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Optimality conditions under generalized convexity revisited

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Abstract - We make some comments and generalizations of a result of J. B. Lasserre (2010), by the use of suitable generalized convex functions.

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1. Introduction

In a recent paper (see [21]), Lasserre has considered the following nonlinear programming problem

(P) $$\min_x \{f(x),\ x \in K\}, \text{ where}$$

$$K = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, ...m\}, \tag{1.1}$$

$f,\ g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, f and every g_i are differentiable on some open convex set $S \subset \mathbb{R}^n$, f is convex on S and the feasible set $K \subset \mathbb{R}^n$ is convex. If $x^0 \in K$ we denote by

$$I(x^0) = \left\{i : g_i(x^0) = 0\right\}$$

the set of the *active constraints at* x^0.

Lasserre proves the following result.

Theorem 1.1. (see [21]) *Consider the nonlinear programming problem (P); let $x^0 \in K, \nabla g_i(x^0) \neq 0, \forall i \in I(x^0)$ and let the following Slater's condition hold:*

$$\exists \bar{x} \in K : g_i(\bar{x}) < 0, \forall i \in I(x^0). \tag{1.2}$$

Then x^0 is a global minimum point for (P) if and only if x^0 satisfies the following Karush-Kuhn-Tucker (KKT) conditions for (P)*:*

$$\nabla f(x^0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^0) = 0, \tag{1.3}$$

$$\lambda_i g_i(x^0) = 0,\ \ i = 1, ...m, \tag{1.4}$$

$$\lambda_i \geq 0,\ \ i = 1, ...m. \tag{1.5}$$

This theorem can be deduced from the pioneering papers of Arrow and Enthoven [1] and Arrow, Hurwicz and Uzawa [3]. Moreover, it can be better enlightened by making reference to appropriate generalized convex functions. With regard to this last point, Lasserre remarks that the set

$$K = \left\{ x \in \mathbb{R}^2 : 1 - x_1 x_2 \leq 0, x \geq 0 \right\}$$

is convex, but the function $g(x_1, x_2) = 1 - x_1 x_2$ is not convex on $\mathbb{R}^2_+$. Indeed, this function is *quasiconvex* on $\mathbb{R}^2_+$.

The present note is organized as follows. In Section 2 we make some comments and generalizations concerning the main result of Lasserre in [21]. In Section 3 we recall some possible applications of suitable generalized convex functions in obtaining saddle points conditions for the Lagrangian function for (P), also without any differentiability assumptions.

We recall some basic definitions and properties.

Definition 1.1. *Let $S \subset \mathbb{R}^n$ be a convex set; a function $f : S \longrightarrow \mathbb{R}$ is* ***quasiconvex*** *on S if the lower-level set*

$$L_{\leq \alpha} = \{x \in S : f(x) \leq \alpha\}$$

is convex for each $\alpha \in \mathbb{R}$.

If f is differentiable on the open and convex set S, then f is quasiconvex on S (see [1]) if and only if

$$x,\ y \in S,\ f(y) \leq f(x) \Longrightarrow \nabla f(x)(y - x) \leq 0. \tag{1.6}$$

Definition 1.2. *A function $f : S \longrightarrow \mathbb{R}$, differentiable on the open (and convex) set $S \subset \mathbb{R}^n$ is* ***pseudoconvex*** *on S if*

$$x,\ y \in S,\ \nabla f(x)(y - x) \geq 0 \Longrightarrow f(y) \geq f(x). \tag{1.7}$$

The function f is then called *quasiconvex at x^0* (respectively, *pseudoconvex at x^0*), with respect to S, if relation (1.6) (respectively, relation (1.7)) holds at a fixed point $x^0 \in S$, for each $y \in S$. In this last case ("generalized convexity at a point") the set S is no longer required to be convex, but it is required only to be *star-shaped* at x^0. A set $S \subset \mathbb{R}^n$ is star-shaped at $x^0 \in S$ if $\lambda x^0 + (1 - \lambda)x \in S, \forall x \in S, \forall \lambda \in [0, 1]$.

