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Weak convergence theorems for two finite families of asymptotically quasi-nonexpansive type mappings in Banach spaces

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Abstract - The aim of this paper is to establish some weak convergence theorems of finite step iteration process for two finite families of non-Lipschitzian asymptotically quasi-nonexpansive type mappings to converge to common fixed point in the framework of uniformly convex Banach spaces.

Key words and phrases : asymptotically quasi-nonexpansive type mapping, finite-step iteration process, common fixed point, weak convergence, uniformly convex Banach space.

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1. Introduction and preliminaries

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$;

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \tag{1.2}$$

for all $x \in K$ and $p \in F(T)$;

(3) asymptotically nonexpansive [6] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\| \tag{1.3}$$

for all $x, y \in K$ and $n \geq 1$;

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \quad \leq \quad k_n \|x - p\| \tag{1.4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(5) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \quad \leq \quad L \|x - y\| \tag{1.5}$$

for all $x, y \in K$ and $n \geq 1$;

(6) asymptotically nonexpansive type [8], if

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in K} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \right\} \quad \leq \quad 0; \tag{1.6}$$

(7) asymptotically quasi-nonexpansive type [14], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x\in K,\ p\in F(T)} \Big(\|T^n x - p\| - \|x - p\| \Big) \right\} \quad \leq \quad 0. \tag{1.7}$$

Remark 1.1. It is easy to see that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping are the special cases of asymptotically quasi-nonexpansive type mappings.

The class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [6] in 1972 as an important generalization of the class of nonexpansive self-mappings, who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Since then, iteration processes for asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in Banach spaces have studied extensively by many authors (see [2], [5], [7]-[20]). In 2002, Xu and Noor [23] introduced and studied a three-step iteration scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. Cho et al. [3] extended the work of Xu and Noor to a three-step iterative scheme with errors in Banach space and proved the weak and strong convergence theorems for asymptotically nonexpansive mappings. In 2003, Sahu and Jung [14] studied Ishikawa and Mann iteration process in Banach spaces and they proved some weak and strong convergence theorems for asymptotically quasi-nonexpansive type mapping. In 2006, Shahzad and Udomene [20] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space. In 2009, Sitthikul and Saejung [21] introduced and studied a finite-step iteration scheme for a finite family of nonexpansive and asymptotically nonexpansive mappings and proved some weak and strong convergence theorems in the setting of Banach spaces. Recently, Chen and Guo [1] introduced and studied a new finite-step iteration scheme with errors for two finite families of asymptotically nonexpansive mappings as follows:

Let K be a nonempty convex subset of a Banach space E with $K + K \subset K$. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N\colon K \to K$ be $2N$ asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

$$\begin{aligned}
x_1 &= x \in K,\\
x_n^{(0)} &= x_n,\\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1-\alpha_n^{(1)}) S_1^n x_n + u_n^{(1)},\\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1-\alpha_n^{(2)}) S_2^n x_n + u_n^{(2)},\\
&\ \ \vdots\\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1-\alpha_n^{(N-1)}) S_{N-1}^n x_n + u_n^{(N-1)},\\
x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1-\alpha_n^{(N)}) S_N^n x_n + u_n^{(N)},\\
x_{n+1} &= x_n^{(N)},\ \forall\, n \geq 1,
\end{aligned} \tag{1.8}$$

where $\{\alpha_n^{(i)}\} \subset [0,1]$ and $\{u_n^{(i)}\}$ are bounded sequences in K for all $i \in I = \{1, 2, \dots, N\}$, and the weak and strong convergence theorems are proved, which improve and generalize some results in [21].

Letting $u_n^{(i)} = 0$ for all $n \geq 1$, $i \in I$ in (1.8). We have the following:

$$\begin{aligned}
x_1 &= x \in K, \\
x_n^{(0)} &= x_n, \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1 - \alpha_n^{(1)}) S_1^n x_n, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1 - \alpha_n^{(2)}) S_2^n x_n, \\
&\vdots \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - \alpha_n^{(N-1)}) S_{N-1}^n x_n, \\
x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1 - \alpha_n^{(N)}) S_N^n x_n, \\
x_{n+1} &= x_n^{(N)}, \ \forall\, n \geq 1, \qquad (1.9)
\end{aligned}$$

where $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $i \in I$ and the author [1] proved weak convergence theorem of iteration scheme (1.9).

