Annals of the University of Bucharest (mathematical series)
4 (LXII) (2013), 455–478

On the continuity of superposition operators between higher-order Sobolev spaces in the supercritical case

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Abstract - This paper is a completion of our previous work [6, 7] on superposition operators between higher-order Sobolev spaces, where sufficient conditions which ensure the well-definedness, the continuity, the boundedness, and the validity of the higher-order chain rule for such operators were given. We prove the continuity of these superposition operators in the supercritical case (see Remark 1.1).

Key words and phrases : Sobolev spaces, superposition operators, higher-order chain rule, continuity.

Mathematics Subject Classification (2010) **:** 47H30, 46E35, 26B05, 26A46.

1. Introduction

Let $g : \mathbb{R} \to \mathbb{R}$ be a function, let $n \in \mathbb{N}^*$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The superposition (or Nemytskij) operator generated by the function g is, by definition, the operator denoted by N_g that associates to each function $u : \Omega \to \mathbb{R}$ the function $N_g u : \Omega \to \mathbb{R}$ defined by

$$(N_g u)(x) = (g \circ u)(x) = g(u(x)), \quad x \in \Omega.$$

For each $k \in \mathbb{N}^*$, we denote by $\mathcal{L}^k$ the k-dimensional Lebesgue measure. It is well-known that if $g : \mathbb{R} \to \mathbb{R}$ is continuous and $u : \Omega \to \mathbb{R}$ is $\mathcal{L}^n$-measurable, then $N_g u = g \circ u$ is $\mathcal{L}^n$-measurable as well. The same conclusion remains valid even when the hypothesis "$g : \mathbb{R} \to \mathbb{R}$ is continuous" is replaced with the weaker hypothesis "$g : \mathbb{R} \to \mathbb{R}$ is Borel measurable".

Within a series of papers, Marcus and Mizel [9, 10, 11] obtained necessary and sufficient conditions for a function g to generate a superposition operator N_g having the following properties: N_g is well defined from a Sobolev space $W^{1,p}(\Omega)$ into another Sobolev space $W^{1,q}(\Omega)$, with $1 \le q \le p < \infty$, N_g is bounded, continuous and satisfies in addition the first-order chain rule

$$\partial_i (g \circ u) = (g' \circ u)\, \partial_i u \quad \mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } i = 1, \ldots, n, \tag{1.1}$$

for all $u \in W^{1,p}(\Omega)$. Here and throughout this paper, ∂_i denotes the weak derivative with respect to x_i. Part of Marcus and Mizel's results are also reproduced in Appell [1].

Another essential result in the study of superposition operators between Sobolev spaces is due to Bourdaud. In [2], he obtained necessary and sufficient conditions on a function g such that $N_g : W^{m,p}(\mathbb{R}^n) \to W^{m,p}(\mathbb{R}^n)$ is well defined, with $m \in \mathbb{N}$, $m \geq 2$, and $1 \leq p < \infty$. In their survey [3], the authors noticed that when $p = 1$ and $m = 2 < n$, or $mp = n$, it is not proved yet that those conditions on g also ensure the continuity and the boundedness of $N_g : W^{m,p}(\mathbb{R}^n) \to W^{m,p}(\mathbb{R}^n)$. Moreover, those conditions do not generally ensure the validity of the higher-order chain rule for N_g up to order m inclusive, except for the degeneracy case $g'' \equiv 0$.

In [5], motivated by the intention to obtain a generalization of the well-known Pohožaev identity, the authors generalized the results of Marcus and Mizel. They obtained sufficient conditions for a function g to generate a superposition operator N_g having the following properties: N_g is well defined from a Sobolev space $W^{m,p}(\Omega)$ into another Sobolev space $W^{1,q}(\Omega)$, with $1 \leq q, p < \infty$, $m \in \mathbb{N}^*$, N_g is bounded, continuous and satisfies in addition the chain rule (1.1). In the supercritical case (see Remark 1.1 below), this result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, having the cone property, let $m \in \mathbb{N}^*$, and let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function.*

(i) *If $\frac{n}{m} < p < \frac{n}{m-1}$, with $n \geq m$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$), then $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ for all $1 \leq q \leq \frac{np}{n-(m-1)p}$. Moreover, N_g is bounded and the chain rule* (1.1) *holds for all $u \in W^{m,p}(\Omega)$, where the product $(g' \circ u)\,\partial_i u$ is to be interpreted in the sense of de la Vallée Poussin, namely it is considered to be zero whenever $\partial_i u(x) = 0$, irrespective of whether $(g' \circ u)(x)$ is defined.*

(ii) *If $p = \frac{n}{m-1}$, with $m \geq 2$ and $n \geq m-1$, then $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ for all $1 \leq q < \infty$. Moreover, N_g is bounded and* (1.1) *holds for all $u \in W^{m,p}(\Omega)$ (with the usual convention on the product $(g' \circ u)\,\partial_i u$).*

(iii) *If $\frac{n}{m-1} < p < \infty$, with $m \geq 2$ ($1 \leq p < \infty$ when $n \leq m-1$), then $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ for all $1 \leq q \leq \infty$. Moreover, N_g is bounded and* (1.1) *holds for all $u \in W^{m,p}(\Omega)$ (with the usual convention on the product $(g' \circ u)\,\partial_i u$).*

(iv) *If $g^* : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that $g^* = g'$ $\mathcal{L}^1$-a.e. in $\mathbb{R}$, then in all cases* (i)-(iii) *the chain rule* (1.1) *can be rewritten as*

$$\partial_i (g \circ u) = (g^* \circ u)\,\partial_i u \quad \mathcal{L}^n\text{-a.e in } \Omega, \text{ for all } i = 1, \ldots, n,$$

the convention on the product $(g^* \circ u)\, \partial_i u$ *being no longer necessary.*

(v) *The hypotheses which ensure the well-definedness of the operator* N_g *from* $W^{m,p}(\Omega)$ *into* $W^{1,q}(\Omega)$, *with* $1 \leq p, q < \infty$, *are sufficient to ensure the continuity of* N_g *in each of the cases* (i)-(iii).

Remark 1.1. According to Bourdaud [2], the Sobolev space $W^{m,p}(\Omega)$ is said to be supercritical if the imbedding $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is valid.

In [7], the authors generalized Theorem 1.1. More specifically, they obtained sufficient conditions for a function g to generate a superposition operator N_g having the following properties: N_g is well defined from a Sobolev space $W^{m,p}(\Omega)$ into another Sobolev space $W^{l,q}(\Omega)$, with $1 \leq q, p < \infty$, $m, l \in \mathbb{N}^*$, $mp > n$, $l \leq m$, N_g is bounded, continuous, and satisfies in addition the higher-order chain rule

$$D^\alpha (g \circ u) = \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \alpha \\ |\alpha^i| \neq 0}} c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^k} u \qquad \mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } \alpha \in \mathbb{N}^n \text{ with } 1 \leq |\alpha| \leq l, \tag{1.2}$$

for all $u \in W^{m,p}(\Omega)$. Here and throughout this paper, $c_{\alpha,k,\alpha^1,\ldots,\alpha^k} \in \mathbb{N}^*$ denotes a combinatorial constant and D^α denotes the weak derivative with respect to the multi-index $\alpha \in \mathbb{N}^n$.

The statement of this result is the following.

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded, having the cone property, let* $m, l \in \mathbb{N}^*$, $l \leq m$, *and let* $g : \mathbb{R} \to \mathbb{R}$ *be a function of class* C^{l-1} *with* $g^{(l-1)} : \mathbb{R} \to \mathbb{R}$ *locally Lipschitz.*

(i) *If* $\frac{n}{m} < p < \frac{n}{m-l}$, *with* $n \geq m - l + 1$ *(*$1 \leq p < \frac{n}{m-l}$ *when* $n \in \{m - l + 1, \ldots, m\}$*), then* $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ *for all* $1 \leq q \leq \frac{np}{n-(m-l)p}$. *Moreover,* N_g *is bounded and the higher-order chain rule* (1.2) *holds for all* $u \in W^{m,p}(\Omega)$, *where the product* $\left(g^{(l)} \circ u\right)$ $\partial_{j_1} u \ldots \partial_{j_l} u$ *is to be interpreted in the sense of de la Vallée Poussin, namely it is considered to be zero whenever one of the factors* $\partial_{j_1} u(x)$, $\ldots, \partial_{j_l} u(x)$ *is zero, irrespective of whether* $\left(g^{(l)} \circ u\right)(x)$ *is defined.*

(ii) *If* $p = \frac{n}{m-l}$, *with* $m \geq l + 1$ *and* $n \geq m - l$, *then* $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ *for all* $1 \leq q < \infty$. *Moreover,* N_g *is bounded and* (1.2) *holds for all* $u \in W^{m,p}(\Omega)$ *(with the usual convention on the product* $\left(g^{(l)} \circ u\right) \partial_{j_1} u \ldots \partial_{j_l} u$*).*