2. Comments on the result of Lasserre

The following theorem of Mangasarian (see [22]) is well-known.

Theorem 2.1. (see [22]) *Let $S \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. Suppose that f is differentiable and pseudoconvex on S and that every g_i, $i = 1, ..., m$, is differentiable and quasiconvex on S. If $x^0 \in K$ satisfies the KKT conditions* (1.2)$-$(1.5)*, then x^0 solves (P).*

Remark 2.1. a) The above result holds also if f is pseudoconvex at x^0, with respect to S or also with respect to K, and every g_i, $i \in I(x^0)$, is quasiconvex at x^0, with respect to S (or also to K).

b) If every g_i, $i = 1, ..., m$, is quasiconvex, obviously the feasible set K is a convex set.

c) The theorem fails if the objective function f is quasiconvex (a counterexample is due to Arrow and Enthoven, see [1]). However, additional assumptions allow to recover also the case of f quasiconvex. The starting point is a theorem of Crouzeix and Ferland in [10], see also the paper by Giorgi [14].

Theorem 2.2. *Let f be a differentiable and quasiconvex function on the open convex set $S \subset \mathbb{R}^n$. Then f is pseudoconvex on S if and only if f has a minimum at $x \in S$ whenever $\nabla f(x) = 0$.*

An immediate consequence is the following result.

Corollary 2.1. *Assume that $S \subset \mathbb{R}^n$ is an open convex set, $f : S \longrightarrow \mathbb{R}$ is differentiable and $\nabla f(x) \neq 0$, for all $x \in S$. Then f is pseudoconvex on S if and only if f is quasiconvex on S.*

See the paper [8] by Cambini and Martein for a version of the above corollary, involving quasiconvexity and pseudoconvexity at a point $x^0 \in S$, where S is star-shaped at x^0.

Now we turn to the first half of the result of Lasserre. First we note that his assumptions on the convexity of f can be relaxed on the grounds of what previously expounded.

Theorem 2.3. *Let in* (P) *f be differentiable and pseudoconvex at $x^0 \in K$, with respect to S, let every g_i, $i \in I(x^0)$, be differentiable at x^0 and let K be a convex set. If x^0 satisfies the KKT conditions* (1.3)$-$(1.5)*, then x^0 solves* (P).

Proof. Being K convex, the point $(1 - \alpha)x^0 + \alpha x$ belongs to K, for every $\alpha \in [0, 1]$ and for every $x, x^0 \in K$, whence

$$\tilde{g}_i(\alpha) =: g_i((1 - \alpha)x^0 + \alpha x) \leq 0, \ \forall \alpha \in [0, 1] .$$

As $\tilde{g}_i(0) = g_i(x^0) = 0$, $\forall i \in I(x^0)$, it results ($\tilde{g}_i'(\alpha)$ denoting $\mathrm{d}\tilde{g}_i(\alpha)/\mathrm{d}\alpha$)

$$\tilde{g}_i'(0) = \nabla g_i(x^0)(x - x^0) \leq 0, \ \forall i \in I(x^0), \forall x \in K. \tag{2.1}$$

This relation, taking into account that x^0 satisfies the KKT conditions (1.3)−(1.5) and that f is pseudoconvex at x^0, allows to state that x^0 solves (P). □

Remark 2.2. a) Note that the above sufficient optimality conditions hold without imposing on the constraint functions g_i, $i \in I(x^0)$, any *nondegeneracy condition* at x^0 (i.e. $\nabla g_i(x^0) \neq 0$, $i \in I(x^0)$).

b) We have already remarked that a sufficient condition for K to be convex is that every g_i, $i = 1, ..., m$, is quasiconvex. Note, moreover, that relation (2.1) is equivalent to state that every g_i, $i \in I(x^0)$, is quasiconvex at x^0, with respect to K.

c) Being x^0 fixed, the previous theorem holds also if the set K is star-shaped at x^0.

Now we turn to the second half of the result of Lasserre. This part holds without any convexity or generalized convexity requirement on the objective function f. This result appears as Corollary 5 in [3]. For the reader's convenience we give a proof.

