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Weighted composition operators on Musielak-Orlicz spaces of Bochner-type

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Abstract - The compact, Fredholm and isometric weighted composition operators are characterized in this paper.

Key words and phrases : weighted composition operator, Orlicz space, Musielak-Orlicz space of Bochner-type, compact operator, isometry and Fredholm operator.

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1. Introduction and preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of reals, non-negative reals and the set of natural numbers respectively. Let $\big(G, \Sigma, \mu\big)$ be a σ-finite measure space and $L^0 = L^0\big(G, \Sigma, \mu\big)$ be the space of all (equivalence classes of) $\mu-$measurable complex-valued functions defined on G. By $\varphi : \mathbb{R} \to [0, \infty]$ we denote an Orlicz function i.e. φ is convex, even, continuous at zero and left hand side continuous in the extended sense (that is infinite limits are not excluded) on $\mathbb{R}_+$ (see [4], [6], [9], [12],[18]). By M we denote a Musielak-Orlicz function, that is $M : G \times \mathbb{R} \to [0, \infty]$ and

1. $M(t, .)$ is an Orlicz function for μ-a.e. $t \in G$,
2. $M(., u) \in L^0$ for any $u \in \mathbb{R}$.

The function M generates on the space L^0 the convex modular

$$\varrho_M(f) = \int_G M(t, |f(t)|)d\mu.$$

The space

$$L_M = \Big\{ f \in L^0 : \varrho_M(\lambda f) < \infty \ \text{ for some } \ \lambda > 0 \Big\}$$

is called the Musielak-Orlicz space generated by M. Its subspace E_M is defined as

$$E_M = \Big\{ f \in L^0 : \varrho_M(\lambda f) < \infty \text{ for any } \lambda > 0 \Big\}.$$

The space L_M endowed with the Luxemberg norm

$$||f||_M = \inf \Big\{ \lambda > 0 : \varrho_M(\frac{f}{\lambda}) \leq 1 \Big\}$$

is a Banach space (see [13], [14]). For every Musielak-Orlicz function M we define complementary function $M^*(t,v)$ as

$$M^*(t,v) = \sup_{u>0} \Big\{ u|v| - M(t,u) : v \geq 0 \text{ and } t \in G \; a.e. \Big\}.$$

It is easy to see that $M^*(t,v)$ is also Musielak-Orlicz function. We say that Musielak-Orlicz function M satisfies Δ_2-condition (write $M \in \Delta_2$) if there exists a constant $k > 2$ and a measurable non-negative function f such that $\varrho_M(f) < \infty$ and

$$M(t, 2u) \leq kM(t,u)$$

for every $u \geq f(t)$ and for $t \in G$ a.e.. For more details see [12]. Throughout this paper we assumed that M satisfies Δ_2-condition.

If T is a non-singular measurable transformation, then the measure μT^{-1} is absolutely continuous with respect to the measure μ. Hence by Radon-Nikodym derivative theorem there exists a positive measurable function f_0 such that $\mu(T^{-1}(E)) = \int_E f_0 d\mu$ for every $E \in \Sigma$. The function f_0 is called the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ. It is denoted by $f_0 = \frac{d\mu T^{-1}}{d\mu}$.

Let (G, Σ, μ) be a $\sigma-$finite measure space and $\Sigma_0 \subset \Sigma$ be a $\sigma-$finite subalgebra. Then the conditional expectation $E(\cdot \big| \Sigma_0)$ is defined as a linear transformation from certain $\Sigma-$measurable function spaces into their Σ_0-measurable counterparts. In particular the conditional expectation with respect to $T^{-1}(\Sigma)$ is a bounded projection from $L^0(G, \Sigma, \mu)$ into the space $L^0(G, T^{-1}(\Sigma), \mu)$. We denote this transformation by E.
The operator E has the following properties:
(i) $E(f.goT) = E(f).(goT)$.
(ii) If $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere.
(iii)$E(1) = 1$.
(iv)$E(f)$ has the form $E(f) = goT$ for exactly one σ-measurable function g. In particular $g = E(f)oT^{-1}$ is a well defined measurable function.
(v) $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$. This is a Cauchy-Schwartz inequality for conditional expectation.
(vi) For $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.
(vii) If ϕ is a convex function, then $\phi(E(f)) \leq E(\phi(f))$ μ-almost everywhere. For deeper study of properties of E see [11].