(iii) *If* $\frac{n}{m-l} < p < \infty$, *with* $m \geq l + 1$ *(*$1 \leq p < \infty$ *when* $n \leq m - l$*), then* $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$ *for all* $1 \leq q \leq \infty$. *Moreover,* N_g *is bounded* *and* (1.2) *holds for all* $u \in W^{m,p}(\Omega)$ *(with the usual convention on the product* $\left(g^{(l)} \circ u\right) \partial_{j_1} u \ldots \partial_{j_l} u$*).*

(iv) *If* $g^* : \mathbb{R} \to \mathbb{R}$ *is a Borel measurable function such that* $g^* = g^{(l)}$ $\mathcal{L}^1$*-a.e. in* $\mathbb{R}$*, then in all cases* (i)-(iii)*, the chain rule* (1.2) *can be rewritten with* g^* *instead of* $g^{(l)}$*, the convention on the product* $\left(g^{(l)} \circ u\right) \partial_{j_1} u \ldots \partial_{j_l} u$ *being no longer necessary.*

(v) *The hypotheses which ensure that the operator* N_g *from* $W^{m,p}(\Omega)$ *into* $W^{l,q}(\Omega)$ *is well defined, with* $1 \leq p, q < \infty$*, are sufficient to ensure the continuity of* N_g *in each of the cases* (i)-(iii)*.*

The proof of this result is given in [7] only for the first four points. The aim of the present paper is to give the proof of the last point of Theorem 1.2, i.e. to prove the continuity of the superposition operator $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$, with $1 \leq p, q < \infty$, in each of the cases (i)-(iii).

We end this section with the following remark.

Remark 1.2. Formula (1.2) is formally identical to the well-known higher-order chain rule used to compute higher partial derivatives of the composite function $g \circ u$ when $g : \mathbb{R} \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ are sufficiently smooth (see e.g. [4, Corollary 2.10, formula (2.9)]). In its turn, the result given by Corollary 2.10 in [4] generalizes the famous Faà di Bruno formula (see [8]). According to our knowledge, there are several equivalent manners to express formula (1.2) or formula (2.9) in [4] but, for our convenience, we prefer to use this form taken from [12, Subsection 5.2.1, formula (6)]. It is worth noticing the extra generality of formula (1.2) over formula (2.9) in [4]. While (2.9) in [4] is obtained for functions g and u of class C^l, for obtaining (1.2), the regularity conditions imposed on g are slightly weakened and those imposed on u are much more general, namely

$$g \text{ is of class } C^{l-1} \text{ and } g^{(l-1)} \text{ is locally Lipschitz,}$$
$$u \in W^{m,p} \text{ with } m \geq l.$$

In [4], interesting applications of formula (2.9) to stochastic processes and multivariate cumulants are given. Due to its greater generality, it is expected that formula (1.2) should allow the enlargement of the field of applications (e.g., boundary value problems for nonlinear partial differential equations).

2. Proof of Theorem 1.2(v)

The following auxiliary result will be needed. It is a simple consequence Hölder's inequality.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $1 \leq p_1, \ldots, p_k, p \leq \infty$ satisfy*

$$\frac{1}{p_1} + \ldots + \frac{1}{p_k} = \frac{1}{p}.$$

For each $i = 1, \ldots, k$, consider a function $u^i \in L^{p_i}(\Omega)$ and a sequence $\left\{u^i_\eta\right\}_\eta \subset L^{p_i}(\Omega)$ such that $\left\|u^i_\eta - u^i\right\|_{L^{p_i}(\Omega)} \overset{\eta\to\infty}{\longrightarrow} 0$. Then

$$\left\|u^1_\eta \cdot \ldots \cdot u^k_\eta - u^1 \cdot \ldots \cdot u^k\right\|_{L^p(\Omega)} \overset{\eta\to\infty}{\longrightarrow} 0.$$

Further on, we use the notation $p^*_k = \frac{np}{n-kp}$, provided that $k, n \in \mathbb{N}^*$, $1 \leq p < \infty$ and $kp < n$. Now, we are able to give the

Proof of Theorem 1.2(v). We will prove the continuity of $N_g : W^{m,p}(\Omega) \to W^{l,q}(\Omega)$, with $1 \leq p, q < \infty$, in each of the cases (i)-(iii).

Let $u \in W^{m,p}(\Omega)$ and $(u_\eta)_{\eta\in\mathbb{N}^*} \subset W^{m,p}(\Omega)$ satisfy $\|u_\eta - u\|_{W^{m,p}(\Omega)} \overset{\eta\to\infty}{\longrightarrow} 0$. We have to show that $\|N_g u_\eta - N_g u\|_{W^{l,q}(\Omega)} \overset{\eta\to\infty}{\longrightarrow} 0$. To this end, we will show that

$$\|D^\alpha (g \circ u_\eta) - D^\alpha (g \circ u)\|_{L^q(\Omega)} \to 0 \quad \text{for all } \alpha \in \mathbb{N}^n \text{ with } 0 \leq |\alpha| \leq l. \tag{2.1}$$

Let us prove (2.1) under the hypotheses of point (i), namely $\frac{n}{m} < p < \frac{n}{m-l}$, with $n \geq m-l+1$ ($1 \leq p < \frac{n}{m-l}$ when $n \in \{m-l+1, \ldots, m\}$), and $1 \leq q \leq \frac{np}{n-(m-l)p}$. We split the proof into four cases (see the proof of point (i) in [7]):

1. $\frac{n}{m} < p < \frac{n}{m-1}$, with $n \geq m$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$),

2. $p = \frac{n}{m-1}$, with $n \geq m-1$,

3. $\frac{n}{m-h} < p < \frac{n}{m-h-1}$, with $n \geq m-h$ and $h \in \{1, \ldots, l-1\}$ ($1 \leq p < \frac{n}{m-h-1}$ when $n = m-h$),

4. $p = \frac{n}{m-h}$, with $n \geq m-h$ and $h \in \{2, \ldots, l-1\}$.

Case 1. $\frac{n}{m} < p < \frac{n}{m-1}$, with $n \geq m$ ($1 \leq p < \frac{n}{n-1}$ when $n = m$).

We have $1 \leq q \leq p^*_{m-l} \leq p^*_{m-1}$. By Theorem 1.1(i,v), we deduce that $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous. Thus, $\|g \circ u_\eta - g \circ u\|_{W^{1,q}(\Omega)} \to 0$, i.e.

$$\|g \circ u_\eta - g \circ u\|_{L^q(\Omega)} \to 0,$$

$$\|\partial_j (g \circ u_\eta) - \partial_j (g \circ u)\|_{L^q(\Omega)} \to 0, \quad j = 1, \ldots, n.$$

Consequently, formula (2.1) is proved for $0 \leq |\alpha| \leq 1$.

According to point (i), $g\circ u$ and $g\circ u_\eta$ satisfy the higher-order chain rule (1.2). Hence, in order to prove (2.1) for $2\le|\alpha|\le l$, it suffices to prove that

$$\left\|\left(g^{(k)}\circ u_\eta\right)D^{\alpha^1}u_\eta\dots D^{\alpha^k}u_\eta-\left(g^{(k)}\circ u\right)D^{\alpha^1}u\dots D^{\alpha^k}u\right\|_{L^q(\Omega)}\to 0 \quad (2.2)$$

for all $\alpha\in\mathbb{N}^n$ with $2\le|\alpha|\le l$, all $k\in\{1,\dots,s=|\alpha|\}$, all $\alpha^1,\dots,\alpha^k\in\mathbb{N}^n$ with $\left|\alpha^i\right|\ne 0$ and $\alpha^1+\ldots+\alpha^k=\alpha$. To this end, we use Theorem 1.1(i,v) and Proposition 2.1.

