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# On the simplest expression of the perturbed Moore–Penrose metric generalized inverse

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**Abstract -** In this paper, by means of certain geometric assumptions of Banach spaces, we first give some equivalent conditions for the Moore–Penrose metric generalized inverse of perturbed operator to have the simplest expression $T^M(I+\delta TT^M)^{-1}$. Then, as applications of our results, we investigate the stability of some operator equations in Banach spaces under certain perturbations.

**Key words and phrases :** Metric generalized inverse, stable perturbation, operator equation.

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## 1. Introduction

Throughout the paper, we always let $X$ and $Y$ be Banach spaces over real field $\mathbb{R}$, and $B(X,Y)$ be the Banach space consisting of all bounded linear operators from $X$ to $Y$. For $T \in B(X,Y)$, let $\mathcal{N}(T)$ (resp. $\mathcal{R}(T)$) denote the kernel (resp. range) of $T$. It is well–known that for $T \in B(X,Y)$, if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are topologically complemented in the spaces $X$ and $Y$, respectively, then there exists a linear projector generalized inverse $T^+ \in B(Y,X)$ defined by

$$T^+Tx = x,\ x \in \mathcal{N}(T)^c \text{ and } T^+y = 0,\ y \in \mathcal{R}(T)^c,$$

where $\mathcal{N}(T)^c$ and $\mathcal{R}(T)^c$ are topologically complemented subspaces of $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. We know that linear projector generalized inverses of bounded linear operators have many important applications in numerical approximation, statistics and optimization et al. (see [3, 20, 24, 28]). But, generally speaking, the linear projector generalized inverse can not deal with the extremal solution, or the best approximation solution of an

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ill–posed operator equation in Banach spaces. In order to solve the best approximation problems for an ill–posed linear operator equation in Banach spaces, Nashed and Votruba [19] introduced the concept of the (set–valued) metric generalized inverse of a linear operator in Banach spaces. Later, in 2003, H. Wang and Y. Wang [27] defined the Moore–Penrose metric generalized inverse for a linear operator with closed range in Banach spaces, and gave some useful characterizations. Then in [21], the author defined and characterized the Moore–Penrose metric generalized inverse for an arbitrary linear operator in a Banach space. From then on, many research papers about the Moore–Penrose metric generalized inverses have appeared in the literature, see [1, 15, 18, 25, 16, 30] for instance.

In his recent thesis [17], H. Ma presented some perturbation results of the Moore–Penrose metric generalized inverses under certain additional assumptions, and also obtained some descriptions of the Moore–Penrose metric generalized inverses in Banach spaces. It is well–known that the perturbation analysis of generalized inverses of linear operators has wide applications and plays an important role in many fields such as computation, control theory, frame theory and nonlinear analysis. While the metric projector on closed subspace in Banach space are no longer linear, and then the linear projector generalized inverse and the Moore–Penrose metric projector generalized inverse of a bounded linear operator in Banach space are quite different. Motivated by many related perturbation results of the linear operator generalized inverses in Hilbert spaces or Banach spaces in [6, 11, 28] and some recent results in [17], in this paper, we further study the following perturbation and representation problems for the Moore–Penrose metric generalized inverses: *Let $T \in B(X,Y)$ such that the Moore–Penrose metric generalized inverse $T^M$ of $T$ exists, what conditions on the small perturbation $\delta T$ can guarantee that the Moore–Penrose metric generalized inverse $\bar{T}^M$ of the perturbed operator $\bar{T} = T + \delta T$ exists? Furthermore, if it exists, when does $\bar{T}^M$ have the simplest expression $(I_X + T^M \delta T)^{-1} T^M$?* Under the geometric assumption that the Banach spaces $X$ and $Y$ are smooth, and by using the generalized orthogonal decomposition theorem [24], we will give a complete answer to these problems.