We now define the types of spaces considered in this paper. For a Banach space $(X, ||.||_X)$, denote by $L^0(X)$, the family of strongly measurable functions $f : G \to X$ identifying functions which are equal μ-almost everywhere in G. Define a new modular $\tilde{\varrho}_M$ on $L^0(X)$ by

$$\tilde{\varrho}_M(f) = \int_G M(t, ||f(t)||) d\mu.$$

Let

$$L_M(G, X) = \Big\{ f \in L^0(X) : ||f(t)|| = ||f(t)||_X \in L_M \Big\}.$$

Then $L_M(G, X)$ becomes a Banach space with the norm

$$||f|| = || \; ||f(t)||_X \; ||_M = \inf \Big\{ \lambda : \tilde{\varrho}_M(\frac{f}{\lambda}) \leq 1 \Big\}$$

and it is called Musielak-Orlicz space of Bochner-type (see [7]). We denote by $B(L_M(G, X))$ the set of all bounded linear operators from $L_M(G, X)$ into itself. Also by $B(X)$ we denote the set of all bounded linear operators on X into itself.

Let $w : G \to B(X)$ be a strongly measurable operator-valued map and $T : G \to G$ be a measurable transformation. Then a bounded linear transformation $S_{w,T} : L_M(G, X) \to L_M(G, X)$ defined by

$$(S_{w,T} f)(t) = w(T(t)) f(T(t)),$$

for every $t \in G$ and for every $f \in L_M(G, X)$ is called a weighted composition operator induced by the pair (w, T). If we take $w(t) = I$, the identity operator on G, we write $S_{w,T}$ as C_T and call it a composition operator induced by T. In case $T(t) = t$ for some $t \in G$, we write $S_{w,T}$ as M_w and call it a multiplication operator induced by w. A weighted composition operator is a product of composition operator and a multiplication operator. Thus the multiplication operators and composition operators are special types of weighted composition operators. In the early 1930's the composition operators were used to study problems in mathematical physics and especially classical mechanics, see Koopman's paper [9]. In those days these operators were known as substitution operators. The systematic study of composition operators has relatively a very short history. It was started by Nordgren in 1968 in his paper [15]. After this, the study of composition operators has been extended in several directions by several mathematicians.

The study of compact weighted composition operators on L^p-spaces $(1 \leq p < \infty)$ was initiated by Takagi [20] in 1992. He also determined the spectra of these operators. In 1993, Takagi [21] characterized the weighted composition operators on $C(X)$ and proved that a weighted composition operator is Fredholm if and only if it is invertible. The same equivalence is true for weighted composition operator on $L^p(\mu)$ spaces $1 \leq p < \infty$, where μ is non-atomic measure. In 1994, Campbell and Hornor [1] used a localized conditional expectation operator to characterize subnormality for the adjoint of a weighted composition operator. In 1996, Hornor and Jamison [5] investigated the criteria for the hypernormality, cohyponormality and normality of weighted composition operators acting on Hilbert spaces of vector-valued functions. For more details on composition and weighted composition operators see [2], [3], [5], [16], [17], [19], [22] and references therein.

By $N^{||f_0||}(w, \epsilon)$ we mean the set

$$\Big\{t \in G : E\big[M\big(I \circ T^{-1}(t), ||w(t)y||\big)\big] f_0(t) \geq M(t, \epsilon||y||) \text{ for } \; y \in X\Big\}.$$

The set $\{\overline{t \in G : w(t) \neq 0}\}$ is called support of w and we shall write it as supp w.

The main purpose of this paper is to characterize the boundedness, compactness, invertibility, Fredholmness and isometry of the weighted composition operators defined on the Musielak-Orlicz function spaces of Bochner-type.

2. Weighted composition operators

The main aim of this section is to characterize boundedness and compactness of weighted composition operators on Musielak-Orlicz spaces of Bochner-type. Before proving the compactness of weighted composition operator, we first prove the necessary and sufficient condition for a weighted composition operator to be bounded away from zero.

Theorem 2.1. *Let $w : G \to B(X)$ be a strongly measurable operator-valued map and let $T : G \to G$ be a measurable transformation. Then $S_{w,T} : L_M(G, X) \to L_M(G, X)$ is a bounded operator if and only if there exists a constant $K > 0$ such that*

$$E\Big[M\big(I \circ T^{-1}(t)\big), ||w(t)y||\Big] f_0(t) \leq M(t, K||y||) \tag{2.1}$$

for every $y \in X$ and for μ-almost all $t \in G$.