The aim of this paper is to establish some weak convergence of the iteration scheme (1.9) to converge to common fixed points for two finite families of uniformly L-Lipschitzian and non-Lipschitzian asymptotically quasi-nonexpansive type mappings in the framework of uniformly convex Banach spaces. The results presented in this paper improve and extend the corresponding results of Chen and Guo [1] and Sitthikul and Saejung [21] to the case of more general class of nonexpansive and asymptotically nonexpansive mappings because both these mappings include in the class of asymptotically quasi-nonexpansive type mappings.

In order to prove the main results of this paper, we need the following concepts and lemmas.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Recall that a Banach space E is said to satisfy Opial's condition [11] if, for any sequence $\{x_n\}$ in E, $x_n \to x$ weakly implies that

$$\limsup_{n \longrightarrow \infty} \|x_n - x\| < \limsup_{n \longrightarrow \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

A Banach space E has the Kadec-Klee property [21] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Lemma 1.1. (see [22]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \quad \leq \quad a_n + b_n, \;\; n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. (see [18]) *Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$. Then it holds $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.3. (see [21]) *Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and p, $q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.*

2. Main result

In this section, we first prove the following lemmas in order to prove our main theorems.

Lemma 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N \colon K \to K$ be $2N$ asymptotically quasi-nonexpansive type mappings with $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.9), where $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $i \in I$. Put*

$$\begin{aligned} A_{in} \;\; = \;\; \max\Big\{ & \sup_{p\in F,\; n\geq 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee \\ & \sup_{p\in F,\; n\geq 1} \Big(\|S_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 : \; 1 \leq i \leq N \Big\} \end{aligned} \tag{2.1}$$

such that $\sum_{n=1}^{\infty} A_{in} < \infty$ for all $i \in I$. Then the limit $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$.

Proof. Let $q \in F$. Then from (1.9) and (2.1), we have

$$\begin{aligned} \left\|x_n^{(1)} - q\right\| &= \left\|\alpha_n^{(1)} T_1^n x_n + (1-\alpha_n^{(1)}) S_1^n x_n - q\right\| \\ &\leq \alpha_n^{(1)} \|T_1^n x_n - q\| + (1-\alpha_n^{(1)}) \|S_1^n x_n - q\| \\ &\leq \alpha_n^{(1)} \Big[\|x_n - q\| + A_{1n} \Big] + (1-\alpha_n^{(1)}) \Big[\|x_n - q\| + A_{1n} \Big] \\ &\leq \|x_n - q\| + A_{1n}. \end{aligned} \tag{2.2}$$

Again using (1.9)-(2.2), we obtain

$$\begin{aligned} \left\|x_n^{(2)} - q\right\| &= \left\|\alpha_n^{(2)} T_2^n x_n^{(1)} + (1-\alpha_n^{(2)}) S_2^n x_n - q\right\| \\ &\leq \alpha_n^{(2)} \left\|T_2^n x_n^{(1)} - q\right\| + (1-\alpha_n^{(2)}) \|S_2^n x_n - q\| \\ &\leq \alpha_n^{(2)} \Big[\left\|x_n^{(1)} - q\right\| + A_{2n} \Big] + (1-\alpha_n^{(2)}) \Big[\|x_n - q\| + A_{2n} \Big] \\ &\leq \alpha_n^{(2)} \Big[\|x_n - q\| + A_{1n} \Big] + (1-\alpha_n^{(2)}) \|x_n - q\| + A_{2n}] \\ &\leq \|x_n - q\| + \alpha_n^{(2)} A_{1n} + A_{2n} \\ &\leq \|x_n - q\| + A_{1n} + A_{2n}. \end{aligned} \tag{2.3}$$

Continuing the above process, we get that

$$\left\|x_n^{(i)} - q\right\| \leq \|x_n - q\| + \sum_{k=1}^{i} A_{kn}. \tag{2.4}$$

In particular,

$$\|x_{n+1} - q\| = \left\|x_n^{(N)} - q\right\| \leq \|x_n - q\| + \sum_{k=1}^{N} A_{kn}. \tag{2.5}$$