Theorem 2.4. *Let x^0 be a local solution of* (P)*, let the feasible set K be convex, let $\nabla g_i(x^0) \neq 0$, $\forall i \in I(x^0)$, and let the Slater's condition* (1.2) *hold. Then x^0 verifies the KKT conditions* (1.3)−(1.5)*.*

Proof. Let us denote by L the *linearizing cone at x°*:

$$L = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^0) y \leq 0, \ i \in I(x^\circ) \right\}.$$

Being K convex, on the same lines of the proof of the previous theorem, we can obtain relation (2.1), i.e. the vector $y = (x - x^\circ)$ belongs to L for all $x \in K$. Since K possesses a nonempty interior, L must possess one also and therefore has the full dimensionality of the entire space. If, for some $i \in I(x^\circ)$, it holds $\nabla g_i(x^0) y = 0$ for all y in L, then, being $\nabla g_i(x^0) \neq 0$, $\forall i \in I(x^0)$, the set

$$H^i = \left\{ y \in L : \nabla g_i(x^0) y = 0 \right\}$$

would belong to a hyperplane and therefore $int(L) = \varnothing$. Therefore, it must exist $y^i \in \mathbb{R}^n$ such that $\nabla g_i(x^0) y^i < 0$, $i \in I(x^\circ)$. Hence, also the vector $\bar{y} = \sum_{i \in I(x^\circ)} y^i$ is an element of L and it holds $\nabla g_i(x^0) \bar{y} < 0, i \in I(x^\circ)$.

This last system of inequalities is just the Mangasarian-Fromovitz constraint qualification (see [22]), previously considered by Cottle in [9] and by Arrow, Hurwicz and Uzawa in [3]. This assures the validity of the KKT conditions (1.3)−(1.5) for the point x°. □

Note that if in Theorem 2.4 the assumption on the convexity of K is replaced by: every g_i, $i \in I(x^0)$, is quasiconvex at x^0, with respect to the open and star-shaped set S, then the assumption $\nabla g_i(x^0) \neq 0, \forall i \in I(x^0)$, entails that every g_i, $i \in I(x^0)$, is pseudoconvex at x^0, with respect to S. Therefore, if the Slater's condition (1.2) holds, the point x° verifies the KKT conditions (1.3)$-$(1.5), see [22].

Now we want to stress that a powerful generalization of convexity (for the differentiable case) is the notion of *invexity*, which can be used to obtain necessary and sufficient optimality conditions for (P), see, e.g., the paper [25] by Mishra and Giorgi.

Definition 2.1. *A differentiable function f defined on an open set X of $\mathbb{R}^n$ is said to be **invex** if there exists a vector function $\eta : X \times X \longrightarrow \mathbb{R}^n$ such that $f(y) \geq f(x) + \nabla f(x)\eta(y,x)$, $\forall x, y \in X$.*

Theorem 2.5. (see [6]) *A differentiable function f defined on an open set $X \subset \mathbb{R}^n$ is invex if and only if every statiom point is a global minimum point.*

Since a stationary point is a global minimum point for pseudoconvex functions, the class of pseudoconvex functions is contained in the class of invex functions. On the other hand, there is only a partial overlapping between the class of invex functions and the class of quasiconvex functions. For example, $f(x) = x^3$ is quasiconvex on $\mathbb{R}$ (and quasiconcave), but not invex, since its stationary point $x = 0$ is not a minimum point; $f(x, y) = x^2y^2$ is invex on $\mathbb{R}^2$, but not quasiconvex (and so, not pseudoconvex), see also the paper [6] by Ben-Israel and Mond and Giorgi's paper [14].

Following Martin (see [23]), let us introduce the following notion of *Kuhn-Tucker invexity* (KT-invexity) for (P): there exists $\eta : X \times X \longrightarrow \mathbb{R}^n$ such that, for any $x, x^\circ \in K$ we have

$$\begin{cases} f(x) - f(x^\circ) - \nabla f(x^\circ)\eta(x, x^\circ) \geq 0 \\ -\nabla g_i(x^\circ)\eta(x, x^\circ) \geq 0, \; i \in I(x^0). \end{cases}$$

Martin then proved the following result.