Proof. Suppose the condition (2.1) is true. Then for every $f \in L_M(G, X)$, we have

$$\begin{aligned}\int_G M\Big(t, \frac{||(S_{w,T}f)(t)||}{K||f||}\Big) d\mu(t) &= \int_G M\Big(t, \frac{||w(T(t))f(T(t))||}{K||f||}\Big) d\mu(t) \\ &= \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{||w(t)f(t)||}{K||f||}\Big)\Big] f_0(t) d\mu(t) \\ &\leq \int_G M\Big(t, \frac{||f(t)||}{||f||}\Big) d\mu(t) \\ &\leq 1.\end{aligned}$$

This shows that

$$||S_{w,T}f|| \leq K||f||.$$

Conversely, suppose that the condition (2.1) is not true. Then for every positive integer k, there exists a measurable set $G_k \subset G$ and some $y_k \in X$ such that

$$E\big[M(I \circ T^{-1}(t), ||w(t)y_k||)\big] f_0(t) \geq M(t, K||y_k||)$$

for μ-almost every $t \in G_k$. Choose a measurable subset F_k of G_k such that $\chi_{F_k} \in L_M(G, X)$. Let $f_k = y_k \chi_{F_k}$. Then

$$\begin{aligned}
\int_G M\Big(t, \frac{||Kf_k(t)||}{||S_{w,T}f_k||}\Big) d\mu(t) &= \int_{F_k} M\Big(t, \frac{||Ky_k||}{||S_{w,T}f_k||}\Big) d\mu(t) \\
&\leq \int_{F_k} E\Big[M\Big(I \circ T^{-1}(t), \frac{||w(t)y_k||}{||S_{w,T}f_k||}\Big)\Big] f_0(t) d\mu(t) \\
&= \int_G M\Big(t, \frac{||(S_{w,T}f_k)(t)||}{||S_{w,T}f_k||}\Big) d\mu(t) \\
&\leq 1.
\end{aligned}$$

This implies that

$$||S_{w,T}f_k|| \geq K||f_k||.$$

This contradicts the boundedness of $S_{w,T}$. Hence the condition (2.1) must be true. □

Theorem 2.2. *Let $w : G \to B(X)$ be a strongly measurable operator-valued map and $T : G \to G$ be a measurable transformation. Then $S_{w,T} : L_M(G, X) \to L_M(G, X)$ is bounded away from zero if and only if*

$$E\Big[M\Big(I \circ T^{-1}(t), ||w(t)g||\Big)\Big] f_0(t) \geq M(t, \delta||g||) \tag{2.2}$$

for each $t \in G$ and $g \in X$.

Proof. We first suppose that the condition (2.2) is true. Then for every $f \in L_M(G, X)$, we have

$$\begin{aligned}
\int_G M\Big(t, \frac{||\delta f(t)||}{||S_{w,T}f||}\Big) d\mu(t) &\leq \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{||w(t)f(t)||}{||S_{w,T}f||}\Big)\Big] f_0(t) d\mu(t) \\
&= \int_G M\Big(t, \frac{||(S_{w,T}f)(t)||}{||S_{w,T}f||}\Big) d\mu(t) \\
&\leq 1.
\end{aligned}$$

Therefore, $||S_{w,T}f|| \geq \delta||f||$ for all $f \in L_M(G, X)$. This shows that $S_{w,T}$ is bounded away from zero.

Conversely, suppose that the condition (2.2) is not true. Then for every integer k, there exists $g_k \in X$ and a measurable subset G_k of G such that

$$E\Big[M(I \circ T^{-1}(t), ||w(t)g_k||)\Big] f_0(t) \leq M\Big(t, \frac{||g_k||}{K}\Big).$$

Choose a measurable subset F_k of G_k such that $\chi_{F_k} \in L_M(G, X)$. Let $f_k = K\chi_{F_k}$. Then

$$\begin{aligned}\int_G M\Big(t, \frac{||K(S_{w,T}f_k)(t)||}{||f_k||}\Big) d\mu(t) &= \int_{F_k} E\Big[M\Big(I \circ T^{-1}(t), \frac{||Kw(t)g_k||}{||f_k||}\Big)\Big] f_0(t) d\mu(t) \\ &< \int_{F_k} M\Big(t, \frac{||g_k||}{||f_k||}\Big) d\mu(t) \\ &= \int_G M\Big(t, \frac{||f_k(t)||}{||f_k||}\Big) d\mu(t) \\ &\leq 1.\end{aligned}$$