Firstly, we fix $\alpha\in\mathbb{N}^n$ with $2\le|\alpha|=s\le l-1$, we fix $k\in\{1,\dots,s\}$ and we fix $\alpha^1,\dots,\alpha^k\in\mathbb{N}^n$ with $\left|\alpha^i\right|\ne 0$ and $\alpha^1+\ldots+\alpha^k=\alpha$. By Theorem 1.1(i,v), we infer that $N_{g^{(k)}}:W^{m,p}(\Omega)\to W^{1,p^*_{m-1}}(\Omega)$ is continuous. Thus, $\left\|g^{(k)}\circ u_\eta-g^{(k)}\circ u\right\|_{W^{1,p^*_{m-1}}(\Omega)}\to 0$, whence

$$\left\|g^{(k)}\circ u_\eta-g^{(k)}\circ u\right\|_{L^{p^*_{m-1}}(\Omega)}\to 0, \quad (2.3)$$

$$\left\|\left(g^{(k+1)}\circ u_\eta\right)\partial_j u_\eta-\left(g^{(k+1)}\circ u\right)\partial_j u\right\|_{L^{p^*_{m-1}}(\Omega)}\to 0,\quad j=1,\dots,n. \quad (2.4)$$

It follows from $\|u_\eta-u\|_{W^{m,p}(\Omega)}\to 0$ that $\left\|D^{\alpha^i}u_\eta-D^{\alpha^i}u\right\|_{W^{m-|\alpha^i|,p}(\Omega)}\to 0$. On the other hand, since $\left(m-\left|\alpha^i\right|\right)p\le(m-1)\,p<n$, the Sobolev imbedding

$$W^{m-|\alpha^i|,p}(\Omega)\hookrightarrow L^{p^*_{m-|\alpha^i|}}(\Omega),$$

is valid. Therefore

$$\left\|D^{\alpha^i}u_\eta-D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)}\to 0,\quad i=1,\dots,k. \quad (2.5)$$

It is not difficult to show that

$$\frac{1}{p^*_{m-1}}+\sum_{i=1}^{k}\frac{1}{p^*_{m-|\alpha^i|}}<\frac{1}{q} \quad (2.6)$$

(see the proof of point (i), case 1, in [7]). By using Proposition 2.1, formulas (2.3), (2.5), (2.6), and the fact that Ω is bounded, we obtain the validity of formula (2.2) for $2\le|\alpha|\le l-1$.

It remains to be shown that (2.2) is valid in the case $|\alpha|=l$ as well. We fix $\alpha\in\mathbb{R}^n$ with $|\alpha|=l$. There is $\beta\in\mathbb{R}^n$ and $j\in\{1,\dots,n\}$ such that $|\beta|=l-1$ and $D^\alpha=\partial_j D^\beta$. According to point (i), we have

$$D^\beta(g\circ v)=\sum_{k=1}^{l-1}\sum_{\substack{\alpha^1+\ldots+\alpha^k=\beta\\ |\alpha^i|\ne 0}}c_{\beta,k,\alpha^1,\dots,\alpha^k}\left(g^{(k)}\circ v\right)D^{\alpha^1}v\dots D^{\alpha^k}v$$

$$\mathcal{L}^n\text{-a.e. in }\Omega,\ \text{for all } v\in W^{m,p}(\Omega).$$

It follows that

$$\begin{aligned} & D^{\alpha}\left(g\circ v\right)=\partial_j D^{\beta}\left(g\circ v\right) \\ &=\sum_{k=1}^{l-1}\sum_{\substack{\alpha^1+\ldots+\alpha^k=\beta \\ |\alpha^i|\neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k}\left[\left(g^{(k+1)}\circ v\right)\partial_j v D^{\alpha^1}v\ldots D^{\alpha^k}v\right. \\ &\left.+\left(g^{(k)}\circ v\right)\left(\partial_j D^{\alpha^1}vD^{\alpha^2}v\ldots D^{\alpha^k}v+\ldots+D^{\alpha^1}v\ldots D^{\alpha^{k-1}}v\partial_j D^{\alpha^k}v\right)\right] \\ & \mathcal{L}^n\text{-a.e. in } \Omega,\ \text{for all } v\in W^{m,p}\left(\Omega\right). \end{aligned}$$

Hence, in order to prove (2.2) in the case $|\alpha|=l$, it suffices to show that

$$\begin{aligned} &\Big\|\left(g^{(k+1)}\circ u_\eta\right)\partial_j u_\eta D^{\alpha^1}u_\eta\ldots D^{\alpha^k}u_\eta \\ &\quad-\left(g^{(k+1)}\circ u\right)\partial_j u D^{\alpha^1}u\ldots D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \end{aligned}\tag{2.7}$$

$$\begin{aligned} &\Big\|\left(g^{(k)}\circ u_\eta\right)\partial_j D^{\alpha^1}u_\eta D^{\alpha^2}u_\eta\ldots D^{\alpha^k}u_\eta \\ &\quad-\left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \end{aligned}\tag{2.8.1}$$

$$\begin{aligned} &\Big\|\left(g^{(k)}\circ u_\eta\right)D^{\alpha^1}u_\eta\partial_j D^{\alpha^2}u_\eta\ldots D^{\alpha^k}u_\eta \\ &\quad-\left(g^{(k)}\circ u\right)D^{\alpha^1}u\partial_j D^{\alpha^2}u\ldots D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \end{aligned}\tag{2.8.2}$$

$$\vdots$$

$$\begin{aligned} &\Big\|\left(g^{(k)}\circ u_\eta\right)D^{\alpha^1}u_\eta\ldots D^{\alpha^{k-1}}u_\eta\partial_j D^{\alpha^k}u_\eta \\ &\quad-\left(g^{(k)}\circ u\right)D^{\alpha^1}u\ldots D^{\alpha^{k-1}}u\partial_j D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \end{aligned}\tag{2.8.k}$$

for all $k\in\{1,\ldots,l-1\}$ and all $\alpha^1,\ldots,\alpha^k\in\mathbb{N}^n$ with $\left|\alpha^i\right|\neq 0$ and $\alpha^1+\ldots+\alpha^k=\beta$. Fix $k\in\{1,\ldots,l-1\}$ and fix $\alpha^1,\ldots,\alpha^k\in\mathbb{N}^n$ with $\left|\alpha^i\right|\neq 0$ and $\alpha^1+\ldots+\alpha^k=\beta$.

Formula (2.7) is a direct consequence of Proposition 2.1, formulas (2.4), (2.5), (2.6), and the fact that Ω is bounded.

It follows from $\|u_\eta-u\|_{W^{m,p}(\Omega)}\to 0$ that

$$\left\|\partial_j D^{\alpha^1}u_\eta-\partial_j D^{\alpha^1}u\right\|_{W^{m-|\alpha^1|-1,p}(\Omega)}\to 0.$$

On the other hand, since $(m-1)\,p<n$, the Sobolev imbedding

$$W^{m-|\alpha^1|-1,p}\left(\Omega\right)\hookrightarrow L^{p^*_{m-|\alpha^1|-1}}\left(\Omega\right),$$

is valid. Therefore,

$$\left\|\partial_j D^{\alpha^1}u_\eta-\partial_j D^{\alpha^1}u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)}\to 0.\tag{2.9}$$

According to the above, since $1 \le k \le l-1$, we have

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{W^{1,p^*_{m-1}}(\Omega)} \to 0.$$

On the other hand, by $mp > n$, we get $p^*_{m-1} > n$. Thus, the Sobolev imbedding

$$W^{1,p^*_{m-1}}(\Omega) \hookrightarrow L^\infty(\Omega),$$

is valid. Consequently,

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \to 0. \tag{2.10}$$

Simple computations show that

$$\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i=2}^{k} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q} \tag{2.11}$$

(see the proof of point (i), case 1, in [7]). Now, (2.8.1) is a direct consequence of Proposition 2.1, formulas (2.5), (2.9), (2.10), (2.11), and of the fact that Ω is bounded. Formulas (2.8.2), …, (2.8.k) can be proved in the same way as (2.8.1). Consequently, formula (2.2) is proved in the case $|\alpha| = l$ as well.

Formula (2.1) is now proved under the hypotheses of point(i), case 1.

Case 2. $p = \frac{n}{m-1}$, with $n \ge m-1$.

We have $1 \le q \le p^*_{m-l} < \infty$. By Theorem 1.1(ii,v), we deduce that $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous. Thus, $\|g \circ u_\eta - g \circ u\|_{W^{1,q}(\Omega)} \to 0$, i.e.

$$\|g \circ u_\eta - g \circ u\|_{L^q(\Omega)} \to 0,$$

$$\|\partial_j (g \circ u_\eta) - \partial_j (g \circ u)\|_{L^q(\Omega)} \to 0, \quad j = 1, \ldots, n.$$

Consequently, formula (2.1) is proved for $0 \le |\alpha| \le 1$.

According to point (i), $g \circ u$ and $g \circ u_\eta$ satisfy the higher-order chain rule (1.2). Hence, in order to prove (2.1) for $2 \le |\alpha| \le l$, it suffices to prove (2.2) for all $\alpha \in \mathbb{N}^n$ with $2 \le |\alpha| \le l$, all $k \in \{1, \ldots, s = |\alpha|\}$, all $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \ne 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$. To this end, we use Theorem 1.1(ii,v) and Proposition 2.1.