Perturbation analysis for the extremal solution of the linear operator equation $Tx = y$ by using the linear generalized inverses has been made by many authors (cf. [6, 12, 23]). It is well–known that the theory of the Moore–Penrose metric generalized inverses has its genetic in the context of the so–called "ill–posed" linear problems. So, as applications of our result, in the last section of this paper, we will investigate the stability of the solutions of the operator equation $Tx = y$ and the best approximate solutions of the operator equation $\|Tx - b\| = \inf_{y \in X} \|Ty - b\|$ in Banach spaces under some different conditions.

## 2. Preliminaries

In this section, we will recall some concepts and results frequently used in this paper. We first recall some related concepts about homogeneous operators and the geometry of Banach spaces. For more information about the geometric properties of Banach spaces, such as strict convexity, reflexivity and complemented subspaces, we refer to [2, 10].

Let $X, Y$ be Banach spaces, let $T : X \to Y$ be a mapping and $D \subset X$ be a subset of $X$. Recall from [1, 26] that a subset $D$ in $X$ is called to be homogeneous if $\lambda x \in D$ whenever $x \in D$ and $\lambda \in \mathbb{R}$; a mapping $T\colon X \to Y$ is called to be a bounded homogeneous operator if $T$ maps every bounded set in $X$ into a bounded set in $Y$ and $T(\lambda x) = \lambda T(x)$ for every $x \in X$ and every $\lambda \in \mathbb{R}$. Let $H(X, Y)$ denote the set of all bounded homogeneous operators from $X$ to $Y$. Equipped with the usual linear operations on $H(X, Y)$ and norm on $T \in H(X, Y)$ defined by $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1, x \in X\}$, we can easily prove that $(H(X, Y), \|\cdot\|)$ is a Banach space (cf. [24, 26]). For a bounded homogeneous operator $T \in H(X, Y)$, we always denote by $\mathcal{D}(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the domain, the null space and respectively, the range of an operator $T$. Obviously, we have $B(X, Y) \subset H(X, Y)$.

**Definition 2.1.** *Let $M \subset X$ be a subset and let $T : X \to Y$ be a mapping. Then we called $T$ is quasi–additive on $M$ if $T$ satisfies*

$$T(x + z) = T(x) + T(z), \qquad \forall\, x \in X,\ \forall\, z \in M.$$

*For a homogeneous operator $T \in H(X, X)$, if $T$ is quasi–additive on $\mathcal{R}(T)$, then we will simply call $T$ is a quasi–linear operator.*

**Definition 2.2.** (See [16, 24]) *Let $P \in H(X, X)$. If $P^2 = P$, then we call $P$ is a homogeneous projector. In addition, if $P$ is also quasi–additive on $\mathcal{R}(P)$, i.e., for any $x \in X$ and any $z \in \mathcal{R}(P)$,*

$$P(x + z) = P(x) + P(z) = P(x) + z,$$

*then we call $P$ is a quasi–linear projector.*

Now we recall the definition of dual mapping for Banach spaces.

**Definition 2.3.** (See [2]) *The set–valued mapping $F_X : X \to X^*$ defined by*

$$F_X(x) = \{f \in X^* \mid f(x) = \|x\|^2 = \|f\|^2\}, \qquad \forall\, x \in X$$

*is called the dual mapping of $X$, where $X^*$ is the dual space of $X$.*

It is well known that the dual mapping $F_X$ is a homogeneous set–valued mapping; $F_X$ is surjective if and only if $X$ is reflexive; $F_X$ is injective or strictly monotone if and only if $X$ is strictly convex; $F_X$ is single–valued if and only if $X$ is smooth, that is, for any $x \in X$ with $\|x\| = 1$, there is a unique $f \in X^*$ such that $f(x) = 1 = \|f\|$. Clearly, if $X^*$ is strictly convex, then $X$ is smooth. We will need these properties of $F_X$ to prove our main results for the Moore–Penrose metric generalized inverse. Please see [2] for more information about the mapping $F_X$.

**Definition 2.4.** (See [22]) *Let $G \subset X$ be a subset of $X$. The set–valued mapping $P_G : X \to G$ defined by*

$$P_G(x) = \{s \in G \mid \|x - s\| = \operatorname{dist}(x, G)\}, \qquad \forall x \in X$$

*is called the set–valued metric projection, where* $\operatorname{dist}(x, G) = \inf_{z \in X} \|x - z\|$.