Hence

$$||S_{w,T}f_k|| \leq \frac{1}{K}||f_k||,$$

which shows that $S_{w,T}$ is not bounded away from zero. Hence the condition of the theorem must be true. □

Theorem 2.3. *Suppose $S_{w,T} \in B\big(L_M(G, X)\big)$. Then $S_{w,T}$ is compact if and only if the space $L_M\Big[(G, X), N^{||f_0||}(w, \epsilon)\Big]$ is finite dimensional for each $\epsilon > 0$, where*

$$L_M[(G, X), N^{||f_0||}(w, \epsilon)] = \Big\{f \in L_M(G, X) : f(t) = 0, \forall\ \ t \notin N^{||f_0||}(w, \epsilon)\Big\}.$$

Proof. We first suppose that the space $L_M\Big[(G, X), N^{||f_0||}(w, \frac{1}{n})\Big]$ is finite dimensional for each $n = 1, 2, 3, \cdots$. Define

$$w_n(t) = \begin{cases} w(t), & \text{if} \quad t \in N^{||f_0||}\big(w, \frac{1}{n}\big) \\ 0, & \text{if} \quad t \notin N^{||f_0||}\big(w, \frac{1}{n}\big). \end{cases}$$

Then $S_{w_n,T}$ is a compact operator for each $n \in \mathbb{N}$. Thus,

$$\begin{aligned}
\int_G M\Big(t,\frac{||n(S_{w_n,T}-S_{w,T})f(t)||}{||f||}\Big)d\mu(t) & \\
&= \int_G E\Big[M\Big(I\circ T^{-1}(t),\frac{||n\big(w_n(t)-w(t)\big)f(t)||}{||f||}\Big)\Big]f_0(t)d\mu(t)\\
&= \int_{\big(N^{||f_0||}\big(w,\frac{1}{n}\big)\big)'} E\Big[M\Big(I\circ T^{-1}(t),\frac{||nw(t)f(t)||}{||f||}\Big)\Big]f_0(t)d\mu(t)\\
&< \int_{\big(N^{||f_0||}\big(w,\frac{1}{n}\big)\big)'} M\Big(t,\frac{||f(t)||}{||f||}\Big)d\mu(t)\\
&\leq \int_G M\Big(t,\frac{||f(t)||}{||f||}\Big)d\mu(t)\\
&\leq 1.
\end{aligned}$$

Hence

$$\begin{aligned}
||(S_{w_n,T}-S_{w,T})f|| &\leq \frac{1}{n}||f||\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

This proves that $S_{w,T}$ is a compact operator.
Conversely, suppose that $L_M\Big[(G,X),N^{||f_0||}(w,\epsilon)\Big]$ is infinite dimensional for some $\epsilon>0$. Then the closed unit ball of $L_M\Big[(G,X),N^{||f_0||}(w,\epsilon)\Big]$ is not compact. Therefore, there exists a bounded sequence $\{f_n\}$ in the closed unit ball of $L_M\Big[(G,X),N^{||f_0||}(w,\epsilon)\Big]$ such that it has a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ for which $||f_{n_k}-f_{n_j}||\geq\delta$ for some $\delta>0$. Now

$$\begin{aligned}
1 &\geq \int_G M\Big(t,\frac{||(S_{w,T}f_{n_k}-S_{w,T}f_{n_j})(t)||}{\epsilon||S_{w,T}f_{n_k}-S_{w,T}f_{n_j}||}\Big)d\mu(t)\\
&= \int_G E\Big[M\Big(I\circ T^{-1}(t),\frac{||w(t)f_{n_k}(t)-w(t)f_{n_j}(t)||}{||S_{w,T}f_{n_k}-S_{w,T}f_{n_j}||}\Big)\Big]f_0(t)d\mu(t)\\
&> \int_{N^{||f_0||}(w,\epsilon)} M\Big(t,\frac{||\epsilon\big(f_{n_k}(t)-f_{n_j}(t)\big)||}{||S_{w,T}f_{n_k}-S_{w,T}f_{n_j}||}\Big)d\mu(t)\\
&= \int_G M\Big(t,\frac{||\epsilon(f_{n_k}(t)-f_{n_j}(t))||}{||S_{w,T}f_{n_k}-S_{w,T}f_{n_j}||}\Big)d\mu(t).
\end{aligned}$$