Firstly, we fix $\alpha \in \mathbb{N}^n$ with $2 \le |\alpha| = s \le l-1$, we fix $k \in \{1, \ldots, s\}$ and we fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \ne 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

As in the proof of point (i), case 2 (see [7]) denote

$$\begin{aligned} I_1 &= \left\{ i \in \{1, \ldots, k\} : \left|\alpha^i\right| = 1 \right\}, \\ I_2 &= \left\{ i \in \{1, \ldots, k\} : 2 \le \left|\alpha^i\right| \le s \right\}. \end{aligned}$$

We have

$$\begin{aligned} \left(m - \left|\alpha^i\right|\right) p &= n \quad \text{if } i \in I_1, \\ \left(m - \left|\alpha^i\right|\right) p &< n \quad \text{if } i \in I_2, \end{aligned}$$

whence we deduce the following Sobolev imbeddings

$$W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } 1 \le r < \infty, \text{ if } i \in I_1,$$

$$W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^i|}}(\Omega) \quad \text{if } i \in I_2.$$

On the other hand, it follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{W^{m-|\alpha^i|,p}(\Omega)} \to 0.$$

Therefore

$$\begin{gathered} \left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{L^r(\Omega)} \to 0 \quad \text{for all } 1 \le r < \infty, \text{ if } i \in I_1, \\ \left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \to 0 \quad \text{if } i \in I_2. \end{gathered} \tag{2.12}$$

By Theorem 1.1(ii,v), we infer that $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ is continuous for all $1 \le r < \infty$. Thus, $\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \to 0$, whence

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^r(\Omega)} \to 0, \quad 1 \le r < \infty, \tag{2.13}$$

$$\left\| \left(g^{(k+1)} \circ u_\eta \right) \partial_j u_\eta - \left(g^{(k+1)} \circ u \right) \partial_j u \right\|_{L^r(\Omega)} \to 0,\ 1 \le r < \infty,\ j = 1, \dots, n. \tag{2.14}$$

It is not difficult to show that

$$\sum_{i \in I_2} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \tag{2.15}$$

(see the proof of point (i), case 2, in [7]). By using Proposition 2.1 and formulas (2.12), (2.13), and (2.15), we obtain the validity of formula (2.2) for $2 \le |\alpha| \le l - 1$.

It remains to be shown that (2.2) is valid in the case $|\alpha| = l$ as well. We fix $\alpha \in \mathbb{R}^n$ with $|\alpha| = l$. There is $\beta \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$ such that $|\beta| = l - 1$ and $D^\alpha = \partial_j D^\beta$. According to point (i), we have

$$D^\beta (g \circ v) = \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1 + \dots + \alpha^k = \beta \\ |\alpha^i| \ne 0}} c_{\beta,k,\alpha^1,\dots,\alpha^k} \left(g^{(k)} \circ v \right) D^{\alpha^1} v \dots D^{\alpha^k} v$$

$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } v \in W^{m,p}(\Omega).$$

It follows that

$$D^{\alpha}(g\circ v)=\partial_j D^{\beta}(g\circ v)$$
$$=\sum_{k=1}^{l-1}\sum_{\substack{\alpha^1+\ldots+\alpha^k=\beta\\ |\alpha^i|\neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k}\Big[\Big(g^{(k+1)}\circ v\Big)\partial_j v D^{\alpha^1}v\ldots D^{\alpha^k}v$$
$$+\Big(g^{(k)}\circ v\Big)\Big(\partial_j D^{\alpha^1}vD^{\alpha^2}v\ldots D^{\alpha^k}v+\ldots+D^{\alpha^1}v\ldots D^{\alpha^{k-1}}v\partial_j D^{\alpha^k}v\Big)\Big]$$
$$\mathcal{L}^n\text{-a.e. in }\Omega,\text{ for all }v\in W^{m,p}(\Omega).$$

Hence, in order to prove (2.2) in the case $|\alpha|=l$, it suffices to show that

$$\Big\|\big(g^{(k+1)}\circ u_\eta\big)\partial_j u_\eta D^{\alpha^1}u_\eta\ldots D^{\alpha^k}u_\eta - \big(g^{(k+1)}\circ u\big)\partial_j u D^{\alpha^1}u\ldots D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \tag{2.16}$$

$$\Big\|\big(g^{(k)}\circ u_\eta\big)\partial_j D^{\alpha^1}u_\eta D^{\alpha^2}u_\eta\ldots D^{\alpha^k}u_\eta - \big(g^{(k)}\circ u\big)\partial_j D^{\alpha^1}uD^{\alpha^2}u\ldots D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \tag{2.17.1}$$

$$\Big\|\big(g^{(k)}\circ u_\eta\big)D^{\alpha^1}u_\eta\partial_j D^{\alpha^2}u_\eta\ldots D^{\alpha^k}u_\eta - \big(g^{(k)}\circ u\big)D^{\alpha^1}u\partial_j D^{\alpha^2}u\ldots D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \tag{2.17.2}$$

$$\vdots$$

$$\Big\|\big(g^{(k)}\circ u_\eta\big)D^{\alpha^1}u_\eta\ldots D^{\alpha^{k-1}}u_\eta\partial_j D^{\alpha^k}u_\eta - \big(g^{(k)}\circ u\big)D^{\alpha^1}u\ldots D^{\alpha^{k-1}}u\partial_j D^{\alpha^k}u\Big\|_{L^q(\Omega)}\to 0, \tag{2.17.k}$$

for all $k\in\{1,\ldots,l-1\}$ and all $\alpha^1,\ldots,\alpha^k\in\mathbb{N}^n$ with $|\alpha^i|\neq 0$ and $\alpha^1+\ldots+\alpha^k=\beta$. Fix $k\in\{1,\ldots,l-1\}$ and fix $\alpha^1,\ldots,\alpha^k\in\mathbb{N}^n$ with $|\alpha^i|\neq 0$ and $\alpha^1+\ldots+\alpha^k=\beta$.

Formula (2.16) is a direct consequence of Proposition 2.1 and formulas (2.12), (2.14), (2.15), and the fact that Ω is bounded.

It follows from $\|u_\eta-u\|_{W^{m,p}(\Omega)}\to 0$ that

$$\Big\|\partial_j D^{\alpha^1}u_\eta-\partial_j D^{\alpha^1}u\Big\|_{W^{m-|\alpha^1|-1,p}(\Omega)}\to 0.$$

On the other hand, since $(m-1)p=n$, the Sobolev imbedding

$$W^{m-|\alpha^1|-1,p}(\Omega)\hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega),$$

is valid. Therefore,

$$\Big\|\partial_j D^{\alpha^1}u_\eta-\partial_j D^{\alpha^1}u\Big\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)}\to 0. \tag{2.18}$$

According to the above, since $1 \leq k \leq l-1$, we have

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \to 0 \quad \text{for all } 1 \leq r < \infty.$$

On the other hand, the Sobolev imbedding

$$W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega),$$

is valid for all $n < r < \infty$. Consequently,

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \to 0. \tag{2.19}$$

It is not difficult to show that

$$\begin{aligned}
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} &\leq \frac{1}{q} \quad \text{if } 1 \in I_2 \text{ and } I_1 = \phi, \\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} &< \frac{1}{q} \quad \text{if } 1 \in I_2 \text{ and } I_1 \neq \phi, \\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} &\leq \frac{1}{q} \quad \text{if } I_1 = \{1\}, \\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} &< \frac{1}{q} \quad \text{if } I_1 \supsetneq \{1\}.
\end{aligned} \tag{2.20}$$

(see the proof of point (i), case 2, in [7]). Now, (2.17.1) is a direct consequence of Proposition 2.1, formulas (2.12), (2.18), (2.19), (2.20), and of the fact that Ω is bounded. Formulas (2.17.2), ..., (2.17.k) can be proved in the same way as (2.17.1). Consequently, formula (2.2) is proved in the case $|\alpha| = l$ as well.

Formula (2.1) is now proved under the hypotheses of point (i), case 2.

Case 3. $\frac{n}{m-h} < p < \frac{n}{m-h-1}$, with $n \geq m-h$ and $h \in \{1, \ldots, l-1\}$ ($1 \leq p < \frac{n}{m-h-1}$ when $n = m-h$).

We have $1 \leq q \leq p^*_{m-l} < \infty$. By Theorem 1.1(iii,v), we deduce that $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous. Thus, $\|g \circ u_\eta - g \circ u\|_{W^{1,q}(\Omega)} \to 0$, i.e.

$$\|g \circ u_\eta - g \circ u\|_{L^q(\Omega)} \to 0,$$

$$\|\partial_j (g \circ u_\eta) - \partial_j (g \circ u)\|_{L^q(\Omega)} \to 0, \quad j = 1, \ldots, n.$$

Consequently, formula (2.1) is proved for $0 \leq |\alpha| \leq 1$.