For a subset $G \subset X$, if $P_G(x) \neq \emptyset$ for each $x \in X$, then $G$ is said to be proximinal; if $P_G(x)$ is at most a singleton for each $x \in X$, then $G$ is said to be semi–Chebyshev; if $G$ is simultaneously proximinal and semi–Chebyshev set, then $G$ is called a Chebyshev set. We denote by $\pi_G$ any selection for the set-valued mapping $P_G$, i.e., any single–valued mapping $\pi_G : \mathcal{D}(\pi_G) \to G$ with the property that $\pi_G(x) \in P_G(x)$ for any $x \in \mathcal{D}(\pi_G)$, where $\mathcal{D}(\pi_G) = \{x \in X : P_G(x) \neq \emptyset\}$. For the particular case, when $G$ is a Chebyshev set, then $\mathcal{D}(\pi_G) = X$ and $P_G(x) = \{\pi_G(x)\}$. In this case, the mapping $\pi_G$ is called the metric projector from $X$ onto $G$.

**Remark 2.1.** Let $G \subset X$ be a closed convex subset. It is well–known that if $X$ is reflexive, then $G$ is a proximal set; if $X$ is a strictly convex, then $G$ is a semi–Chebyshev set. Thus, every closed convex subset in a reflexive strictly convex Banach space is a Chebyshev set.

The following lemma gives some important properties of the metric projectors.

**Lemma 2.1.** (See [22]) *Let $X$ be a Banach space and $L$ be a subspace of $X$. Then*

(1) $\pi_L^2(x) = \pi_L(x)$ *for any $x \in \mathcal{D}(\pi_L)$, i.e., $\pi_L$ is idempotent;*

(2) $\|x - \pi_L(x)\| \leq \|x\|$ *for any $x \in \mathcal{D}(\pi_L)$, i.e., $\|\pi_L\| \leq 2$.*

*In addition, If $L$ is a semi–Chebyshev subspace, then*

(3) $\pi_L(\lambda x) = \lambda \pi_L(x)$ *for any $x \in X$ and $\lambda \in \mathbb{R}$, i.e., $\pi_L$ is homogeneous;*

(4) $\pi_L(x + z) = \pi_L(x) + \pi_L(z) = \pi_L(x) + z$ *for any $x \in \mathcal{D}(\pi_L)$ and $z \in L$, i.e., $\pi_L$ is quasi–additive on $L$.*

The following so called generalized orthogonal decomposition theorem in Banach space is a main tool in this paper.

**Lemma 2.2.** (Generalized Orthogonal Decomposition Theorem, see [15, 24]) *Let $G \subset X$ be a proximinal subspace. Then for any $x \in X$, we have*

(1) $x = x_1 + x_2$ *with* $x_1 \in G$ *and* $x_2 \in F_X^{-1}(G^\perp)$*;*

(2) *Furthermore, if $G \subset X$ is a Chebyshev subspace, then the decomposition in (1) is unique such that $x = \pi_G(x) + x_2$. In this case, we can write $X = G \dotplus F_X^{-1}(G^\perp)$,*

*where $G^\perp = \{f \in X^* | f(x) = 0, \forall x \in G\}$ and $F_X^{-1}(G^\perp) = \{x \in X | F_X(x) \cap G^\perp \neq \emptyset\}$.*

Now we give the definition of the Moore–Penrose metric generalized for $T \in B(X, Y)$.