Since by assumption of the theorem $\sum_{n=1}^{\infty} A_{in} < \infty$ for all $n \geq 1$ and $i \in I$, it follows by Lemma 1.1, we have that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Lemma 2.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N \colon K \to K$ be $2N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.9), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$. Put*

$$\begin{aligned} A_{in} &= \max\Big\{ \sup_{p\in F,\ n\geq 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee \\ &\quad \sup_{p\in F,\ n\geq 1} \Big(\|S_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 :\ 1 \leq i \leq N \Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} A_{in} < \infty$ *for all* $i \in I$. *Then* $\lim_{n\to\infty} \left\| S_i^n x_n - T_i^n x_n^{(i-1)} \right\| = 0$ *for all* $i \in I$.

Proof. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. So we can assume that

$$\lim_{n\to\infty} \|x_n - q\| = d \tag{2.6}$$

for all $q \in F$, where $d \geq 0$ is nonnegative number. It follows from (2.4) and (2.6) that

$$\limsup_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| \leq d \tag{2.7}$$

and so

$$\limsup_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - q \right\| \leq d. \tag{2.8}$$

Also,

$$\limsup_{n\to\infty} \|S_N^n x_n - q\| \leq d. \tag{2.9}$$

Further, from (1.9) and (2.6), we have

$$\begin{aligned} d &= \lim_{n\to\infty} \|x_n^{(N)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N)}(T_N^n x_n^{(N-1)} - q) \\ &\quad + (1 - \alpha_n^{(N)})(S_N^n x_n - q)\|. \end{aligned}$$

By Lemma 1.2, we get that

$$\lim_{n\to\infty} \left\| S_N^n x_n - T_N^n x_n^{(N-1)} \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - q \right\| = d.$$

From (2.7), we have

$$\begin{aligned} d &= \liminf_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - q \right\| \\ &\leq \liminf_{n\to\infty} [\left\| x_n^{(N-1)} - q \right\| + A_{Nn}] \\ &= \liminf_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| \leq \limsup_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| \leq d \end{aligned}$$

and so

$$\lim_{n\to\infty} \left\| x_n^{(N-1)} - q \right\| = d. \tag{2.10}$$

It follows from (2.4) and (2.6) that

$$\limsup_{n\to\infty} \left\| x_n^{(N-2)} - q \right\| \leq d.$$

Further, we know that

$$\limsup_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - q \right\| \leq d \tag{2.11}$$

and

$$\limsup_{n\to\infty} \left\| S_{N-1}^n x_n - q \right\| \leq d. \tag{2.12}$$

From (1.9) and (2.10), we have

$$\begin{aligned} d &= \lim_{n\to\infty} \|x_n^{(N-1)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - q) \\ &\quad + (1-\alpha_n^{(N-1)})(S_{N-1}^n x_n - q)\|. \end{aligned} \tag{2.13}$$

It follows from (2.11)-(2.13) and Lemma 1.2 that

$$\lim_{n\to\infty} \left\| S_{N-1}^n x_n - T_{N-1}^n x_n^{(N-2)} \right\| = 0.$$

Continuing the above process, we obtain the result of Lemma 2.2. This completes the proof. □

Lemma 2.3. *Under the assumptions of Lemma* 2.2*, if*

$$\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0 \tag{2.14}$$

for all $i \in I$, then

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \ \forall\, i \in I.$$

Proof. Since $\lim_{n\to\infty} \left\| S_i^n x_n - T_i^n x_n^{(i-1)} \right\| = 0$ for all $i \in I$ by Lemma 2.2. It follows from (2.14) that

$$\lim_{n\to\infty} \left\| x_n - T_i^n x_n^{(i-1)} \right\| = 0 \tag{2.15}$$

for all $i \in I$. Next, from (1.9), we have

$$\|x_n - x_{n+1}\| \ \leq \ \alpha_n^{(N)} \left\| x_n - T_N^n x_n^{(N-1)} \right\| + (1-\alpha_n^{(N)}) \left\| x_n - S_N^n x_n \right\|.$$

Using (2.14) and (2.15), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.16}$$