Hence

$$\begin{aligned}
||S_{w,T}f_{n_k}-S_{w,T}f_{n_j}|| &\geq \epsilon||f_{n_k}-f_{n_j}||\\
&\geq \epsilon\delta.
\end{aligned}$$

This proves that $\{S_{w,T}f_n\}$ cannot have convergent subsequence. Therefore, $S_{w,T}$ is not compact. Thus, the space $L_M\Big[(G,X),N^{||f_0||}(w,\epsilon)\Big]$ must be finite dimensional. □

Corollary 2.1. *Let $S_{w,T} \in B(L_M(G,X))$. Then $S_{w,T}$ is compact if and only if $S_{w,T} = 0$.*

3. Isometric and invertible weighted composition operators on Musielak-Orlicz spaces of Bochner-type

In this section we investigate a necessary and sufficient condition for a weighted composition operator to be invertible and then we make the use of it to characterize Fredholm weighted composition operators. We also make an effort to characterize the isometric weighted composition operator on Musielak-Orlicz spaces of Bochner-type.

Theorem 3.1. *Let $S_{w,T} \in B(L_M(G,X))$. Then $S_{w,T}$ is invertible if and only if*
(i) T is invertible and
(ii) there exists $\delta > 0$ such that

$$E\Big[M\big(I \circ T^{-1}(t), ||w(t)y||\big)\Big] f_0(t) \geq M(t, \delta||y||)$$

for μ-almost all $t \in G$ and $y \in (X, ||.||_X)$.

Proof. Suppose $S_{w,T}$ is invertible. Then clearly T is surjective. If T is not surjective, we can find a measurable subset $F \subset G \setminus T(G)$ for which $\chi_F \in L_M(G,X)$. We see that $S_{w,T}\chi_F = 0$, which shows that $S_{w,T}$ has non-trivial kernel. Hence T must be surjective. Similarly if T is not injective, then $S_{w,T}$ has not dense range. By Hahn-Banach theorem there exists $0 \neq g* \in L^*_M(G,X)$ such that $g^*(S_{w,T}f) = 0$ for all $f \in L_M(G,X)$. Now by the Riesz-Representation theorem for linear functionals there exists $g \in L^*_M(G,X)$ such that $g^*(h) = \int h.gd\mu$. Thus,

$$(S^*_{w,T}g^*)(f) = g^*(S_{w,T}f) = 0.$$

Also

$$(\ker S_{w,T})^* = (\overline{\text{ran} S_{w,T}})^{\perp} \neq \{0\},$$

which shows $\text{ran}S_{w,T}$ is not dense. This contradicts that $S_{w,T}$ has dense range. Hence T must be injective. Thus, T is invertible. Also $S_{w,T}$ is bounded away from zero. Therefore, the condition (2.2) is satisfied.
Conversely, if the conditions (i) and (ii) hold, then $S_{w,T}$ is bounded away from zero and has dense range. Hence $S_{w,T}$ is invertible. □

Theorem 3.2. *Let $S_{w,T} \in B(L_M(G,X))$. Then $S_{w,T}$ is Fredholm if and only if $S_{w,T}$ is invertible.*

Proof. Suppose $S_{w,T}$ is invertible. Then clearly $S_{w,T}$ is Fredholm. Conversely, suppose that $S_{w,T}$ is Fredholm. Then $\ker S_{w,T}$ is finite dimensional. We know that $\ker S_{w,T}$ is either zero dimensional or infinite dimensional. Hence $\ker S_{w,T} = 0$, which shows that $w \circ T \neq 0$ and T is surjective. Next, if $(\overline{\text{ran} S_{w,T}})^{\perp} \neq \{0\}$, then there exists a bounded linear functional $0 \neq g^* \in (L^*_M(G,X)$ such that $g^*(S_{w,T}f) = 0$ that is $(S^*_{w,T}g^*)(f) = 0$. By the Representation theorem for functionals there exists $g \in L_{M^*}(G,X)$ such that $g^*(S_{w,T}f) = \int_G S_{w,T}fgd\mu = 0$. Let $F = \text{supp } g$ and $\{F_n\}$ be a sequence of disjoint measurable sets such that $\cup_{n=1}^{\infty} F_n = F$ and $\chi_{F_n} \in L_M(G,X)$. Take $g^*_n = g^*\chi_{F_n}$. Clearly $S^*_{w,T}g^*_n = 0$ for all $n = 1, 2, 3, \cdots$. This proves that $\text{ran} S_{w,T}$ is dense, since $\text{ran} S_{w,T}$ is closed. Therefore there exists $\delta > 0$ such that