**Definition 2.5.** (See [24, 27]) *Let $T \in B(X, Y)$. Suppose that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of $X$ and $Y$, respectively. If there exists a bounded homogeneous operator $T^M : Y \to X$ such that:*

$$(1)\ TT^M T = T; \quad (2)\ T^M T T^M = T^M;$$
$$(3)\ T^M T = I_X - \pi_{\mathcal{N}(T)}; \quad (4)\ TT^M = \pi_{\mathcal{R}(T)},$$

*then $T^M$ is called the Moore–Penrose metric generalized inverse of $T$, where $\pi_{\mathcal{N}(T)}$ and $\pi_{\mathcal{R}(T)}$ are the metric projectors onto $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively.*

If $T^M$ exists, then it is also unique (cf. [24, 27]). Moreover, if $T^M$ exists, then by Lemma 2.2, the spaces $X$ and $Y$ have the following unique decompositions

$$X = \mathcal{N}(T) \dotplus F_X^{-1}(\mathcal{N}(T)^\perp), \qquad Y = \mathcal{R}(T) \dotplus F_Y^{-1}(\mathcal{R}(T)^\perp),$$

respectively, where $F_X : X \to X^*$ (resp. $F_Y : Y \to Y^*$) is the set–valued dual mapping of $X$ (resp. $Y$). Please see [24] for more information about the Moore–Penrose metric generalized inverses and related knowledge. Here we only need the following result which characterizes the existence of the Moore–Penrose metric generalized inverse.

**Lemma 2.3.** (See [21, 24]) *Let $T \in B(X, Y)$ with $\mathcal{R}(T)$ closed. If $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of $X$ and $Y$, respectively. Then there exists a unique Moore–Penrose metric generalized inverse $T^M$ of $T$.*

## 3. The simplest expression of the Moore–Penrose metric generalized inverse of the perturbed operator

In order to prove the main result in the paper, we need the following three lemmas.

**Lemma 3.1.** (See [4]) *Let $T \in B(X,Y)$ such that $T^M$ exists and let $\delta T \in B(X,Y)$ such that $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$ and $\|T^M \delta T\| < 1$. Put $\bar{T} = T + \delta T$. Then*

(1) *$I_X + T^M \delta T$ and $I_Y + \delta T T^M$ are invertible in $B(X,X)$ and $H(Y,Y)$, respectively;*

(2) *$\Phi = (I_X + T^M \delta T)^{-1} T^M = T^M (I_Y + \delta T T^M)^{-1} \in H(Y,X)$;*

(3) *$\bar{T}\Phi\bar{T} = \bar{T}$ and $\Phi\bar{T}\Phi = \Phi$ when $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$.*

**Lemma 3.2.** *Let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed and $\mathcal{N}(T)$ (resp. $\mathcal{R}(T)$) Chebyshev in $X$ (resp. $Y$). Let $\delta T \in B(X,Y)$ with $T^M$ quasi–additive on $\mathcal{R}(\delta T)$ and $\|T^M \delta T\| < 1$. Put $\bar{T} = T + \delta T$ and $\Phi = T^M (I_Y + \delta T T^M)^{-1} = (I_X + T^M \delta T)^{-1} T^M$. Then $\mathcal{R}(\Phi) = F_X^{-1}(\mathcal{N}(T)^\perp)$ and $\mathcal{N}(\Phi) = F_Y^{-1}(\mathcal{R}(T)^\perp)$.*

**Proof.** From Lemma 3.1, we see $\Phi$ is well defined. Then, according to the expressions of $\Phi$, we have $\mathcal{R}(\Phi) = \mathcal{R}(T^M)$ and $\mathcal{N}(\Phi) = \mathcal{N}(T^M)$. Then from Lemma 2.2 and Definition 2.5, we can get that $\mathcal{R}(T^M) = F_X^{-1}(\mathcal{N}(T)^\perp)$ and $\mathcal{N}(T^M) = F_Y^{-1}(\mathcal{R}(T)^\perp)$. □

For convenience, we recall the concept of smoothness of Banach space. Let $X^*$ be the dual space of $X$. Let $S(X)$ and $S(X^*)$ be the unit spheres of $X$ and $X^*$, respectively. We say $X$ is smooth if for each point $x \in S(X)$ there exists a unique $f \in S(X^*)$ such that $f(x) = 1$. Please see [10] for more information about this important concept and related topics. We have indicated that if $X$ is smooth, then the dual mapping $F_X$ (see Definition 2.3) is single-valued.

