \to 0$, we get

$$\widehat{H}\left(F(x_1), F(x_2)\right) \leq \theta \left\| x_1 - x_2 \right\|, \quad \forall\, x_1, x_2 \in \mathcal{H},$$

where,

$$\begin{aligned} \theta &= t(\rho) + \rho\left(\kappa + \vartheta\right), \\ t(\rho) &= \sqrt{1 - 2\rho\zeta + \rho^2\alpha^2\lambda^2}, \\ \kappa &= \alpha\lambda\sqrt{1 - 2\sigma + \tau^2}, \\ \vartheta &= \tau(\beta\mu + \gamma\delta) + (\omega + \varrho). \end{aligned}$$

It follows from (3.1) that $\theta < 1$, hence F is a set valued contraction mapping, by Lemma 2.2, it has a fixed point in $\mathcal{H}$, that is, the mapping $x \mapsto z$, defined by (3.14) has a fixed point in $\mathcal{H}$, which is the solution of the generalized variational-like inequality (2.1). This completes the proof. □

4. Algorithm and convergence result

For a given $x_0 \in H$, $w_0 \in Ax_0$, $v_0 \in Bx_0$, $u_0 \in Cx_0$, from Theorem 3.1, we know that the auxiliary problem (3.14) has a solution x_1, i.e.

$$\begin{aligned} \langle x_1, y - x_1 \rangle \geq \langle x_0, y - x_1 \rangle - \rho \left\langle N(w_0, v_0, u_0) - f, \eta(y, x_1) \right\rangle \\ - \rho a(x_0, y - x_1) - \rho b(x_0, y) + \rho b(x_0, x_1), \end{aligned} \tag{4.1}$$

for all $y \in \mathcal{H}$, where $\rho > 0$ is a constant.
Since $w_0 \in Ax_0 \in CB(\mathcal{H})$, $v_0 \in Bx_0 \in CB(\mathcal{H})$, $u_0 \in Cx_0 \in CB(\mathcal{H})$, by Lemma 2.1 there exist $w_1 \in Ax_1$, $v_1 \in Bx_1$ and $u_1 \in Cx_1$ such that

$$\begin{aligned} \|w_0 - w_1\| &\leq (1+1)\, \widehat{H}(Ax_0, Ax_1) \\ \|v_0 - v_1\| &\leq (1+1)\, \widehat{H}(Bx_0, Bx_1) \\ \|u_0 - u_1\| &\leq (1+1)\, \widehat{H}(Cx_0, Cx_1). \end{aligned}$$

Again by Theorem 3.1, the auxiliary problem (4.1) has a solution $x_2 \in \mathcal{H}$, that is

$$\begin{aligned} \langle x_2, y - x_2 \rangle \geq \langle x_1, y - x_2 \rangle - \rho \left\langle N(w_1, v_1, u_1) - f, \eta(y, x_2) \right\rangle \\ - \rho a(x_1, y - x_2) - \rho b(x_1, y) + \rho b(x_1, x_2), \end{aligned} \tag{4.2}$$

for all $y \in \mathcal{H}$, where $\rho > 0$ is a constant.

For $w_1 \in Ax_1$, $v_1 \in Bx_1$, $u_1 \in Cx_1$, by Lemma 2.1 there exist $w_2 \in Ax_2$, $v_2 \in Bx_2$ and $u_2 \in Cx_2$ such that

$$\|w_1 - w_2\| \leq \left(1 + \frac{1}{2}\right) \widehat{H}(Ax_1, Ax_2)$$
$$\|v_1 - v_2\| \leq \left(1 + \frac{1}{2}\right) \widehat{H}(Bx_1, Bx_2)$$
$$\|u_1 - u_2\| \leq \left(1 + \frac{1}{2}\right) \widehat{H}(Cx_1, Cx_2).$$

By induction, we can get the iterative algorithm for solving the problem (2.1), as follows:

