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On a fractional differential inclusion arising from real estate asset securitization and HIV models

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Abstract - We study a class of fractional differential inclusion arising from real estate asset securitization and HIV models and we establish a Filippov type existence result in the case of nonconvex set-valued maps.

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1. Introduction

In this note we study the following problem

$$\begin{array}{l} -D^{\alpha}x(t) \in F(t, x(t), -D^{\beta}x(t)) \quad a.e.\ ([0,1]), \\ D^{\beta}x(0) = D^{\beta+1}x(0) = 0, \quad D^{\beta}x(1) = \int_0^1 D^{\beta}x(s)dA(s), \end{array} \tag{1.1}$$

D^{α} is the standard Riemann-Liouville fractional derivative, $\beta \in (0,1)$, $\alpha \in (2,3]$, $\alpha - \beta > 1$, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map, $A(.)$ is a function of bounded variation and $\int_0^1 D^{\beta}x(s)dA(s)$ denotes the Riemann-Stieltjes integral.

The present note is motivated by a recent paper of Ahmad and Ntouyas ([1]) where existence results for problem (1.1) are established for convex as well as nonconvex set-valued maps. The existence results in [1] are based on a nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory. For the motivation, examples and a consistent bibliography on fractional differential equations and inclusions we refer to [1] and the references therein. We mention, only, that in [5] the authors discussed the existence and uniqueness of solutions for problem (1.1) with F single valued which represents a model for a problem arising from real estate asset securitization. On the other hand, motivated by a model of HIV infection Yang [6] considered the existence of nontrivial solutions for the fractional differential system

$$\begin{array}{l} -D^{\alpha}x(t) = \lambda f(t, x(t), -D^{\beta}x(t), y(t)) \quad -D^{\gamma}y(t) = g(t, x(t)) \quad t \in (0,1), \\ D^{\beta}x(0) = 0,\ D^{\beta}x(1) = \int_0^1 D^{\beta}x(s)dA(s),\ y(0) = 0,\ y(1) = \int_0^1 y(s)dB(s), \end{array}$$

with $\lambda \in \mathbb{R}$, $1 < \gamma < \alpha \leq 2$, $1 < \alpha - \beta < \gamma$, $0 < \beta - 1$ and $A(.), B(.)$ are functions of bounded variation.

The aim of this note is to show that Filippov's ideas ([3]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([3]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained. In this way we improve an existence result for problem (1.1) in [1].

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

Let (X, d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\},\ d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

Let $I = [0, 1]$, we denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $||x(.)||_C = \sup_{t \in I} |x(t)|$ and $L^1(I, \mathbb{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbb{R}$ endowed with the norm $||u(.)||_1 = \int_0^1 |u(t)|dt$.

The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and Γ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-\alpha+n-1} f(s) ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0, \infty)$.

As it is pointed out in [1], taking $x(t) = I^\beta y(t)$, $t \in I$ with $y(.) \in C(I, \mathbb{R})$, the boundary value problem (1.1) is equivalent with the following problem

$$\begin{aligned} &-D^{\alpha-\beta} y(t) \in F(t, I^\beta y(t), -y(t)) \quad a.e.\ ([0,1]), \\ &y(0) = y'(0) = 0, \quad y(1) = \int_0^1 y(s) dA(s), \end{aligned} \tag{2.1}$$

Lemma 2.1. (see [1]) *For a given $f(.) \in L^1(I, \mathbb{R})$, the unique solution $x(.)$ of problem*

$$D^{\alpha-\beta}x(t) + f(t) = 0 \quad a.e. \ ([0,T]),$$
$$y(0) = y'(0) = 0, \quad y(1) = \textstyle\int_0^1 y(s)dA(s),$$

is given by

$$x(t) = \int_0^1 G(t,s)f(s)ds,$$

where $G(.,.)$ is the Green function defined by

$$G(t,s) := \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} [t(1-s)]^{\alpha-\beta-1}, & \text{if} \quad 0 \le t < s \le 1, \\ [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & \text{if} \quad 0 \le s < t \le 1. \end{cases}$$

By Lemma 2.1 the unique solution of the problem

$$D^{\alpha}x(t) = 0, \quad x(0) = x'(0) = 0 \quad x(1) = 1$$

is $t^{\alpha-\beta-1}$. Let $c = \int_0^1 t^{\alpha-\beta-1}dA(t)$ and $G_A(s) = \int_0^1 G(t,s)dA(t)$. It follows (e.g., [7]) that the Green function of the boundary value problem (2.1) is

$$H(t,s) = \frac{t^{\alpha-\beta-1}}{1-c}G_A(s) + G(t,s).$$

Hypothesis 2.2. $A(.)$ is an increasing function of bounded variation such that $G_A(s) \ge 0$ $\forall s \in [0,1]$ and $0 \le c < 1$.

Lemma 2.3. (see [8]) *Let $1 < \alpha - \beta \le 2$ and let Hypothesis 2.2 be satisfied. Then*

$$0 \le H(t,s) \le \frac{1}{(1-c)\Gamma(\alpha-\beta-1)} =: M_1.$$

3. The main result

First we recall a selection result ([2]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 3.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X, L : I \to \mathbb{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In order to prove our results we need the following hypotheses.