**Lemma 3.3.** *Let $M, N \subset X$ be Chebyshev subspaces of $X$. If $X$ is smooth, then $F_X^{-1}(M^\perp) = F_X^{-1}(N^\perp)$ if and only if $M = N$.*

**Proof.** If $M = N$, obviously, we have $F_X^{-1}(M^\perp) = F_X^{-1}(N^\perp)$.

Suppose that $F_X^{-1}(M^\perp) = F_X^{-1}(N^\perp) \triangleq G$. We prove that $M = N$ if $X$ is smooth. In fact, since $M, N$ are Chebyshev subspace of $X$, by the Generalized Orthogonal Decomposition Theorem (cf. Lemma 2.2) in Banach space, we have

$$X = M \dotplus F_X^{-1}(M^\perp) = N \dotplus F_X^{-1}(N^\perp).$$

Then for any $m \in M \backslash N$, we have the unique decomposition $m = m + 0$ with respect to $M$ and $G$. Noting that we also have the unique decomposition

$m = n_1 + n_2$ with respect to $N$ and $G$, where $n_1 \in N$ and $n_2 \in G$. If $M \neq N$, by the uniqueness of the decomposition, we must have $n_2 \neq 0$.

Since $X$ is smooth, then $F_X$ is single–valued. So from $F_X^{-1}(M^\perp) = F_X^{-1}(N^\perp) \triangleq G$, we get that $f = F_X(n_2) \in N^\perp \cap M^\perp$, that is, $f(n_2) = \|f\|^2 = \|n_2\|^2$ and $f(m) = f(n_1) = 0$. Therefore, $\|n_2\|^2 = f(m - n_1) = 0$ and $n_2 = 0$, which is a contradiction. Thus, we have $M = N$. □

Under the geometric assumptions that both $X$ and $Y$ are smooth Banach spaces, now we can prove the following useful result for the perturbation of the Moore–Penrose metric generalized inverse of the perturbed operator.

**Theorem 3.1.** *Let $X, Y$ be smooth Banach spaces and let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Let $\delta T \in B(X,Y)$ and put $\bar{T} = T + \delta T$. Assume that $\mathcal{N}(T)$ and $\mathcal{N}(\bar{T})$ are Chebyshev subspaces of $X$, $\mathcal{R}(T)$ and $\mathcal{R}(\bar{T})$ are Chebyshev subspaces of $Y$. Then the Moore–Penrose metric generalized inverse $T^M$ of $T$ exists. In addition, if $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$ and $I_X + T^M \delta T$ is invertible in $B(X,X)$, then $\Phi = T^M(I_Y + \delta T T^M)^{-1} = (I_X + T^M\delta T)^{-1}T^M$ is well–defined and the following statements are equivalent:*

(1) *$\Phi$ is the Moore–Penrose metric generalized inverse of $\bar{T}$, i.e., $\Phi = \bar{T}^M$;*

(2) *$\mathcal{R}(\bar{T}) = \mathcal{R}(T)$ and $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$;*

(3) *$\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ and $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$.*

**Proof.** Since $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of $X$ and $Y$, respectively, then from Lemma 2.3, we know $T^M$ exists and is unique. If $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$ and $I_X + T^M\delta T$ is invertible in $B(X,X)$, then from Lemma 3.1, we see

$$\Phi = T^M(I_Y + \delta T T^M)^{-1} = (I_X + T^M\delta T)^{-1}T^M$$

is well defined. Now we show that the equivalences hold.

(1) ⇒ (2) Since $\Phi = \bar{T}^M$, then from Lemma 3.2, we get that

$$\mathcal{N}(\bar{T}^M) = \mathcal{N}(\Phi) = F_X^{-1}(\mathcal{R}(T)^\perp), \quad \mathcal{R}(\bar{T}^M) = \mathcal{R}(\Phi) = F_X^{-1}(\mathcal{N}(T)^\perp).$$