Algorithm 4.1. For given $x_0 \in H$, $w_0 \in Ax_0$, $v_0 \in Bx_0$, $u_0 \in Cx_0$, there exist $\{w_n\}$, $\{v_n\}$, $\{u_n\}$ such that

$$w_n \in Ax_n\,, \quad \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \widehat{H}(Ax_n, Ax_{n+1})$$
$$v_n \in Bx_n\,, \quad \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \widehat{H}(Bx_n, Bx_{n+1})$$
$$u_n \in Cx_n\,, \quad \|u_n - u_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \widehat{H}(Cx_n, Cx_{n+1}),$$

and

$$\begin{aligned}\langle x_{n+1}, y - x_{n+1}\rangle \geq \langle x_n, y - x_{n+1}\rangle - \rho\,\langle N(w_n, v_n, u_n) - f, \eta(y, x_{n+1})\rangle \\ - \rho a(x_n, y - x_{n+1}) - \rho b(x_n, y) + \rho b(x_n, x_{n+1}),\end{aligned} \tag{4.3}$$

for all $y \in H$, where $\rho > 0$ is a constant.

We now prove that the sequences $\{x_n\}$, $\{w_n\}$, $\{v_n\}$, $\{u_n\}$ generated by Algorithm 4.1 converge strongly to a solution (x^*, w^*, v^*, u^*) of the problem (2.1), where $x^* \in \mathcal{H}$, $w^* \in A(x^*)$, $v^* \in B(x^*)$ and $u^* \in C(x^*)$.

Theorem 4.1. *Let a, b, A, B, C, N, η be as in Theorem* 3.1. *If Assumption* 2.1 *holds and* (3.1), (3.2) *are satisfied, then the sequences $\{x_n\}$, $\{w_n\}$, $\{v_n\}$, $\{u_n\}$ generated by Algorithm* 4.1 *converge strongly to x^*, w^*, v^*, u^* respectively, and (x^*, w^*, v^*, u^*) is a solution of the problem* (2.1), *where $x^* \in \mathcal{H}$, $w^* \in A(x^*)$, $v^* \in B(x^*)$ and $u^* \in C(x^*)$.*

Proof. It follows from (4.3), that for any $y \in \mathcal{H}$,

$$\begin{aligned}\langle x_n, y - x_n\rangle \geq \langle x_{n-1}, y - x_n\rangle - \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - f, \eta(y, x_n)\rangle \\ - \rho a(x_{n-1}, y - x_n) - \rho b(x_{n-1}, y) + \rho b(x_{n-1}, x_n),\end{aligned} \tag{4.4}$$

and

$$\langle x_{n+1}, y - x_{n+1}\rangle \geq \langle x_n, y - x_{n+1}\rangle - \rho\,\langle N(w_n, v_n, u_n) - f, \eta(y, x_{n+1})\rangle \\ - \rho a(x_n, y - x_{n+1}) - \rho b(x_n, y) + \rho b(x_n, x_{n+1}). \tag{4.5}$$

Taking $y = x_{n+1}$ in (4.4) and $y = x_n$ in (4.5), we get

$$\begin{aligned}\langle x_n, x_{n+1} - x_n\rangle & \\ &\geq \langle x_{n-1}, x_{n+1} - x_n\rangle - \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - f, \eta(x_{n+1}, x_n)\rangle \\ &\quad - \rho a(x_{n-1}, x_{n+1} - x_n) - \rho b(x_{n-1}, x_{n+1}) + \rho b(x_{n-1}, x_n),\end{aligned} \tag{4.6}$$

and

$$\langle x_{n+1}, x_n - x_{n+1}\rangle \geq \langle x_n, x_n - x_{n+1}\rangle - \rho\,\langle N(w_n, v_n, u_n) - f, \eta(x_n, x_{n+1})\rangle \\ - \rho a(x_n, x_n - x_{n+1}) - \rho b(x_n, x_n) + \rho b(x_n, x_{n+1}). \tag{4.7}$$

Adding (4.6) and (4.7), we get

$$\begin{aligned}\langle x_{n+1} - x_n, x_n - x_{n+1}\rangle & \\ &\geq \langle x_n - x_{n-1}, x_n - x_{n+1}\rangle \\ &\quad - \rho\,\langle N(w_n, v_n, u_n) - N(w_{n-1}, v_{n-1}, u_{n-1}), \eta(x_n, x_{n+1})\rangle \\ &\quad - \rho a(x_{n-1}, x_{n+1} - x_n) - \rho a(x_n, x_n - x_{n+1}) \\ &\quad - \rho b(x_{n-1}, x_{n+1}) + \rho b(x_{n-1}, x_n) - \rho b(x_{n-1}, x_{n+1}) + \rho b(x_{n-1}, x_n).\end{aligned}$$