Since $\lim_{n\to\infty} \|x_n - T_1^n x_n\| = 0$ by (2.15) and

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \left\|x_n - T_i^n x_n^{(i-1)}\right\| + \left\|T_i^n x_n^{(i-1)} - T_i^n x_n\right\| \\ &\leq \left\|x_n - T_i^n x_n^{(i-1)}\right\| + L\left\|x_n^{(i-1)} - x_n\right\| \\ &\leq \left\|x_n - T_i^n x_n^{(i-1)}\right\| + L\alpha_n^{(i-1)}\left\|T_{i-1}^n x_n^{(i-2)} - x_n\right\| \\ &\quad + L(1-\alpha_n^{(i-1)})\left\|S_{i-1}^n x_n - x_n\right\| \end{aligned} \tag{2.17}$$

for all $i = 1, 2, \ldots, N$. From (2.14), (2.15) and (2.17), we have

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0 \tag{2.18}$$

for all $i \in I$. It follows from (2.16) and (2.18) that

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\ &\quad + \left\|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\right\| + \left\|T_i^{n+1} x_n - T_i x_n\right\| \\ &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| + L\|x_{n+1} - x_n\| \\ &\quad + L\|T_i^n x_n - x_n\| \\ &\leq (1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T_i^{n+1} x_{n+1}\right\| \\ &\quad + L\|T_i^n x_n - x_n\|. \end{aligned} \tag{2.19}$$

Using (2.16) and (2.18), we get that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0,$$

for all $i \in I$. This completes the proof. □

Lemma 2.4. *Under the assumptions of Lemma* 2.2*, if*

$$\|x - T_i y\| \leq \|S_i x - T_i y\| \tag{2.20}$$

for all x, $y \in K$ *and* $i \in I$*, then*

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \ \forall\, i \in I.$$

Proof. By (2.20), we obtain that

$$\begin{aligned} 0 &\leq \left\|x_n - T_i^n x_n^{(i-1)}\right\| \leq \left\|S_i x_n - T_i^n x_n^{(i-1)}\right\| \\ &\leq \left\|S_i^n x_n - T_i^n x_n^{(i-1)}\right\| \end{aligned} \tag{2.21}$$

for all $i \in I$. It follows from (2.21) and Lemma 2.2 that

$$\lim_{n\to\infty} \left\|S_i x_n - T_i^n x_n^{(i-1)}\right\| = \lim_{n\to\infty} \left\|x_n - T_i^n x_n^{(i-1)}\right\| = 0. \tag{2.22}$$

Since

$$\|x_n - S_i x_n\| \leq \left\|x_n - T_i^n x_n^{(i-1)}\right\| + \left\|T_i^n x_n^{(i-1)} - S_i x_n\right\|. \tag{2.23}$$

Using (2.22) in (2.23), we obtain

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0 \tag{2.24}$$

for all $i \in I$. Also,

$$\|x_n - S_i^n x_n\| \leq \left\|x_n - T_i^n x_n^{(i-1)}\right\| + \left\|T_i^n x_n^{(i-1)} - S_i^n x_n\right\|. \tag{2.25}$$

Using (2.22) and Lemma 2.2 in (2.25), we obtain

$$\lim_{n\to\infty} \|x_n - S_i^n x_n\| = 0 \tag{2.26}$$

for all $i \in I$. Thus $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$ by Lemma 2.3. This completes the proof. □

Theorem 2.1. *Under the assumptions of Lemma* 2.4*, if E satisfying Opial's condition. Assume that the mappings $I - S_i$ and $I - T_i$ for all $i \in I$, where I denotes the identity mapping, are demiclosed at zero. Then the sequence $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N, S_1, S_2, \dots, S_N\}$.*

Proof. Let $q \in F$, from Lemma 2.1 the sequence $\{\|x_n - q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q^* \in K$. From Lemma 2.4, we get that

$$\lim_{n\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0 \text{ and } \lim_{n\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for all $i \in I$. Since the mappings $I - S_i$ and $I - T_i$ for all $i \in I$ are demiclosed at zero, therefore $S_i q^* = q^*$ and $T_i q^* = q^*$, which means $q^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to q^*. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p^* \in K$ and $q^* \neq p^*$. Then by the same method as given above, we can also prove that $p^* \in F$. From Lemma 2.1 the limits $\lim_{n\to\infty} \|x_n - q^*\|$ and