Hypothesis 3.2. i) *$F(.,.) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ measurable.*

ii) *There exists* $L(.) \in L^1(I,(0,\infty))$ *such that, for almost all* $t \in I$, $F(t,.,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2)) \leq L(t)(|x_1-x_2|+|y_1-y_2|)\ \forall\, x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

We use next the following notations

$$M(t) := L(t)(1+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}ds) = L(t)(1+\frac{t^\beta}{\Gamma(\beta+1)}), \quad t \in I, \tag{3.1}$$

$$M_0 = \int_0^1 M(t)dt. \tag{3.2}$$

Theorem 3.3. *Assume that Hypothesis* 2.2 *and Hypothesis* 3.2 *are satisfied and* $M_1M_0 < 1$. *Let* $y(.) \in C(I,\mathbb{R})$ *be such that* $y(0) = y'(0) = 0$, $y(1) = \int_0^1 y(s)dA(s)$ *and there exists* $p(.) \in L^1(I,\mathbb{R}_+)$ *verifying* $d(-D^{\alpha-\beta}y(t), F(t, I^\beta y(t), -y(t))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem* (2.1) *satisfying for all* $t \in I$

$$|x(t)-y(t)| \leq \frac{M_1}{1-M_1M_0}\int_0^1 p(t)dt. \tag{3.3}$$

Proof. The set-valued map $t \to F(t,I^\beta y(t),-y(t))$ is measurable with closed values and

$$F(t,I^\beta y(t),-y(t)) \cap \{-D^{\alpha-\beta}y(t)+p(t)[-1,1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t,I^\beta y(t),-y(t))$ *a.e.* (I) such that

$$|f_1(t)+D^{\alpha-\beta}y(t)| \leq p(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t) = \int_0^1 H(t,s)f_1(s)ds$ and one has

$$|x_1(t)-y(t)| \leq M_1\int_0^1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I,\mathbb{R})$, $f_n(.) \in L^1(I,\mathbb{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_0^1 H(t,s)f_n(s)ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t,I^\beta x_{n-1}(t),-x_{n-1}(t)) \quad a.e.\ (I), \tag{3.6}$$

$$|f_{n+1}(t)-f_n(t)| \leq L(t)(|x_n(t)-x_{n-1}(t)|+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|x_n(s)-x_{n-1}(s)|ds) \tag{3.7}$$

for almost all $t \in I$.

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq M_1(M_1M_0)^n \int_0^1 p(t)dt \quad \forall n \in \mathbb{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^1 |H(t,t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \leq$$

$$M_1 \int_0^1 L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} |x_n(s) - x_{n-1}(s)|ds]dt_1 \leq$$

$$M_1 \int_0^1 L(t_1)(1 + \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} ds)dt_1.M_1^n M_0^{n-1} \int_0^1 p(t)dt =$$

$$= M_1(M_1M_0)^n \int_0^1 p(t)dt$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbb{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbb{R})$. Therefore, by (3.7), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$\begin{aligned} &|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq \\ &M_1 \textstyle\int_0^1 p(t)dt + \sum_{i=1}^{n-1} (M_1 \int_0^1 p(t)dt)(M_1M_0)^i = \frac{M_1 \int_0^1 p(t)dt}{1-M_1M_0}. \end{aligned} \tag{3.8}$$

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all $t \in I$

$$\begin{aligned} &|f_n(t) + D^{\alpha-\beta}y(t)| \leq \textstyle\sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + \\ &+|f_1(t) + D^{\alpha-\beta}y(t)| \leq L(t)\frac{M_1 \int_0^1 p(t)dt}{1-M_1M_0} + p(t). \end{aligned}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I,\mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that $x(.)$ is a solution of (2.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.5)-(3.7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I,\mathbb{R})$ and $f_n(.) \in L^1(I,\mathbb{R})$, $n = 1,2,...N$ satisfying (3.5), (3.7) for $n = 1,2,...N$ and (3.6) for $n = 1,2,...N-1$. The set-valued map $t \to F(t, I^\beta x_N(t), -x_N(t))$ is measurable. Moreover, the map $t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$

$$F(t, I^\beta x_N(t), -x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|x_N(s) - x_{N-1}(s)|ds)[-1,1]\} \neq \emptyset.$$

Lemma 3.1 yields that there exists a measurable selection $f_{N+1}(.)$ of $F(., I^\beta x_N(.), -x_N(.))$ such that

$$|f_{N+1}(t) - f_N(t)| \le L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|x_N(s) - x_{N-1}(s)|ds) \quad a.e.\ (I).$$

We define $x_{N+1}(.)$ as in (3.5) with $n = N+1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete. □

The assumption in Theorem 3.3 is satisfied, in particular, for $y(.) = 0$ and therefore with $p(.) = L(.)$. We obtain the following consequence of Theorem 3.3.

Corollary 3.4. *Assume that Hypothesis* 2.2 *and Hypothesis* 3.2 *are satisfied,* $d(0, F(t,0,0) \le L(t)$ *a.e.* (I) *and* $M_1M_0 < 1$. *Then there exists* $x(.)$ *a solution of problem* (2.1) *satisfying for all* $t \in I$

$$|x(t)| \le \frac{M_1}{1 - M_1M_0}\int_0^1 p(t)dt. \tag{3.9}$$

Remark 3.5. In [1], Theorem 3.5 it is proved that if Hypothesis 2.2 and Hypothesis 3.2 are satisfied, $d(0, F(t,0,0) \le L(t)$ a.e. (I), $F(.,.)$ has compact values and $M_1(1+\frac{1}{\Gamma(\beta+1)})\int_0^1 L(t)dt < 1$ then problem (2.1) has at least one solution.

Obviously, our Corollary 3.4 improves Theorem 3.5 in [1]. On one hand, we do not require that the values of $F(.,.)$ are compact; on the other hand

$$M_0 = \int_0^1 M(t)dt = \int_0^1 L(t)(1 + \frac{t^\beta}{\Gamma(\beta+1)})dt < (1 + \frac{1}{\Gamma(\beta+1)})\int_0^1 L(t)dt.$$

At the same time, the approach in [1] does not provides a priori bounds as in (3.9).

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