and therefore,

$$\begin{aligned}&\langle x_n - x_{n+1}, x_n - x_{n+1}\rangle \\ &\quad \leq \langle x_{n-1} - x_n, x_n - x_{n+1}\rangle \\ &\qquad - \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - N(w_n, v_n, u_n), \eta(x_n, x_{n+1})\rangle \\ &\qquad + \rho a\,(x_n - x_{n-1}, x_n - x_{n+1}) + \rho b\,(x_n - x_{n-1}, x_n - x_{n+1}) \\ &\quad = \langle x_{n-1} - x_n, x_n - x_{n+1}\rangle \\ &\qquad - \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - N(w_n, v_{n-1}, u_{n-1}), \eta(x_n, x_{n+1})\rangle \\ &\qquad - \rho\,\langle N(w_n, v_{n-1}, u_{n-1}) - N(w_n, v_n, u_n), \eta(x_n, x_{n+1})\rangle \\ &\qquad + \rho a\,(x_n - x_{n-1}, x_n - x_{n+1}) + \rho b\,(x_n - x_{n-1}, x_n - x_{n+1}) \\ &\quad = \langle x_{n-1} - x_n, x_n - x_{n+1}\rangle \\ &\qquad - \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - N(w_n, v_{n-1}, u_{n-1}), x_n - x_{n+1}\rangle \\ &\qquad + \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - N(w_n, v_{n-1}, u_{n-1}), x_n - x_{n+1} - \eta(x_n, x_{n+1})\rangle \\ &\qquad - \rho\,\langle N(w_n, v_{n-1}, u_{n-1}) - N(w_n, v_n, u_n), \eta(x_n, x_{n+1})\rangle \\ &\qquad + \rho a\,(x_n - x_{n-1}, x_n - x_{n+1}) + \rho b\,(x_n - x_{n-1}, x_n - x_{n+1}) \\ &\quad = \langle x_{n-1} - x_n - \rho\,(N(w_{n-1}, v_{n-1}, u_{n-1}) - N(w_n, v_{n-1}, u_{n-1}))\,, x_n - x_{n+1}\rangle \\ &\qquad + \rho\,\langle N(w_{n-1}, v_{n-1}, u_{n-1}) - N(w_n, v_{n-1}, u_{n-1}), x_n - x_{n+1} - \eta(x_n, x_{n+1})\rangle \\ &\qquad - \rho\,\langle N(w_n, v_{n-1}, u_{n-1}) - N(w_n, v_n, u_n), \eta(x_n, x_{n+1})\rangle \\ &\qquad + \rho a\,(x_n - x_{n-1}, x_n - x_{n+1}) + \rho b\,(x_n - x_{n-1}, x_n - x_{n+1})\,.\end{aligned} \tag{4.8}$$

It follows from (4.8), that

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&\le \|x_{n-1}-x_n-\rho\,(N(w_{n-1},v_{n-1},u_{n-1}) - N(w_n,v_{n-1},u_{n-1}))\|\,\|x_n - x_{n+1}\| \\
&\quad +\rho\,\|N(w_{n-1},v_{n-1},u_{n-1})-N(w_n,v_{n-1},u_{n-1})\|\,\|x_n-x_{n+1}-\eta(x_n,x_{n+1})\| \\
&\quad +\rho\,\|N(w_n,v_{n-1},u_{n-1}) - N(w_n,v_n,u_n)\|\,\|\eta(x_n,x_{n+1})\| \\
&\quad +\rho\omega\,\|x_n - x_{n-1}\|\,\|x_n - x_{n+1}\| + \rho\varrho\,\|x_n - x_{n-1}\|\,\|x_n - x_{n+1}\|\,. \qquad (4.9)
\end{aligned}$$