$\lim_{n\to\infty}\|x_n - p^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - q^*\| &= \lim_{n_k\to\infty}\|x_{n_k} - q^*\| \\
&< \lim_{n_k\to\infty}\|x_{n_k} - p^*\| \\
&= \lim_{n\to\infty}\|x_n - p^*\| \\
&= \lim_{n_j\to\infty}\left\|x_{n_j} - p^*\right\| \\
&< \lim_{n_j\to\infty}\left\|x_{n_j} - q^*\right\| \\
&= \lim_{n\to\infty}\|x_n - q^*\|
\end{aligned}$$

which is a contradiction so $q^* = p^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$. This completes the proof. □

Lemma 2.5. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N\colon K \to K$ be $2N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.9), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$. Put*

$$\begin{aligned}
A_{in} &= \max\Big\{ \sup_{p\in F,\ n\geq 1}\Big(\|T_i^n x_n - p\| - \|x_n - p\|\Big) \vee \\
&\qquad \sup_{p\in F,\ n\geq 1}\Big(\|S_i^n x_n - p\| - \|x_n - p\|\Big) \vee 0 :\ 1 \leq i \leq N\Big\}
\end{aligned}$$

such that $\sum_{n=1}^\infty A_{in} < \infty$ for all $i \in I$. Then $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all p, $q \in F$ and $t \in [0,1]$.

Proof. By Lemma 2.1, we know that $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 2.1. It, therefore, remains to prove the Lemma 2.5 for $t \in (0,1)$. For all $x \in K$, we define the mapping $R_n\colon K \to K$ by

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n^{(0)} + (1 - \alpha_n^{(1)}) S_1^n x_n, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + (1 - \alpha_n^{(2)}) S_2^n x_n, \\
&\ \vdots \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - \alpha_n^{(N-1)}) S_{N-1}^n x_n, \\
R_n(x) &= \alpha_n^{(N)} T_N^n x^{(N-1)} + (1 - \alpha_n^{(N)}) S_N^n x.
\end{aligned}$$

It is easy to prove

$$\|R_n x - R_n y\| \quad \le \quad \|x - y\| + A_{Nn}, \tag{2.27}$$

for all $x, y \in K$, with $\sum_{n=1}^{\infty} A_{Nn} < \infty$. Setting

$$S_{n,\,m} = R_{n+m-1} R_{n+m-2} \dots R_n, \quad m \ge 1 \tag{2.28}$$

and

$$b_{n,\,m} \quad = \quad \left\| S_{n,\,m}(tx_n + (1-t)p) - (tS_{n,\,m} x_n + (1-t)S_{n,m} q) \right\|. \tag{2.29}$$

From (2.27) and (2.28), we have

$$\|S_{n,\,m} x - S_{n,\,m} y\| \quad \le \quad \|x - y\| + \sum_{k=n}^{n+m-1} A_{kn} \tag{2.30}$$

for all $x, y \in K$ and $S_{n,\,m} x_n = x_{n+m}$, $S_{n,\,m} p = p$ for all $p \in F$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\le b_{n,\,m} + \|S_{n,\,m}(tx_n + (1-t)p) - q\| \\ &\le b_{n,\,m} + a_n(t) + \sum_{k=n}^{n+m-1} A_{kn}. \end{aligned} \tag{2.31}$$

By using [4, Theorem 2.3],we have

$$\begin{aligned} b_{n,\,m} &\le \phi^{-1}(\|x_n - p\| - \|S_{n,\,m} x_n - S_{n,\,m} p\|) \\ &\le \phi^{-1}(\|x_n - p\| - \|x_{n+m} - p + p - S_{n,\,m} p\|) \\ &\le \phi^{-1}\Big(\|x_n - p\| - (\|x_{n+m} - p\| - \|S_{n,\,m} p - p\|)\Big), \end{aligned} \tag{2.32}$$

and so the sequence $\{b_{n,\,m}\}$ converges to 0 as $n \to \infty$ for all $m \ge 1$. Thus, fixing n and letting $m \to \infty$ in (2.32), we have

$$\begin{aligned} \limsup_{m \to \infty} a_{n+m}(t) \quad &\le \quad \phi^{-1}\Big(\|x_n - p\| - \Big(\lim_{m \to \infty} \|x_m - p\| - \|S_{n,\,m} p - p\| \Big)\Big) \\ &\quad + a_n(t) + \sum_{k=n}^{n+m-1} A_{kn}, \end{aligned} \tag{2.33}$$

and again letting $n \to \infty$, we obtain

$$\limsup_{n \to \infty} a_n(t) \quad \le \quad \phi^{-1}(0) + \liminf_{n \to \infty} a_n(t) + 0 = \liminf_{n \to \infty} a_n(t).$$