According to point (i), $g \circ u$ and $g \circ u_\eta$ satisfy the higher-order chain rule (1.2). Hence, in order to prove (2.1) for $2 \leq |\alpha| \leq l$, it suffices to prove (2.2) for all $\alpha \in \mathbb{N}^n$ with $2 \leq |\alpha| \leq l$, all $k \in \{1, \ldots, s = |\alpha|\}$, all $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$. To this end, we use Theorem 1.1(iii,v) and Proposition 2.1.

Firstly, we fix $\alpha \in \mathbb{N}^n$ with $2 \leq |\alpha| = s \leq l-1$, we fix $k \in \{1, \ldots, s\}$ and we fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

As in the proof of point (i), case 3 (see [7]) denote

$$\begin{aligned} I_1 &= \left\{ i \in \{1, \ldots, k\} : 1 \leq \left|\alpha^i\right| \leq h \right\}, \\ I_2 &= \left\{ i \in \{1, \ldots, k\} : h+1 \leq \left|\alpha^i\right| \leq s \right\}. \end{aligned}$$

We have

$$\begin{aligned} \left(m - \left|\alpha^i\right|\right) p &> n \quad \text{if } i \in I_1, \\ \left(m - \left|\alpha^i\right|\right) p &< n \quad \text{if } i \in I_2, \end{aligned}$$

whence we deduce the following Sobolev imbeddings

$$W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \quad \text{if } i \in I_1,$$

$$W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^{p^*_{m-\left|\alpha^i\right|}}(\Omega) \quad \text{if } i \in I_2.$$

On the other hand, it follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{W^{m-\left|\alpha^i\right|,p}(\Omega)} \to 0.$$

Therefore

$$\begin{aligned} \left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{L^{\infty}(\Omega)} &\to 0 \quad \text{if } i \in I_1, \\ \left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{L^{p^*_{m-\left|\alpha^i\right|}}(\Omega)} &\to 0 \quad \text{if } i \in I_2. \end{aligned} \tag{2.21}$$

By Theorem 1.1(iii,v), we infer that $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ is continuous for all $1 \leq r < \infty$. Thus, $\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{W^{1,r}(\Omega)} \to 0$, whence

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^r(\Omega)} \to 0, \quad 1 \leq r < \infty, \tag{2.22}$$

$$\left\| \left(g^{(k+1)} \circ u_\eta\right) \partial_j u_\eta - \left(g^{(k+1)} \circ u\right) \partial_j u \right\|_{L^r(\Omega)} \to 0,\ 1 \leq r < \infty,\ j = 1, \ldots, n. \tag{2.23}$$

Simple computations show that

$$\sum_{i \in I_2} \frac{1}{p^*_{m-\left|\alpha^i\right|}} < \frac{1}{q} \tag{2.24}$$

(see the proof of point (i), case 3, in [7]). By using Proposition 2.1 and formulas (2.21), (2.22), (2.24), we obtain the validity of formula (2.2) for $2 \leq |\alpha| \leq l-1$.

It remains to be shown that (2.2) is valid in the case $|\alpha| = l$ as well. We fix $\alpha \in \mathbb{R}^n$ with $|\alpha| = l$. There is $\beta \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$ such that $|\beta| = l - 1$ and $D^\alpha = \partial_j D^\beta$. According to point (i), we have

$$D^\beta (g \circ v) = \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \beta \\ |\alpha^i| \neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k} \left(g^{(k)} \circ v\right) D^{\alpha^1} v \ldots D^{\alpha^k} v$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } v \in W^{m,p}(\Omega).$$

It follows that

$$D^\alpha (g \circ v) = \partial_j D^\beta (g \circ v)$$
$$= \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \beta \\ |\alpha^i| \neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k} \left[\left(g^{(k+1)} \circ v\right) \partial_j v D^{\alpha^1} v \ldots D^{\alpha^k} v \right.$$
$$\left. + \left(g^{(k)} \circ v\right) \left(\partial_j D^{\alpha^1} v D^{\alpha^2} v \ldots D^{\alpha^k} v + \ldots + D^{\alpha^1} v \ldots D^{\alpha^{k-1}} v \partial_j D^{\alpha^k} v\right)\right]$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } v \in W^{m,p}(\Omega).$$

Hence, in order to prove (2.2) in the case $|\alpha| = l$, it suffices to show that

$$\begin{aligned}\Big\| &\left(g^{(k+1)} \circ u_\eta\right) \partial_j u_\eta D^{\alpha^1} u_\eta \ldots D^{\alpha^k} u_\eta \\ &- \left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \ldots D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.25}$$

$$\begin{aligned}\Big\| &\left(g^{(k)} \circ u_\eta\right) \partial_j D^{\alpha^1} u_\eta D^{\alpha^2} u_\eta \ldots D^{\alpha^k} u_\eta \\ &- \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.26.1}$$

$$\begin{aligned}\Big\| &\left(g^{(k)} \circ u_\eta\right) D^{\alpha^1} u_\eta \partial_j D^{\alpha^2} u_\eta \ldots D^{\alpha^k} u_\eta \\ &- \left(g^{(k)} \circ u\right) D^{\alpha^1} u \partial_j D^{\alpha^2} u \ldots D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.26.2}$$

$$\vdots$$

$$\begin{aligned}\Big\| &\left(g^{(k)} \circ u_\eta\right) D^{\alpha^1} u_\eta \ldots D^{\alpha^{k-1}} u_\eta \partial_j D^{\alpha^k} u_\eta \\ &- \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.26.k}$$

for all $k \in \{1, \ldots, l-1\}$ and all $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \beta$. Fix $k \in \{1, \ldots, l-1\}$ and fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \beta$.

Formula (2.25) is a direct consequence of Proposition 2.1 and formulas (2.21), (2.23), (2.24), and the fact that Ω is bounded.

It follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\|\partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u\right\|_{W^{m-|\alpha^1|-1,p}(\Omega)} \to 0.$$

On the other hand, since $\frac{n}{m-h} < p < \frac{n}{m-h-1}$, we have the Sobolev imbeddings

$$W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega) \quad \text{if } 1 \in I_2,$$

$$W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega) \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| = h,$$

$$W^{m-\left|\alpha^1\right|-1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| \le h-1.$$

Therefore,

$$\begin{gathered}
\left\|\partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \to 0 \quad \text{if } 1 \in I_2,\\
\left\|\partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u\right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \to 0 \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| = h,\\
\left\|\partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u\right\|_{L^{\infty}(\Omega)} \to 0 \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| \le h-1.
\end{gathered} \tag{2.27}$$

According to the above, since $1 \le k \le l-1$, we have

$$\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{W^{1,r}(\Omega)} \to 0 \quad \text{for all } 1 \le r < \infty.$$

On the other hand, the Sobolev imbedding

$$W^{1,r}(\Omega) \hookrightarrow L^{\infty}(\Omega),$$

is valid for all $n < r < \infty$. Consequently,

$$\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{L^{\infty}(\Omega)} \to 0. \tag{2.28}$$

It is not difficult to show that

$$\begin{gathered}
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q}, \quad \text{if } 1 \in I_2,\\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_2} \frac{1}{p^*_{m-|\alpha^i|}} \le \frac{1}{q}, \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| = h,\\
\sum_{i \in I_2} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q}, \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| \le h-1
\end{gathered} \tag{2.29}$$

(see the proof of point (i), case 3, in [7]). Now, (2.26.1) is a direct consequence of Proposition 2.1, formulas (2.21), (2.27), (2.28), (2.29) and of the fact that Ω is bounded. Formulas (2.26.2), …, (2.26.k) can be proved in the same way as (2.26.1). Consequently, formula (2.2) is proved in the case $|\alpha| = l$ as well.

Formula (2.1) is now proved under the hypotheses of point (i), case 3.

Case 4. $p = \frac{n}{m-h}$, with $n \geq m-h$ and $h \in \{2, \ldots, l-1\}$.

We have $1 \leq q \leq p^*_{m-l} < \infty$. By Theorem 1.1(iii,v), we deduce that $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous. Thus, $\|g \circ u_\eta - g \circ u\|_{W^{1,q}(\Omega)} \to 0$, i.e.

$$\|g \circ u_\eta - g \circ u\|_{L^q(\Omega)} \to 0,$$

$$\|\partial_j (g \circ u_\eta) - \partial_j (g \circ u)\|_{L^q(\Omega)} \to 0, \quad j = 1, \ldots, n.$$

Consequently, formula (2.1) is proved for $0 \leq |\alpha| \leq 1$.

According to point (i), $g \circ u$ and $g \circ u_\eta$ satisfy the higher-order chain rule (1.2). Hence, in order to prove (2.1) for $2 \leq |\alpha| \leq l$, it suffices to prove (2.2) for all $\alpha \in \mathbb{N}^n$ with $2 \leq |\alpha| \leq l$, all $k \in \{1, \ldots, s = |\alpha|\}$, all $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $|\alpha^i| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$. To this end, we use Theorem 1.1(iii,v) and Proposition 2.1.