Using strong monotonicity and Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the first argument with constants ζ and α respectively, and $\widehat{H}-$Lipschitz continuity of A with constant λ, we have

$$\begin{aligned}
&\|x_{n-1} - x_n - \rho\,(N(w_{n-1},v_{n-1},u_{n-1}) - N(w_n,v_{n-1},u_{n-1}))\|^2 \\
&= \|x_{n-1}-x_n\|^2-2\rho\,\langle N(w_{n-1},v_{n-1},u_{n-1})-N(w_n,v_{n-1},u_{n-1}),x_{n-1}-x_n\rangle \\
&\quad + \rho^2\,\|N(w_{n-1},v_{n-1},u_{n-1}) - N(w_n,v_{n-1},u_{n-1})\|^2 \\
&\quad \le (1-2\rho\zeta)\,\|x_{n-1} - x_n\|^2 + \rho^2\alpha^2\,\|w_{n-1} - w_n\|^2 \\
&\quad \le (1-2\rho\zeta)\,\|x_{n-1} - x_n\|^2 + \rho^2\alpha^2\left(1+\frac{1}{n}\right)^2 \widehat{H}(A(x_{n-1}),A(x_n))^2 \\
&\quad \le \left(1-2\rho\zeta+\rho^2\alpha^2\lambda^2\left(1+\frac{1}{n}\right)^2\right)\|x_{n-1} - x_n\|^2\,. \qquad (4.10)
\end{aligned}$$

Using the strong monotonicity and Lipschitz continuity of η with constants σ and τ respectively, we have

$$\begin{aligned}
&\|x_n - x_{n+1} - \eta(x_n,x_{n+1})\|^2 \\
&\quad = \|x_n - x_{n+1}\|^2 - 2\,\langle\eta(x_n,x_{n+1}),x_n - x_{n+1}\rangle + \|\eta(x_n,x_{n+1})\|^2 \\
&\quad \le (1-2\sigma+\tau^2)\,\|x_n - x_{n+1}\|^2\,. \qquad (4.11)
\end{aligned}$$

Using the Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the first argument with constant α and $\widehat{H}-$Lipschitz continuity of A with constant α, we get

$$\begin{aligned}
\|N(w_{n-1},v_{n-1},u_{n-1}) - N(w_n,v_{n-1},u_{n-1})\| &\le \alpha\,\|w_{n-1} - w_n\| \\
&\le \alpha\left(1+\frac{1}{n}\right)\widehat{H}(Ax_{n-1},Ax_n) \\
&\le \alpha\lambda\left(1+\frac{1}{n}\right)\|x_{n-1} - x_n\|\,. \qquad (4.12)
\end{aligned}$$

By using Lipschitz continuity of $N(\cdot,\cdot,\cdot)$ in the second and third argument with constants β, γ respectively, $\widehat{H}-$Lipschitz continuity of B with constant

μ and $\widehat{H}-$Lipschitz continuity of C with constant δ, we get

$$\begin{aligned}
&\|N(w_n, v_{n-1}, u_{n-1}) - N(w_n, v_n, u_n)\| \\
&\quad \leq \|N(w_n, v_{n-1}, u_{n-1}) - N(w_n, v_n, u_{n-1})\| \\
&\qquad + \|N(w_n, v_n, u_{n-1}) - N(w_n, v_n, u_n)\| \\
&\quad \leq \beta \|v_{n-1} - v_n\| + \gamma \|u_{n-1} - u_n\| \\
&\quad \leq \beta \left(1 + \frac{1}{n}\right) \widehat{H}(Bx_{n-1}, Bx_n) + \gamma \left(1 + \frac{1}{n}\right) \widehat{H}(Cx_{n-1}, Cx_n) \\
&\quad \leq (\beta\mu + \gamma\delta) \left(1 + \frac{1}{n}\right) \|x_{n-1} - x_n\| . \qquad (4.13)
\end{aligned}$$

Substituting (4.10), (4.11), (4.12) and (4.13) in (4.9), we get

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&\quad \leq \sqrt{1 - 2\rho\zeta + \rho^2\alpha^2\lambda^2 \left(1 + \frac{1}{n}\right)^2} \|x_{n-1} - x_n\| \|x_n - x_{n+1}\| \\
&\qquad + \rho\alpha\lambda \left(1 + \frac{1}{n}\right) \sqrt{1 - 2\sigma + \tau^2} \|x_{n-1} - x_n\| \|x_n - x_{n+1}\| \\
&\qquad + \rho\tau(\beta\mu + \gamma\delta) \left(1 + \frac{1}{n}\right) \|x_{n-1} - x_n\| \|x_n - x_{n+1}\| \\
&\qquad + \rho\,(\omega + \varrho) \|x_n - x_{n-1}\| \|x_n - x_{n+1}\| . \qquad (4.14)
\end{aligned}$$