This shows that $\lim_{n \to \infty} a_n(t)$ exists, that is,

$$\lim_{n \to \infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0, 1]$. This completes the proof. □

Theorem 2.2. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K be a nonempty closed convex subset of E. Let $\{S_i\}_{i=1}^N$, $\{T_i\}_{i=1}^N : K \to K$ be $2N$ uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.9), where $\{\alpha_n^{(i)}\} \subset [a, 1-a]$ for some $a \in (0,1)$ and all $i \in I$. Put*

$$\begin{aligned} A_{in} &= \max\Big\{ \sup_{p\in F,\ n\geq 1} \Big(\|T_i^n x_n - p\| - \|x_n - p\| \Big) \vee \\ &\qquad \sup_{p\in F,\ n\geq 1} \Big(\|S_i^n x_n - p\| - \|x_n - p\| \Big) \vee 0 : 1 \leq i \leq N \Big\} \end{aligned}$$

such that $\sum_{n=1}^\infty A_{in} < \infty$ for all $i \in I$. If the mappings $I - S_i$ and $I - T_i$ for all $i \in I$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$.

Proof. By Lemma 2.1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. By Lemma 2.4 we get that

$$\lim_{n\to\infty} \left\|x_{n_j} - S_i x_{n_j}\right\| = 0 \text{ and } \lim_{n\to\infty} \left\|x_{n_j} - T_i x_{n_j}\right\| = 0$$

for all $i \in I$. Since the mappings $I - S_i$ and $I - T_i$ for all $i \in I$ are demiclosed at zero, therefore $S_i p = p$ and $T_i p = p$ for all $i \in I$ which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and p, $q \in w_w(x_n)$. By Lemma 2.5, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 1.3. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

Remark 2.1. Our results extend and improve the corresponding results of [1] to the case of more general class of asymptotically nonexpansive mappings considered in this paper.

Remark 2.2. Our results also extend and improve the corresponding results of [21] to the case of more general class of nonexpansive and asymptotically nonexpansive mappings considered in this paper.

Example 2.1. Let $E = [-\pi, \pi]$ and let T be defined by

$$Tx = x \cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |x||\cos x| \leq |x| = |x - z|,$$

and T is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. Hence by Remark 1.1, T is asymptotically quasi-nonexpansive type mapping. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$|Tx - Ty| = \Big|\frac{\pi}{2}\cos\frac{\pi}{2} - \pi\cos\pi\Big| = \pi,$$

whereas

$$|x - y| = \Big|\frac{\pi}{2} - \pi\Big| = \frac{\pi}{2}.$$

Example 2.2. Let $E = \mathbb{R}$ and let T be defined by

$$T(x) = \begin{cases} \frac{x}{2}\cos\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\cos\frac{1}{x}$. Thus $2 = \cos\frac{1}{x}$. This is impossssible. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |\frac{x}{2}||\cos\frac{1}{x}| \leq \frac{|x|}{2} < |x| = |x - z|,$$

and T is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. Hence by Remark 1.1, T is asymptotically quasi-nonexpansive type mapping. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$|Tx - Ty| = \Big|\frac{1}{3\pi}\cos\frac{3\pi}{2} - \frac{1}{2\pi}\cos\pi\Big| = \frac{1}{2\pi},$$

whereas

$$|x - y| = \Big|\frac{2}{3\pi} - \frac{1}{\pi}\Big| = \frac{1}{3\pi}.$$

3. Conclusion

By Remark 1.1 it is clear that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings are asymptotically quasi-nonexpansive type mappings, thus the results presented in this paper are good improvement and generalization of corresponding results of [1, 21] and many others from the current literature.

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