Theorem 2.6. *Every Karush-Kuhn-Tucker point of problem* (P)*, i.e. every point satisfying relations* (1.3)$-$(1.5)*, is a global minimizer if and only if* (P) *is KT-invex.*

However, there is an open question. KT-invexity is trivially satisfied at every solution x° of problem (P), by letting $\eta(x, x^\circ) = 0$. This one is a tautological condition: it is coincident with the definition of a solution of (P). The question is overcome by Hanson and Mond (see [15]) who studied the problem of finding necessary optimality conditions of invex type that are not trivial, i.e. with $\eta(x, x^\circ)$ not identically zero for each feasible vector x. These authors introduced the *type I invexity*, a pointwise notion of invexity for (P), we here denote by *HM-invexity*:

(P) is HM-invex at $x^\circ \in K$ if there exists $\eta : X \longrightarrow \mathbb{R}^n$ such that, for every $x \in K$ we have

$$\begin{cases} f(x) - f(x^\circ) - \nabla f(x^\circ)\eta(x) \geq 0 \\ -\nabla g_i(x^\circ)\eta(x) \geq 0, \; i \in I(x^0). \end{cases}$$

The results of Hanson and Mond in [15] are summarized in the following proposition.

Theorem 2.7. *If $x^\circ \in K$ is a Karush-Kuhn-Tucker point for* (P) *and $card(I(x^\circ)) < n$, then x° solves* (P) *if and only if* (P) *is HM-invex at x°, with respect to a vector function $\eta(x)$, which is not identically zero, for each $x \in K$.*

A further weakening of the results of Hanson and Mond is given by Mishra and Giorgi in [25]. For the case $\eta(x, x^\circ) = x - x^\circ$, the result of Martin in [23] has been restated by Ivanov (see [18]), under suitable assumptions. This author calls the problem (P) *KT-pseudoconvex* if:

$$\left.\begin{array}{c} x, x^\circ \in S \\ f(x) < f(x^0) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \nabla f(x^0)(x - x^0) < 0 \\ \nabla g_i(x^0)(x - x^0) \leq 0, i \in I(x^0). \end{array}\right.$$

Theorem 2.8. (see [18]) *Let f and g_i, $i = 1, ..., m$, be quasiconvex on the open convex set $S \subset \mathbb{R}^n$. Then every Karush-Kuhn-Tucker point of problem (P) is a global minimizer if and only if (P) is KT-pseudoconvex.*

Finally, we note that if the feasible set K (not necessarily given by a finite number of inequalities as in (1.1)) is convex and the objective function is pseudoconvex, then we can establish an equivalence between the problem (P) and its "linearized" version

$$(\mathrm{P}_1) \qquad \min_x \left\{\nabla f(x^0)x, \;\; x \in K\right\}.$$

Theorem 2.9. *Let $f : S \longrightarrow \mathbb{R}$ be pseudoconvex on the open convex set $S \subset \mathbb{R}^n$ and let be given the following problem*

$$(\mathrm{P}_0) \qquad \min_x \{f(x), \;\; x \in K\},$$

where $K \subset S$ is a convex set. Then $x^0 \in K$ is a solution of (P_0) *if and only if x^0 is solution of* (P_1).

Proof. If x^0 is solution of (P_0), being f pseudoconvex on the convex set $K \subset S$, f is also quasiconvex on K and, being $f(x^0) \leq f(x)$, $\forall x \in K$, we have $\nabla f(x^0)(x - x^0) \leq 0$, $\forall x \in K$, i.e. x^0 is optimal for (P_1). Conversely, if x^0 is solution for (P_1), i.e. $\nabla f(x^0)x^0 \leq \nabla f(x^0)x$, $\forall x \in K$, then by the pseudoconvexity of f we have $f(x) \geq f(x^0), \forall x \in K$, i.e. x^0 is solution of (P_0). □

Obviously, Theorem 2.9 holds also if K is given by (1.1), i.e. (P_0) ≡ (P) and the functions g_i, $i = 1, ..., m$, are quasiconvex on the open convex set $S \subset \mathbb{R}^n$. Moreover, in this case, we have the following result.