It follows from (4.14) that

$$\|x_n - x_{n+1}\| \leq \theta_n \|x_n - x_{n-1}\| , \qquad (4.15)$$

where

$$\begin{aligned}
\theta_n &= t(\rho_n) + \rho\,(\kappa_n + \vartheta_n) , \\
t(\rho_n) &= \sqrt{1 - 2\rho\zeta + \rho^2\alpha^2\lambda^2 \left(1 + \frac{1}{n}\right)^2}, \\
\kappa_n &= \alpha\lambda \left(1 + \frac{1}{n}\right) \sqrt{1 - 2\sigma + \tau^2}, \\
\vartheta_n &= \tau(\beta\mu + \gamma\delta) \left(1 + \frac{1}{n}\right) + (\omega + \varrho).
\end{aligned}$$

We can see that $\theta_n \to \theta$, where

$$\begin{aligned}
\theta &= t(\rho) + \rho\,(\kappa + \vartheta) \ , \\
t(\rho) &= \sqrt{1 - 2\rho\zeta + \rho^2\alpha^2\lambda^2} \, , \\
\kappa &= \alpha\lambda \sqrt{1 - 2\sigma + \tau^2} \, , \\
\vartheta &= \tau(\beta\mu + \gamma\delta) + (\omega + \varrho).
\end{aligned}$$

By (3.1) and (3.2), we get that $\theta < 1$. Hence, there is a number $\theta_0 < 1$ and an integer $n_0 \geq 1$ such that $\theta_n \leq \theta_0 < 1$ for all $n \geq n_0$. Therefore, it follows from (4.15) that $\{x_n\}$ is a Cauchy sequence and we may assume that $\{x_n\}$ converges to some $x^* \in \mathcal{H}$. Since the set-valued mapping A, B, C are $\widehat{H}-$Lipschitz continuous, from Algorithm 4.1, we get

$$\begin{aligned} \|w_n - w_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \widehat{H}(Ax_n, Ax_{n+1}) \leq \left(1 + \frac{1}{n+1}\right) \lambda \|x_n - x_{n_1}\| \\ \|v_n - v_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \widehat{H}(Bx_n, Bx_{n+1}) \leq \left(1 + \frac{1}{n+1}\right) \mu \|x_n - x_{n_1}\| \\ \|u_n - u_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \widehat{H}(Cx_n, Cx_{n+1}) \leq \left(1 + \frac{1}{n+1}\right) \delta \|x_n - x_{n_1}\| . \end{aligned}$$

Therefore $\{w_n\}$, $\{v_n\}$ and $\{u_n\}$ are Cauchy sequences in $\mathcal{H}$. Let $w_n \to w^*$, $v_n \to v^*$ and $u_n \to u^*$ strongly as $n \to \infty$.

Since $w_n \in Ax_n$, we have

$$\begin{aligned} d(w^*, Ax^*) &\leq \|w^* - w_n\| + d(w_n, Ax_n) + \widehat{H}(Ax_n, Ax^*) \\ &\|w^* - w_n\| + \lambda \|x_n - x^*\| \\ &\to 0 , \quad \text{as } \ n \to \infty. \end{aligned}$$

Hence we must have $w^* \in Ax^*$. Similarly, we can obtain $v^* \in Bx^*$ and $u^* \in Cx^*$.

Auxiliary inequality (4.3) is rewritten as

$$\begin{aligned} &\langle x_{n+1} - x_n, y - x_{n+1}\rangle + \rho \langle N(w_n, v_n, u_n) - f, \eta(y, x_{n+1})\rangle \\ &\quad + \rho \left[a(x_n, y - x_{n+1}) + b(x_n, y) - b(x_n, x_{n+1})\right] \geq 0. \end{aligned} \tag{4.16}$$

Since $x_n \to x^*$ as $n \to \infty$, we have

$$|\langle x_{n+1} - x_n,\, y - x_{n+1}\rangle| \leq \|x_{n+1} - x_n\| \, \|y - x_{n+1}\| \to 0, \quad \text{as } n \to \infty. \tag{4.17}$$

Since $x_n \to x^*$, $w_n \to w^*$, $v_n \to v^*$, $u_n \to u^*$, $w_n \in A(x_n)$, $v_n \in B(x_n)$ and