Firstly, we fix $\alpha \in \mathbb{N}^n$ with $2 \leq |\alpha| = s \leq l-1$, we fix $k \in \{1, \ldots, s\}$ and we fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $|\alpha^i| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

As in the proof of point (i), case 4 (see [7]) denote

$$\begin{aligned} I_1 &= \left\{ i \in \{1, \ldots, k\} : 1 \leq |\alpha^i| < h \right\}, \\ I_2 &= \left\{ i \in \{1, \ldots, k\} : |\alpha^i| = h \right\}, \\ I_3 &= \left\{ i \in \{1, \ldots, k\} : h < |\alpha^i| \leq s \right\}. \end{aligned}$$

We have

$$\begin{aligned} \left(m - |\alpha^i|\right) p &> n \quad \text{if } i \in I_1, \\ \left(m - |\alpha^i|\right) p &= n \quad \text{if } i \in I_2, \\ \left(m - |\alpha^i|\right) p &< n \quad \text{if } i \in I_3, \end{aligned}$$

whence we deduce the following Sobolev imbeddings

$$W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{if } i \in I_1,$$

$$W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } 1 \leq r < \infty, \text{ if } i \in I_2,$$

$$W^{m-|\alpha^i|,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^i|}}(\Omega) \quad \text{if } i \in I_3.$$

On the other hand, it follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{W^{m-|\alpha^i|,p}(\Omega)} \to 0.$$

Therefore

$$\begin{gathered}\left\|D^{\alpha^i}u_\eta - D^{\alpha^i}u\right\|_{L^\infty(\Omega)} \to 0 \quad \text{if } i \in I_1, \\ \left\|D^{\alpha^i}u_\eta - D^{\alpha^i}u\right\|_{L^r(\Omega)} \to 0 \quad \text{for all } 1 \le r < \infty, \text{ if } i \in I_2, \\ \left\|D^{\alpha^i}u_\eta - D^{\alpha^i}u\right\|_{L^{p^*_{m-|\alpha^i|}}(\Omega)} \to 0 \quad \text{if } i \in I_3.\end{gathered} \tag{2.30}$$

By Theorem 1.1(iii,v), we infer that $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ is continuous for all $1 \le r < \infty$. Thus, $\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{W^{1,r}(\Omega)} \to 0$, whence

$$\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{L^r(\Omega)} \to 0, \quad 1 \le r < \infty, \tag{2.31}$$

$$\left\|\left(g^{(k+1)} \circ u_\eta\right)\partial_j u_\eta - \left(g^{(k+1)} \circ u\right)\partial_j u\right\|_{L^r(\Omega)} \to 0, \quad 1 \le r < \infty,\ j = 1, \ldots, n. \tag{2.32}$$

It is not difficult to show that

$$\sum_{i \in I_3} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \tag{2.33}$$

(see the proof of point (i), case 4, in [7]). By using Proposition 2.1 and formulas (2.30), (2.31), (2.33), we obtain the validity of formula (2.2) for $2 \le |\alpha| \le l-1$.

It remains to be shown that (2.2) is valid in the case $|\alpha| = l$ as well. We fix $\alpha \in \mathbb{R}^n$ with $|\alpha| = l$. There is $\beta \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$ such that $|\beta| = l-1$ and $D^\alpha = \partial_j D^\beta$. According to point (i), we have

$$D^\beta(g \circ v) = \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \beta \\ |\alpha^i| \ne 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k}\left(g^{(k)} \circ v\right) D^{\alpha^1}v \ldots D^{\alpha^k}v$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } v \in W^{m,p}(\Omega).$$

It follows that

$$\begin{aligned} & D^\alpha(g \circ v) = \partial_j D^\beta(g \circ v) \\ &= \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \beta \\ |\alpha^i| \ne 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k}\left[\left(g^{(k+1)} \circ v\right)\partial_j v D^{\alpha^1}v \ldots D^{\alpha^k}v\right. \\ &\left. + \left(g^{(k)} \circ v\right)\left(\partial_j D^{\alpha^1}v D^{\alpha^2}v \ldots D^{\alpha^k}v + \ldots + D^{\alpha^1}v \ldots D^{\alpha^{k-1}}v \partial_j D^{\alpha^k}v\right)\right] \\ & \mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } v \in W^{m,p}(\Omega). \end{aligned}$$

Hence, in order to prove (2.2) in the case $|\alpha| = l$, it suffices to show that

$$\begin{aligned}\Big\| \left(g^{(k+1)} \circ u_\eta\right) \partial_j u_\eta D^{\alpha^1} u_\eta \dots D^{\alpha^k} u_\eta \\ - \left(g^{(k+1)} \circ u\right) \partial_j u D^{\alpha^1} u \dots D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.34}$$

$$\begin{aligned}\Big\| \left(g^{(k)} \circ u_\eta\right) \partial_j D^{\alpha^1} u_\eta D^{\alpha^2} u_\eta \dots D^{\alpha^k} u_\eta \\ - \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \dots D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.35.1}$$

$$\begin{aligned}\Big\| \left(g^{(k)} \circ u_\eta\right) D^{\alpha^1} u_\eta \partial_j D^{\alpha^2} u_\eta \dots D^{\alpha^k} u_\eta \\ - \left(g^{(k)} \circ u\right) D^{\alpha^1} u \partial_j D^{\alpha^2} u \dots D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.35.2}$$

$$\vdots$$

$$\begin{aligned}\Big\| \left(g^{(k)} \circ u_\eta\right) D^{\alpha^1} u_\eta \dots D^{\alpha^{k-1}} u_\eta \partial_j D^{\alpha^k} u_\eta \\ - \left(g^{(k)} \circ u\right) D^{\alpha^1} u \dots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \Big\|_{L^q(\Omega)} \to 0,\end{aligned} \tag{2.35.k}$$

for all $k \in \{1, \dots, l-1\}$ and all $\alpha^1, \dots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \dots + \alpha^k = \beta$. Fix $k \in \{1, \dots, l-1\}$ and fix $\alpha^1, \dots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \dots + \alpha^k = \beta$.

Formula (2.34) is a direct consequence of Proposition 2.1 and formulas (2.30), (2.32), (2.33).

It follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{W^{m-|\alpha^1|-1,p}(\Omega)} \to 0.$$

On the other hand, since $p = \frac{n}{m-h}$, we have the Sobolev imbeddings

$$W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega) \quad \text{if } 1 \in I_3,$$

$$W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^{p^*_{m-|\alpha^1|-1}}(\Omega) \quad \text{if } 1 \in I_2,$$

$$W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for all } 1 \le r < \infty, \text{ if } 1 \in I_1 \text{ and } \left|\alpha^1\right| = h-1,$$

$$W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| \le h-2.$$

Therefore,

$$\begin{gathered}\left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \to 0 \quad \text{if } 1 \in I_3, \\ \left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^{p^*_{m-|\alpha^1|-1}}(\Omega)} \to 0 \quad \text{if } 1 \in I_2, \\ \left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^r(\Omega)} \to 0 \text{ for all } 1 \le r < \infty, \text{ if } 1 \in I_1 \text{ and } \left|\alpha^1\right| = h-1, \\ \left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^\infty(\Omega)} \to 0 \quad \text{if } 1 \in I_1 \text{ and } \left|\alpha^1\right| \le h-2.\end{gathered} \tag{2.36}$$

According to the above, since $1 \leq k \leq l-1$, we have

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \to 0 \quad \text{for all } 1 \leq r < \infty.$$

On the other hand, the Sobolev imbedding

$$W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega),$$

is valid for all $n < r < \infty$. Consequently,

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \to 0. \tag{2.37}$$

It is not difficult to show that

$$\begin{gathered}
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_3 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} \leq \frac{1}{q} \quad \text{if } 1 \in I_3 \text{ and } I_2 = \phi, \\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_3 \setminus \{1\}} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \quad \text{if } 1 \in I_3 \text{ and } I_2 \neq \phi, \\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_3} \frac{1}{p^*_{m-|\alpha^i|}} \leq \frac{1}{q} \quad \text{if } 1 \in I_2 = \{1\}, \\
\frac{1}{p^*_{m-|\alpha^1|-1}} + \sum_{i \in I_3} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \quad \text{if } 1 \in I_2 \supsetneq \{1\}, \\
\sum_{i \in I_3} \frac{1}{p^*_{m-|\alpha^i|}} < \frac{1}{q} \quad \text{if } 1 \in I_1
\end{gathered} \tag{2.38}$$

(see the proof of point (i), case 4, in [7]). Now, (2.35.1) is a direct consequence of Proposition 2.1, formulas (2.30), (2.36), (2.37), (2.38) and of the fact that Ω is bounded. Formulas (2.35.2), ..., (2.35.k) can be proved in the same way as (2.35.1). Consequently, formula (2.2) is proved in the case $|\alpha| = l$ as well.