Since $\mathcal{N}(\bar{T})$ and $\mathcal{R}(\bar{T})$ are Chebyshev subspaces of $X$ and $Y$, respectively, it follows from Lemma 2.2 and Definition 2.5 that $\mathcal{R}(\bar{T}^M) = F_X^{-1}(\mathcal{N}(\bar{T})^\perp)$ and $\mathcal{N}(\bar{T}^M) = F_Y^{-1}(\mathcal{R}(\bar{T})^\perp)$. Consequently,

$$\begin{aligned} F_X^{-1}(\mathcal{N}(\bar{T})^\perp) &= \mathcal{R}(\bar{T}^M) = F_X^{-1}(\mathcal{N}(T)^\perp); \\ F_Y^{-1}(\mathcal{R}(\bar{T})^\perp) &= \mathcal{N}(\bar{T}^M) = F_X^{-1}(\mathcal{R}(T)^\perp). \end{aligned} \tag{3.1}$$

Noting that $X$ and $Y$ are smooth Banach spaces, so we have $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$ and $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$ from Lemma 3.3 and (3.1).

(2) $\Rightarrow$ (3) Let $x \in \mathcal{N}(T) = \mathcal{N}(\bar{T})$. Then $Tx = 0$ and $Tx + \delta Tx = 0$. So $\delta Tx = 0$, that is, $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$. Let $y \in \mathcal{R}(\delta T)$, then there exists some $x \in X$ such that $y = \delta Tx = \bar{T}x - Tx$. Noting that $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$, we have $y \in \mathcal{R}(T)$, that is, $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$.

(3) $\Rightarrow$ (1) From $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ and $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$, we get that $\pi_{\mathcal{R}(T)}\delta T = \delta T$ and $\delta T \pi_{\mathcal{N}(T)} = 0$, that is, $TT^M\delta T = \delta T = \delta TT^MT$. Consequently,

$$\bar{T} = T + \delta T = T(I_X + T^M\delta T) = (I_Y + \delta TT^M)T. \tag{3.2}$$

Since $I_X + T^M\delta T$ is invertible in $B(X,X)$ and $I_Y + \delta TT^M$ is invertible in $H(Y,Y)$ by Lemma 3.1, we have $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$ and $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$ by (3.2). Thus $\bar{T}\Phi\bar{T} = \bar{T}$ and $\Phi\bar{T}\Phi = \Phi$ by Lemma 3.1 and moreover,

$$\begin{aligned}
\bar{T}\Phi &= (T + \delta T)T^M(I_Y + \delta TT^M)^{-1} = TT^M(I_Y + \delta TT^M)(I_Y + \delta TT^M)^{-1} \\
&= TT^M = \pi_{\mathcal{R}(T)} = \pi_{\mathcal{R}(\bar{T})}; \\
\Phi\bar{T} &= (I_X + T^M\delta T)^{-1}T^M(T + \delta T) = (I_X + T^M\delta T)^{-1}(I_X + T^M\delta T)T^MT \\
&= T^MT = I_X - \pi_{\mathcal{N}(T)} = I_X - \pi_{\mathcal{N}(\bar{T})}.
\end{aligned}$$

Therefore, $\Phi$ is the Moore–Penrose metric generalized inverse of $\bar{T}$, i.e., $\Phi = \bar{T}^M$. □

**Remark 3.1.** We should remark that, some related results of Theorem 3.1 have been proved in [17]. In [17, Theorem 4.3.3], under the assumptions that

(1) $\mathcal{N}(T)$ and $\mathcal{N}(\bar{T})$ are Chebyshev subspaces of $X$;

(2) $\mathcal{R}(T)$ and $\mathcal{R}(\bar{T})$ are Chebyshev subspaces of $Y$;

(3) $\|T^M\delta T\| < 1$, $\mathcal{N}(T) \subset \mathcal{N}(\delta T)$ and $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$;

(4) $F_X^{-1}(\mathcal{N}(T)^\perp)$ is a linear subspace of $X$ and $\mathcal{R}(T)$ is approximatively compact,

the author proved that $\bar{T}^M$ exists and has the representations

$$\bar{T}^M = T^M(I_Y + \delta TT^M)^{-1} = (I_X + T^M\delta T)^{-1}T^M.$$

Thus, Theorem 3.1 gives some generalization of the above results. We also note that our proof is more concise. Please see [17] for more related results.