Theorem 2.10. *Let f be differentiable and pseudoconvex on the open convex set $S \subset \mathbb{R}^n$ and every g_i, $i = 1, ..., m$, differentiable and quasiconvex on the same set S. We assume a suitable constraint qualification which assures the validity of the KKT conditions at $x^0 \in K$ (for example, $\nabla g_i(x^0) \neq 0, i \in I(x^0)$, and Slater's condition* (1.2)*). Then, x^0 is solution of* (P) *if and only if x^0 is solution of*

$$(P_2) \qquad \min_x \left\{ \nabla f(x^0)x \mid \nabla g_i(x^0)(x - x^0) \leq 0,\ i \in I(x^0) \right\}$$

or of

$$(P_3) \qquad \min_x \left\{ f(x) \mid \nabla g_i(x^0)(x - x^0) \leq 0,\ i \in I(x^0) \right\}.$$

Proof. Denote by K_2 the feasible set of (P_2). If x^0 is solution of (P), then, thanks to the constraint qualification, the KKT conditions hold at x^0:

$$\nabla f(x^0) = -\sum_{i \in I(x^0)} \lambda_i \nabla g_i(x^0),\ \lambda_i \geq 0, i \in I(x^0).$$

Evidently $x^0 \in K_2$ and for any $x \in K_2$ we have

$$\nabla f(x^0)x^0 = -\sum_{i \in I(x^0)} \lambda_i \nabla g_i(x^0)x^0 \leq -\sum_{i \in I(x^0)} \lambda_i \nabla g_i(x^0)x = \nabla f(x^0)x.$$

Therefore x^0 is optimal for (P_2). Now, let x^0 optimal for (P_2). We have $K \subset K_2$, by the quasiconvexity of $g_i, i \in I(x^0)$. Then we have $\nabla f(x^0)x^0 \leq \nabla f(x^0)x, \forall x \in K_2$ and as this inequality holds also for every $x \in K$, we get $f(x^0) \leq f(x), \forall x \in K$, i.e. x^0 is optimal for (P). According to Theorem 2.9, problems (P_2) and (P_3), under our assumptions, are equivalent. So, the proof is finished. □

Note that, in proving the equivalence between (P) and (P_2), in the necessity part of the proof it is possible to assume no generalized convexity of f and g_i, $i = 1, ...m$, but only a suitable constraint qualification which does not involve any generalized convexity assumption. In the sufficiency part, the theorem holds also with the weaker assumption that f is pseudoconvex at x^0 with respect to K and that every g_i, $i \in I(x^0)$, is quasiconvex at x^0 with respect to K. Note that (P_2) is a linear programming problem, therefore the equivalence between (P) and (P_2) is a "linear test" for the optimality of x^0 in the nonlinear programming problem (P).

3. Generalized Convexity and Lagrangian Saddle Point Conditions

A fundamental result of nonlinear programming relates the solution of a convex problem to the saddle point conditions of its Lagrangian function. Given the problem (P) and the associated Lagrangian function $L(x, \lambda) = f(x) + \lambda g(x)$, the point (x^0, λ^0) is called a *saddle point* of L if

$$L(x^0, \lambda) \leq L(x^0, \lambda^0) \leq L(x, \lambda^0),\ \forall x \in S, \forall \lambda \in \mathbb{R}^m,\ \lambda \geq 0. \tag{3.1}$$

It is well-known that if f and every g_i, $i = 1, ..., m$, are convex on the convex set $S \subset \mathbb{R}^n$, then there are interesting relationships between (P), the saddle point conditions (3.1) and the KKT conditions. These relationships generally do not longer hold if f is pseudoconvex and every g_i, $i = 1, ..., m$, is quasiconvex or even pseudoconvex. However, some results can be obtained, with regard to the saddle point problem, under suitable generalized convexity of the functions involved in (P).