Formula (2.1) is now proved under the hypotheses of point (i), case 4.

Next, we prove (2.1) under the hypotheses of point (ii), namely $p = \frac{n}{m-l}$, with $m \geq l+1$ and $n \geq m-l$, and $1 \leq q < \infty$. By Theorem 1.1(ii,iii,v), we deduce that $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous. Thus, $\|g \circ u_\eta - g \circ u\|_{W^{1,q}(\Omega)} \to 0$, i.e.

$$\|g \circ u_\eta - g \circ u\|_{L^q(\Omega)} \to 0,$$

$$\|\partial_j (g \circ u_\eta) - \partial_j (g \circ u)\|_{L^q(\Omega)} \to 0, \quad j = 1, \ldots, n.$$

Consequently, formula (2.1) is proved for $0 \leq |\alpha| \leq 1$.

According to point (ii), $g \circ u$ and $g \circ u_\eta$ satisfy the higher-order chain rule (1.2). Hence, in order to prove (2.1) for $2 \leq |\alpha| \leq l$, it suffices to prove (2.2) for all $\alpha \in \mathbb{N}^n$ with $2 \leq |\alpha| \leq l$, all $k \in \{1, \ldots, s = |\alpha|\}$, all $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$

with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$. To this end, we use Theorem 1.1(ii,iii,v) and Proposition 2.1.

Firstly, we fix $\alpha \in \mathbb{N}^n$ with $2 \leq |\alpha| = s \leq l-1$, we fix $k \in \{1, \ldots, s\}$ and we fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

As in the proof of point (ii) (see [7]) denote

$$\begin{aligned} I_1 &= \left\{i \in \{1, \ldots, k\} : 1 \leq \left|\alpha^i\right| \leq l-2\right\}, \\ I_2 &= \left\{i \in \{1, \ldots, k\} : \left|\alpha^i\right| = l-1\right\}. \end{aligned}$$

Since

$$\left(m - \left|\alpha^i\right|\right) p \geq (m-s)\, p > (m-l)\, p = n,$$

the following Sobolev imbedding holds

$$W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^\infty(\Omega).$$

On the other hand, it follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\|D^{\alpha^i} u_\eta - D^{\alpha^i} u\right\|_{W^{m-\left|\alpha^i\right|,p}(\Omega)} \to 0.$$

Therefore

$$\left\|D^{\alpha^i} u_\eta - D^{\alpha^i} u\right\|_{L^\infty(\Omega)} \to 0. \tag{2.39}$$

By Theorem 1.1(ii,iii,v), we infer that $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ is continuous for all $1 \leq r < \infty$. Thus, $\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{W^{1,r}(\Omega)} \to 0$, whence

$$\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{L^r(\Omega)} \to 0, \quad 1 \leq r < \infty, \tag{2.40}$$

$$\left\|\left(g^{(k+1)} \circ u_\eta\right) \partial_j u_\eta - \left(g^{(k+1)} \circ u\right) \partial_j u\right\|_{L^r(\Omega)} \to 0,\ 1 \leq r < \infty,\ j = 1, \ldots, n. \tag{2.41}$$

By using Proposition 2.1 and formulas (2.39), (2.40), we obtain the validity of formula (2.2) for $2 \leq |\alpha| \leq l-1$.

It remains to be shown that (2.2) is valid in the case $|\alpha| = l$ as well. We fix $\alpha \in \mathbb{R}^n$ with $|\alpha| = l$. There is $\beta \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$ such that $|\beta| = l-1$ and $D^\alpha = \partial_j D^\beta$. According to point (ii), we have

$$D^\beta (g \circ v) = \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1 + \ldots + \alpha^k = \beta \\ \left|\alpha^i\right| \neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k} \left(g^{(k)} \circ v\right) D^{\alpha^1} v \ldots D^{\alpha^k} v$$
$$\mathcal{L}^n\text{-a.e. in } \Omega, \text{ for all } v \in W^{m,p}(\Omega).$$

It follows that

$$D^{\alpha}(g\circ v)=\partial_j D^{\beta}(g\circ v)$$
$$=\sum_{k=1}^{l-1}\sum_{\substack{\alpha^1+\ldots+\alpha^k=\beta\\ \left|\alpha^i\right|\neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k}\left[\left(g^{(k+1)}\circ v\right)\partial_j v D^{\alpha^1}v\ldots D^{\alpha^k}v\right.$$
$$\left.+\left(g^{(k)}\circ v\right)\left(\partial_j D^{\alpha^1}vD^{\alpha^2}v\ldots D^{\alpha^k}v+\ldots+D^{\alpha^1}v\ldots D^{\alpha^{k-1}}v\partial_j D^{\alpha^k}v\right)\right]$$
$$\mathcal{L}^n\text{-a.e. in } \Omega,\ \text{for all } v\in W^{m,p}(\Omega).$$

Hence, in order to prove (2.2) in the case $|\alpha|=l$, it suffices to show that

$$\left\|\left(g^{(k+1)}\circ u_\eta\right)\partial_j u_\eta D^{\alpha^1}u_\eta\ldots D^{\alpha^k}u_\eta - \left(g^{(k+1)}\circ u\right)\partial_j u D^{\alpha^1}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\to 0, \tag{2.42}$$

$$\left\|\left(g^{(k)}\circ u_\eta\right)\partial_j D^{\alpha^1}u_\eta D^{\alpha^2}u_\eta\ldots D^{\alpha^k}u_\eta - \left(g^{(k)}\circ u\right)\partial_j D^{\alpha^1}u D^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\to 0, \tag{2.43.1}$$

$$\left\|\left(g^{(k)}\circ u_\eta\right)D^{\alpha^1}u_\eta\partial_j D^{\alpha^2}u_\eta\ldots D^{\alpha^k}u_\eta - \left(g^{(k)}\circ u\right)D^{\alpha^1}u\partial_j D^{\alpha^2}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)}\to 0, \tag{2.43.2}$$

$$\vdots$$

$$\left\|\left(g^{(k)}\circ u_\eta\right)D^{\alpha^1}u_\eta\ldots D^{\alpha^{k-1}}u_\eta\partial_j D^{\alpha^k}u_\eta - \left(g^{(k)}\circ u\right)D^{\alpha^1}u\ldots D^{\alpha^{k-1}}u\partial_j D^{\alpha^k}u\right\|_{L^q(\Omega)}\to 0, \tag{2.43.k}$$

for all $k\in\{1,\ldots,l-1\}$ and all $\alpha^1,\ldots,\alpha^k\in\mathbb{N}^n$ with $\left|\alpha^i\right|\neq 0$ and $\alpha^1+\ldots+\alpha^k=\beta$. Fix $k\in\{1,\ldots,l-1\}$ and fix $\alpha^1,\ldots,\alpha^k\in\mathbb{N}^n$ with $\left|\alpha^i\right|\neq 0$ and $\alpha^1+\ldots+\alpha^k=\beta$.

Formula (2.42) is a direct consequence of Proposition 2.1 and formulas (2.39), (2.41).

It follows from $\|u_\eta-u\|_{W^{m,p}(\Omega)}\to 0$ that

$$\left\|\partial_j D^{\alpha^1}u_\eta-\partial_j D^{\alpha^1}u\right\|_{W^{m-\left|\alpha^1\right|-1,p}(\Omega)}\to 0.$$

On the other hand, since $p=\frac{n}{m-l}$, we have the Sobolev imbeddings

$$W^{m-\left|\alpha^1\right|-1,p}(\Omega)\hookrightarrow L^\infty(\Omega)\quad \text{if } 1\in I_1,$$

$$W^{m-\left|\alpha^1\right|-1,p}(\Omega)\hookrightarrow L^r(\Omega)\quad \text{for all } 1\leq r<\infty,\ \text{if } 1\in I_2$$

Therefore,

$$\begin{aligned} \left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^\infty(\Omega)} &\to 0 \quad \text{if } 1 \in I_1, \\ \left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^r(\Omega)} &\to 0 \quad \text{for all } 1 \le r < \infty, \text{ if } 1 \in I_2. \end{aligned} \tag{2.44}$$

According to the above, since $1 \le k \le l-1$, we have

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \to 0 \quad \text{for all } 1 \le r < \infty.$$

On the other hand, the Sobolev imbedding

$$W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega),$$

is valid for all $n < r < \infty$. Consequently,

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \to 0. \tag{2.45}$$

Now, (2.43.1) is a direct consequence of Proposition 2.1, formulas (2.39), (2.44), (2.45), and the fact that Ω is bounded. Formulas (2.43.2), …, (2.43.k) can be proved in the same way as (2.43.1). Consequently, formula (2.2) is proved in the case $|\alpha| = l$ as well.