From Theorem 3.1, it is easy to get the following perturbation result which is Theorem 3.1 of [11] for the Moore–Penrose orthogonal projection generalized inverses of bounded linear operators on Hilbert spaces.

**Corollary 3.1.** *Let $H, K$ be Hilbert spaces. Let $T \in B(H,K)$ have the Moore–Penrose generalized inverse $T^\dagger \in B(K,H)$. Let $\delta T \in B(H,K)$ with $\|T^\dagger \delta T\| < 1$. Then $G = (I_X + T^\dagger \delta T)^{-1} T^\dagger$ is the Moore–Penrose generalized inverse of $\bar{T} = T + \delta T$ if and only if $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$ and $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$.*

**Proof.** Since $H$ and $K$ are Hilbert spaces, then from Definition 2.4, we see that the metric projector is just the linear orthogonal projector. Now from Definition 2.5, we see obviously that the Moore–Penrose metric generalized inverse $T^M$ of $T$ is indeed the Moore–Penrose orthogonal projection generalized inverse $T^\dagger$ of $T$ under usual sense. It is well–known that Hilbert spaces are smooth and the condition $\|T^\dagger \delta T\| < 1$ implies $I_X + T^\dagger \delta T$ invertible, hence we can get the assertion by using Theorem 3.1. □

**Remark 3.2.** Let $X, Y$ be Banach spaces and let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Let $\delta T \in B(X,Y)$ and put $\bar{T} = T + \delta T$. Assume that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of $X$ and $Y$, respectively. If $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$ and $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$, then from Lemma 2.2 and Definition 2.5, we see clearly that $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$. Recall from [6] that for $T \in B(X,Y)$ with a bounded linear generalized inverse $T^+ \in B(Y,X)$, we say that the operator $\bar{T} = T + \delta T \in B(X,Y)$ is a stable perturbation of $T$ if $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^+) = \{0\}$. Now for $T \in B(X,Y)$ with $T^M \in H(Y,X)$, we also say that $\bar{T} = T + \delta T \in B(X,Y)$ is a stable perturbation of $T$ if $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$. From related results in [28], we know that the concept of stable perturbation is important and useful for us to study the perturbation problems for generalized inverses. So from Theorem 3.1 and Lemma 3.1, it is nature to ask the following questions:

*If $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then what additional conditions can guarantee that $\bar{T}^M$ exists? If $\bar{T}^M$ exists, what is its expression and can we give some error estimations of the upper bound of $\|\bar{T}^M - T^M\|$?*

In our forthcoming paper [5], we will make a further study on these problems in reflexive strictly convex Banach spaces, specially, in the Banach space $L^p(\Omega, \mu)$ with $1 < p < +\infty$.

## 4. On the stability of some operator equations in Banach spaces

Throughout this section, we assume that the operator $T \in B(X,Y)$ has closed range and both $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of $X$ and $Y$, respectively, so that the corresponding Moore–Penrose metric generalized inverse $T^M$ of $T$ is well–defined as a bounded homogeneous operator. We also let $\delta T \in B(X,Y)$ such that $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$ and $\|T^M\|\|\delta T\| < 1$ in this section, so that $I_X + T^M \delta T$ and $I_Y + \delta T T^M$ are invertible in $B(X,X)$ and $H(Y,Y)$, respectively.

Suppose that the operator equation

$$Tx = b \tag{4.1}$$

is perturbed to the following consistent linear operator equation

$$\bar{T}z = \bar{b}, \tag{4.2}$$

where $\bar{T} = T + \delta T$ and $\bar{b} = b + \delta b \in \mathcal{R}(\bar{T})$. Denoted by $S(T, b)$ and (resp. $S(\bar{T}, \bar{b})$) the solution set of the equation $Tx = b$ (resp. $\bar{T}z = \bar{b}$). For convenience, in this section, we always let $\epsilon_b = \dfrac{\|\delta b\|}{\|b\|}$, $\epsilon_T = \dfrac{\|\delta T\|}{\|T\|}$, and let $\kappa = \|T^M\|\|T\|$ denote the condition number of $T$.