1) A basic result, due to Kuhn and Tucker (see [20]) states that if (x^0, λ^0) satisfies the KKT conditions (1.3)$-$(1.5) and f and every g_i, $i = 1, ..., m$, are convex on the convex set $S \subset \mathbb{R}^n$, then (x^0, λ^0) is a saddle point of L. As already said, this is no longer true under generalized convexity assumptions, as the positive linear combination of pseudoconvex (or of quasiconvex) functions need not be a pseudoconvex (resp. a quasiconvex) function. However, the result of Kuhn and Tucker holds if $L(x, \lambda)$ is pseudoconvex in x: in this case the KKT conditions mean that (x^0, λ^0) is a stationary point of the Lagrangian function and the result is recovered. The said result can be obtained also under the concept of *invex function* (see Definition 2.1).

Due to the fact that the positive linear combination of n functions $f_1, f_2, ..., f_n$, all invex with respect to the *same* vector function η, is an invex function, we can state the following result.

Theorem 3.1. *Assume that* $x^0 \in K$ *satisfies the KKT conditions* (1.3)$-$(1.5)*. If* f *and every* g_i, $i = 1, ..., m$ *are invex with respect to the same* η*, then there exists* $\lambda^0 \geq 0$ *such that* (3.1) *holds. (Moreover, we have the classical complementary conditions* $\lambda^0 g(x^0) = 0$*).*

Recently Martinez-Legaz (see [24]) and Mishra and Giorgi (see [25]) have proved the following result.

Theorem 3.2. *Let* $f_1, f_2, ..., f_n$ *be differentiable functions defined on an open set* $X \subset \mathbb{R}^n$*. The following statements are equivalent.*

(i) *The functions* $f_1, f_2, ..., f_n$ *are invex with respect to the same* η*.*

(ii) *The functions* $\sum_{i=1}^n \lambda_i f_i$, $\lambda_1 \geq 0, ..., \lambda_n \geq 0$*, are invex with respect to the same* η*.*

(iii) *The functions* $\sum_{i=1}^n \lambda_i f_i$, $\lambda_1 \geq 0, ..., \lambda_n \geq 0$*, are invex.*

(iv) For every $\lambda_1 \geq 0, ..., \lambda_n \geq 0$, every stationary point of $\sum_{i=1}^{n} \lambda_i f_i$ is a global minimum point.

2) Another approach to the saddle point problem (3.1), by means of some generalized convex function, is given by the assumption of *preinvexity* of the functions involved in problem (P). This assumption does not require differentiability; see the paper [27] by Weir and Jeyakumar and the paper [28] by Weir and Mond.

Definition 3.1. *A subset X of $\mathbb{R}^n$ is said to be η-**invex** with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ if*

$$x, y \in X,\ \lambda \in [0,1] \Longrightarrow y + \lambda\eta(x,y) \in X.$$

Therefore, $\eta-$invexity of a set is a sort of "connectedness" property. It is obvious that the above definition is a generalization of the notion of a convex set.

Definition 3.2. *Let $f : X \longrightarrow \mathbb{R}$ be defined on the η-invex set $X \subset \mathbb{R}^n$; f is said to be **preinvex on** X with respect to η if*

$$f\left[y + \lambda\eta(x,y)\right] \leq \lambda f(x) + (1-\lambda) f(y),\ \forall x, y \in X, \forall \lambda \in [0,1].$$

The class of convex functions is strictly contained in the class of preinvex functions. Moreover, if $f_1, f_{2,}..., f_n$ are preinvex functions with respect to the same η, then $\sum_{i=1}^{n} \lambda_i f_i$ is preivex, with respect to η, where $\lambda_i \geq 0$, $i = 1, ..., n$. Ben-Israel and Mond have proved (see [6]) that if f is differentiable and preinvex, then f is invex.