Formula (2.1) is now proved under the hypotheses of point (ii).

Finally, we prove (2.1) under the hypotheses of point (iii), namely $\frac{n}{m-l} < p < \infty$, with $m \ge l+1$ ($1 \le p < \infty$ when $n \le m-l$), and $1 \le q < \infty$. By Theorem 1.1(iii,v), we deduce that $N_g : W^{m,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous. Thus, $\|g \circ u_\eta - g \circ u\|_{W^{1,q}(\Omega)} \to 0$, i.e.

$$\|g \circ u_\eta - g \circ u\|_{L^q(\Omega)} \to 0,$$

$$\|\partial_j (g \circ u_\eta) - \partial_j (g \circ u)\|_{L^q(\Omega)} \to 0, \quad j = 1, \dots, n.$$

Consequently, formula (2.1) is proved for $0 \le |\alpha| \le 1$.

According to point (iii), $g \circ u$ and $g \circ u_\eta$ satisfy the higher-order chain rule (2.1). Hence, in order to prove (2.1) for $2 \le |\alpha| \le l$, it suffices to prove (2.2) for all $\alpha \in \mathbb{N}^n$ with $2 \le |\alpha| \le l$, all $k \in \{1, \dots, s = |\alpha|\}$, all $\alpha^1, \dots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \ne 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$. To this end, we use Theorem 1.1(iii,v) and Proposition 2.1.

Firstly, we fix $\alpha \in \mathbb{N}^n$ with $2 \le |\alpha| = s \le l-1$, we fix $k \in \{1, \dots, s\}$ and we fix $\alpha^1, \dots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \ne 0$ and $\alpha^1 + \ldots + \alpha^k = \alpha$.

It follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\| D^{\alpha^i} u_\eta - D^{\alpha^i} u \right\|_{W^{m-|\alpha^i|,p}(\Omega)} \to 0.$$

On the other hand, since

$$\left(m-\left|\alpha^i\right|\right)p \geq (m-s)\,p > (m-l)\,p > n,$$

the following Sobolev imbedding holds

$$W^{m-\left|\alpha^i\right|,p}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Therefore

$$\left\|D^{\alpha^i}u_\eta - D^{\alpha^i}u\right\|_{L^\infty(\Omega)} \to 0. \tag{2.46}$$

By Theorem 1.1(iii,v), we infer that $N_{g^{(k)}} : W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ is continuous for all $1 \leq r < \infty$. Thus, $\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{W^{1,r}(\Omega)} \to 0$, whence

$$\left\|g^{(k)} \circ u_\eta - g^{(k)} \circ u\right\|_{L^r(\Omega)} \to 0, \quad 1 \leq r < \infty, \tag{2.47}$$

$$\left\|\left(g^{(k+1)} \circ u_\eta\right)\partial_j u_\eta - \left(g^{(k+1)} \circ u\right)\partial_j u\right\|_{L^r(\Omega)} \to 0,\ 1 \leq r < \infty,\ j=1,\ldots,n. \tag{2.48}$$

By using Proposition 2.1 and formulas (2.46), (2.47), we obtain the validity of formula (2.2) for $2 \leq |\alpha| \leq l-1$.

It remains to be shown that (2.2) is valid in the case $|\alpha| = l$ as well. We fix $\alpha \in \mathbb{R}^n$ with $|\alpha| = l$. There is $\beta \in \mathbb{R}^n$ and $j \in \{1,\ldots,n\}$ such that $|\beta| = l-1$ and $D^\alpha = \partial_j D^\beta$. According to point (iii), we have

$$D^\beta(g\circ v) = \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1+\ldots+\alpha^k=\beta \\ \left|\alpha^i\right|\neq 0}} c_{\beta,k,\alpha^1,\ldots,\alpha^k}\left(g^{(k)}\circ v\right) D^{\alpha^1}v \ldots D^{\alpha^k}v$$
$$\mathcal{L}^n\text{-a.e. in } \Omega,\ \text{for all } v \in W^{m,p}(\Omega).$$

It follows that

$$\begin{aligned} D^\alpha(g\circ v) &= \partial_j D^\beta(g\circ v) \\ = \sum_{k=1}^{l-1} \sum_{\substack{\alpha^1+\ldots+\alpha^k=\beta \\ \left|\alpha^i\right|\neq 0}} & c_{\beta,k,\alpha^1,\ldots,\alpha^k}\left[\left(g^{(k+1)}\circ v\right)\partial_j v D^{\alpha^1}v \ldots D^{\alpha^k}v\right. \\ &\left.+\left(g^{(k)}\circ v\right)\left(\partial_j D^{\alpha^1}vD^{\alpha^2}v\ldots D^{\alpha^k}v + \ldots + D^{\alpha^1}v\ldots D^{\alpha^{k-1}}v\partial_j D^{\alpha^k}v\right)\right] \\ &\mathcal{L}^n\text{-a.e. in } \Omega,\ \text{for all } v \in W^{m,p}(\Omega). \end{aligned}$$

Hence, in order to prove (2.2) in the case $|\alpha| = l$, it suffices to show that

$$\begin{aligned} &\left\|\left(g^{(k+1)}\circ u_\eta\right)\partial_j u_\eta D^{\alpha^1}u_\eta \ldots D^{\alpha^k}u_\eta \right. \\ &\quad \left. - \left(g^{(k+1)}\circ u\right)\partial_j u D^{\alpha^1}u\ldots D^{\alpha^k}u\right\|_{L^q(\Omega)} \to 0, \end{aligned} \tag{2.49}$$

$$\left\| \left(g^{(k)} \circ u_\eta\right) \partial_j D^{\alpha^1} u_\eta D^{\alpha^2} u_\eta \ldots D^{\alpha^k} u_\eta - \left(g^{(k)} \circ u\right) \partial_j D^{\alpha^1} u D^{\alpha^2} u \ldots D^{\alpha^k} u \right\|_{L^q(\Omega)} \to 0, \tag{2.50.1}$$

$$\left\| \left(g^{(k)} \circ u_\eta\right) D^{\alpha^1} u_\eta \partial_j D^{\alpha^2} u_\eta \ldots D^{\alpha^k} u_\eta - \left(g^{(k)} \circ u\right) D^{\alpha^1} u \partial_j D^{\alpha^2} u \ldots D^{\alpha^k} u \right\|_{L^q(\Omega)} \to 0, \tag{2.50.2}$$

$$\vdots$$

$$\left\| \left(g^{(k)} \circ u_\eta\right) D^{\alpha^1} u_\eta \ldots D^{\alpha^{k-1}} u_\eta \partial_j D^{\alpha^k} u_\eta - \left(g^{(k)} \circ u\right) D^{\alpha^1} u \ldots D^{\alpha^{k-1}} u \partial_j D^{\alpha^k} u \right\|_{L^q(\Omega)} \to 0, \tag{2.50.k}$$

for all $k \in \{1, \ldots, l-1\}$ and all $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \beta$. Fix $k \in \{1, \ldots, l-1\}$ and fix $\alpha^1, \ldots, \alpha^k \in \mathbb{N}^n$ with $\left|\alpha^i\right| \neq 0$ and $\alpha^1 + \ldots + \alpha^k = \beta$.

Formula (2.49) is a direct consequence of Proposition 2.1 and formulas (2.46), (2.48).

It follows from $\|u_\eta - u\|_{W^{m,p}(\Omega)} \to 0$ that

$$\left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{W^{m-|\alpha^1|-1,p}(\Omega)} \to 0.$$

On the other hand, since $p > \frac{n}{m-l}$, we have the Sobolev imbedding

$$W^{m-|\alpha^1|-1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{if } 1 \in I_1.$$

Therefore,

$$\left\| \partial_j D^{\alpha^1} u_\eta - \partial_j D^{\alpha^1} u \right\|_{L^\infty(\Omega)} \to 0. \tag{2.51}$$

According to the above, since $1 \leq k \leq l-1$, we have

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{W^{1,r}(\Omega)} \to 0 \quad \text{for all } 1 \leq r < \infty.$$

On the other hand, the Sobolev imbedding

$$W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega),$$

is valid for all $n < r < \infty$. Consequently,

$$\left\| g^{(k)} \circ u_\eta - g^{(k)} \circ u \right\|_{L^\infty(\Omega)} \to 0. \tag{2.52}$$

Now, (2.50.1) is a direct consequence of Proposition 2.1, formulas (2.46), (2.51), (2.52) and of the fact that Ω is bounded. Formulas (2.50.2), ..., (2.50.k) can be proved in the same way as (2.50.1). Consequently, formula (2.2) is proved in the case $|\alpha| = l$ as well.

Formula (2.1) is now proved under the hypotheses of point (iii).

Theorem 1.2 is completely proved. □

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