$u_n \in C(x_n)$, we have

$$\begin{aligned}
&|\langle N(w_n, v_n, u_n) - f,\, \eta(y, x_{n+1})\rangle \,-\, \langle N(w^*, v^*, u^*) - f,\, \eta(y, x^*)\rangle| \\
&\quad \le |\langle N(w_n, v_n, u_n) - N(w^*, v^*, u^*),\, \eta(y, x_{n+1})\rangle| \\
&\qquad + |\langle N(w^*, v^*, u^*) - f,\, \eta(y, x_{n+1}) - \eta(y, x^*)\rangle| \\
&\quad \le \|N(w_n, v_n, u_n) - N(w^*, v^*, u^*)\|\, \|\eta(y, x_{n+1})\| \\
&\qquad + \|N(w^*, v^*, u^*) - f\|\, \|\eta(y, x_{n+1}) - \eta(y, x^*)\| \\
&\quad \le \{\|N(w_n, v_n, u_n) - N(w^*, v_n, u_n)\| + \|N(w^*, v_n, u_n) - N(w^*, v^*, u_n)\| \\
&\qquad + \|N(w^*, v^*, u_n) - N(w^*, v^*, u_n)\|\}\, \|\eta(y, x_{n+1})\| \\
&\qquad + \|N(w^*, v^*, u^*) - f\|\, \|\eta(x_{n+1}, x^*)\| \\
&\quad \le \tau\, \{\alpha\, \|w_n - w^*\| + \beta\, \|v_n - v^*\| + \gamma\, \|u_n - u^*\|\}\, \|y - x_{n+1}\| \\
&\qquad + \tau\, \|N(w^*, v^*, u^*) - f\|\, \|x_{n+1} - x^*\| \\
&\quad \to 0, \quad \text{as } \; n \to \infty.
\end{aligned}$$

Furthermore, from the property of b, it follows that

$$\begin{aligned}
|b(x_n, x_{n+1}) - b(x^*, x^*)| &\le |b(x_n, x_{n+1}) - b(x_n, x^*)| + |b(x_n, x^*) - b(x^*, x^*)| \\
&\le |b(x_n,\, x_{n+1} - x^*)| + |b(x_n - x^*,\, x^*)| \\
&\le \varrho\, \|x_n\|\, \|x_{n+1} - x^*\| + \varrho\, \|x_n - x^*\|\, \|x^*\| \\
&\to 0\,, \quad \text{as } \; n \to \infty. \qquad (4.18)
\end{aligned}$$

Also,

$$|b(x_n, y) - b(x^*, y)| \le \varrho\, \|x_n - x^*\|\, \|y\| \to 0, \;\; \text{as } \; n \to \infty.$$

Hence $b(x_n, x_{n+1}) \to b(x^*, x^*)$ and $b(x_n, y) \to b(x^*, y)$ as $n \to \infty$.

From property of a, it follows that

$$\begin{aligned}
a(x_n, y - x_{n+1}) &= a(x_n - x^*, y - x_{n+1}) + a(x^*, y - x_{n+1}) \\
&= a(x_n - x^*, y - x^*) + a(x_n - x^*, x^* - x_{n+1}) \\
&\quad + a(x^*, y - x^*) + a(x^*, x^* - x_{n+1})
\end{aligned}$$

this gives that

$$\begin{aligned}
a(x_n, y - x_{n+1}) - a(x^*, y - x^*) &= a(x_n - x^*, y - x^*) + a(x_n - x^*, x^* - x_{n+1}) \\
&\quad + a(x^*, x^* - x_{n+1}) \\
&= a(x_n - x^*, y - x^*) + a(x_n, x^* - x_{n+1})
\end{aligned}$$

$$\begin{aligned}
|a(x_n, y - x_{n+1}) - a(x^*, y - x^*)| &\le |a(x_n - x^*, y - x^*)| + |a(x_n, x^* - x_{n+1})| \\
&\le \|x_n - x^*\|\, \|y - x^*\| + \|x_n\|\, \|x^* - x_{n+1}\| \\
&\to 0, \quad \text{as } \; n \to \infty.
\end{aligned}$$

Letting $n \to \infty$ in (4.16), we obtain

$$\langle N(w^*, v^*, u^*) - f,\, \eta(y, x^*)\rangle + a(x^*, y - x^*) + b(x^*, y) - b(x^*, x^*) \geq 0.$$

Therefore, (x^*, w^*, v^*, u^*) is a solution of the problem (2.1).
This completes the proof. □

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