Theorem 3.3. (see [28, 27]) *Let be given the problem* (P)*, where all the functions involved are preinvex, with respect to the same η, on the η-invex set $S \subset \mathbb{R}^n$. If $x^0 \in K$ is a solution of* (P) *and if the Slater's condition* (1.2) *holds, then there exists $\lambda^0 \geq 0$ such that (x^0, λ^0) is a saddle point for* (P).

3) Another possibility to obtain saddle points result for a not necessarily convex problem (P) is to try to convert (P) into an equivalent convex problem. This can be done (when it is possible!) by means of the so-called *convex range transformations.* Let $f : S \longrightarrow \mathbb{R}$ be defined on the convex set $S \subset \mathbb{R}^n$ and denote by $I_f(S)$ the range of f.

Definition 3.3. *$f : S \longrightarrow \mathbb{R}$ is said to be **convex range transformable** or briefly **F-convex** if there exists a continuous strictly increasing function $F : I_f(S) \longrightarrow \mathbb{R}$ such that $F\left[f(x)\right]$ is convex over S, i.e. for any $x, y \in S$ and any $\lambda \in [0,1]$ we have*

$$f(\lambda x + (1-\lambda)y) \leq F^{-1}\left[\lambda F(f(x)) + (1-\lambda) F(f(y))\right].$$

It is easy to prove that every F-convex function on a convex set S is also quasiconvex on S. Then, if in (P) all the functions involved are defined on the convex set $S \subset \mathbb{R}^n$ and are all F-convex, it is clear that the transformed problem

$$\min_x \{F[f(x)] \mid F_i[g_i(x)] \leq 0,\ i = 1, ..., m\}$$

is equivalent to (P). Convex transformable functions were introduced by de Finetti (see [11]). In [4], Avriel discusses the properties of an important class of F-convex functions, see also the paper [5] by Avriel and others.

4) We mention the possibility to weaken the convexity of the vector-valued function $(f, g) : \mathbb{R}^n \longrightarrow \mathbb{R}^{m+1}$ by means of the definition of *convexlike function* and its generalizations.

Definition 3.4. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is **convexlike** on the nonempty set $X \subset \mathbb{R}^n$ if for any $x^1, x^2 \in X$ and for any $\lambda \in [0, 1]$ there exists $x^3 \in X$ such that*

$$\lambda f(x^1) + (1 - \lambda) f(x^2) - f(x^3) \in \mathbb{R}^p_+.$$

Note that in the above definition the point x^3, which usually depends from x^1, x^2 and λ, is not necessarily given by a convex combination of x^1 and x^2, but it can be just any point of X. Therefore, there is no need to require the convexity of X; note, moreover, thar any real-valued function is convexlike. It is quite immediate to show that the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is convexlike on $X \subset \mathbb{R}^n$ if and only if the set $f(X) + \mathbb{R}^p_+$ is convex. Obviously, all convex functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ are convexlike; the convexity of the function f is only a sufficient condition. For other sufficient conditions for a vector-valued function to be convexlike, see the paper [12] by Elster and Nehse.

For applications of convexlike functions and their generalizations to saddle point problems, alternative theorems, duality theorems, etc., see the papers [7] by Cambini, [16] by Hayashi and Komiya, [19] by Jeyakumar, [13] by Frenk and Kassay, [17] by Illes and Kassay. For the reader's convenience we report only the main result of Jeyakumar (see [19]) concerning a generalized saddle point theorem.

Theorem 3.4. (see [19]) *For the problem* (P) *assume that the pair (f, g) is convexlike with respect to $\mathbb{R}^{m+1}$ and that the Slater's condition* (1.2) *holds. If $x^\circ \in K$ is a solution of* (P)*, then there exists a vector $\lambda^\circ \in \mathbb{R}^m_+$ such that the pair (x°, λ°) satisfies relation* (3.1)*.*

5) Yet another approach to obtain saddle point results for a nonconvex programming problem, is given by considering an "augmented Lagrangian function", i.e. a suitable modification of the usual Lagrangian function $L(x, \lambda)$. We quote only (also for the bibliographical references) the papers of Arrow, Gould and Howe (see [2]) and Rockafellar (see [26]).

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