$$E\Big[M\big(I \circ T^{-1}(t), ||w(t)y||\big)\Big] f_0(t) \geq M(t, \delta||y||)$$

for μ-almost all $t \in G$ and for all $y \in (X, ||.||_X)$. This proves that $S_{w,T}$ is invertible. □

Theorem 3.3. *Let $S_{w,T} \in B(L_M(G,X))$ and*

$$E\Big[M\big(I \circ T^{-1}(t), ||w(t)y||\big)\Big] f_0(t) = M(t, ||y||)$$

for μ-almost all $t \in G$ and $y \in X$. Then $S_{w,T}$ is an isometry if and only if $||w(t)|| = 1$ a.e.

Proof. Suppose $||w(t)|| = 1$ for μ-almost all $t \in G$. Then for $f \in L_M(G,X)$, we have

$$\begin{aligned}
\int_G M\Big(t, \frac{||(S_{w,T}f)(t)||}{||f||}\Big) d\mu(t) &= \int_G M\Big(t, \frac{||w(t)f(t)||}{||f||}\Big) d\mu(t) \\
&= \int_G E\Big[\Big(I \circ T^{-1}(t), \frac{||w(t)f(t)||}{||f||}\Big)\Big] f_0(t) d\mu(t) \\
&= \int_G M\Big(t, \frac{||f(t)||}{||f||}\Big) d\mu(t) \\
&\leq 1.
\end{aligned}$$

Therefore,

$$||S_{w,T}f|| \leq ||f||. \tag{3.1}$$

Again

$$\begin{aligned}
\int_G M\Big(t, \frac{||f(t)||}{||S_{w,T}f||}\Big) d\mu(t) &= \int_G M\Big(t, \frac{||w(t)f(t)||}{||S_{w,T}f||}\Big) d\mu(t) \\
&= \int_G E\Big[\Big(I \circ T^{-1}(t), \frac{||w(t)f(t)||}{||S_{w,T}f||}\Big)\Big] f_0(t) d\mu(t) \\
&= \int_G M\Big(t, \frac{||(S_{w,T}f)(t)||}{||S_{w,T}f||}\Big) d\mu(t) \\
&\leq 1.
\end{aligned}$$

Therefore,

$$||S_{w,T}f|| \geq ||f||. \tag{3.2}$$

Hence

$$||S_{w,T}f|| = ||f||.$$

This proves that $S_{w,T}$ is an isometry.
Conversely, suppose that the condition of the theorem is not true. Then $||w(t)|| \neq 1$ a.e. Suppose $||w(t)|| < 1$ a.e.. Then the set $F = \{t \in G : ||w(t)|| < 1 - \epsilon\}$ is of positive measure for some $\epsilon > 0$. We can choose a subset A of F such that $\chi_A \in L_M(G, X)$. Now

$$\begin{aligned}\int_G M\Big(t, \frac{||(S_{w,T}\chi_A)(t)||}{(1-\epsilon)||\chi_A||}\Big)d\mu(t) &= \int_A E\Big[M\Big(I \circ T^{-1}(t), \frac{||w(t)\chi_A(t)||}{(1-\epsilon)||\chi_A||}\Big)\Big]f_0(t)d\mu(t)\\ &= \int_G M\Big(t, \frac{||w(t)\chi_A(t)||}{(1-\epsilon)||\chi_A||}\Big)d\mu(t)\\ &\leq \ 1.\end{aligned}$$

Therefore,

$$||S_{w,T}\chi_A|| \leq (1-\epsilon)||\chi_A||,$$

which is a contradiction. Similarly, for $||w(t)|| > 1$ a.e. we again get a contradiction. Hence $||w(t)|| = 1$ a.e.. □

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