**Theorem 4.1.** *Let $T$, $\delta T \in B(X,Y)$ such that $\|T^M\|\|\delta T\| < 1$ and $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$. If $T^M$ is also quasi–additive on $\mathcal{R}(T)$, then for any solution $z \in S(\bar{T}, \bar{b})$, there exists a solution $x \in S(T, b)$ such that*

$$\frac{1}{1+\kappa\epsilon_T}\left(\frac{\|T^M\delta b\|}{\|T^M b\| + 2\|z\|} - \kappa\epsilon_T\right) \le \frac{\|z-x\|}{\|x\|} \le \frac{\kappa(\epsilon_b+\epsilon_T)}{1-\kappa\epsilon_T}.$$

**Proof.** Let $z \in \hat{S}(\bar{T}, \bar{b})$ and put $x = T^M b + (I_X - T^M T)z$. Then, we can check that $x \in S(T, b)$. Noting that $T^M$ is quasi–additive on $\mathcal{R}(T)$, then

$$z - x = T^M Tz - T^M b = T^M(Tz - b) \in R(T^M) = F_X^{-1}(\mathcal{N}(T)^\perp). \tag{4.3}$$

It follows that $\pi_{\mathcal{N}(T)}(z-x) = 0$ and then

$$T^M T(z-x) = (I_X - \pi_{\mathcal{N}(T)})(z-x) = z - x.$$

Now from $\bar{T}z = \bar{b}$ and $Tx = b$, we can check that

$$\begin{aligned}(I_X + T^M\delta T)(z-x) &= T^M(T+\delta T)(z-x)\\ &= T^M\bar{T}(z-x) = T^M(\bar{T}z - Tx - \delta Tx)\\ &= T^M(\delta b - \delta Tx).\end{aligned} \tag{4.4}$$

From Lemma 3.1, we know that $(I_X + T^M\delta T)$ is invertible. Thus, from (4.4), we get

$$z - x = (I_X + T^M\delta T)^{-1}T^M(\delta b - \delta Tx).$$

So by using above equation, we can obtain

$$\begin{aligned}\frac{\|z-x\|}{\|x\|} &= \frac{\|(I_X + T^M\delta T)^{-1}T^M(\delta b - \delta Tx)\|}{\|x\|}\\ &\le \|(I_X + T^M\delta T)^{-1}\|\frac{\|T^M(\delta b - \delta Tx)\|}{\|x\|}\\ &\le \frac{\kappa(\epsilon_b+\epsilon_T)}{1-\kappa\epsilon_T}.\end{aligned} \tag{4.5}$$

Noting that $\|x\| \le \|T^M b\| + \|\pi_{\mathcal{N}(T)} z\| \le \|T^M b\| + 2\|z\|$, thus, we also have

$$\frac{\|z-x\|}{\|x\|} \ge \frac{\|T^M(\delta b - \delta T x)\|}{\|I_X + T^M \delta T\|\|x\|} \ge \frac{1}{1+\kappa\epsilon_T}\left(\frac{\|T^M \delta b\|}{\|T^M b\| + 2\|z\|} - \kappa\epsilon_T\right). \quad (4.6)$$

Now, our result follows from (4.5) and (4.6). □

**Corollary 4.1.** *Let $T$, $\delta T \in B(X,Y)$ such that $\|T^M\|\|\delta T\| < 1$ and $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$. Assume that $\mathcal{R}(\bar{T})$ is a Chebychev subspace in $Y$ and $T^M$ is quasi–additive on $\mathcal{R}(T)$. If $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$, then $\bar{T}^M$ exists, and moreover, for the equations* (4.1) *and* (4.2), *we have*

$$\frac{1}{1+\kappa\epsilon_T}\left(\frac{\|T^M \delta b\|}{\|T^M b\|} - \kappa\epsilon_T\right) \le \frac{\|\bar{T}^M \bar{b} - T^M b\|}{\|T^M b\|} \le \frac{\kappa(\epsilon_b + \epsilon_T)}{1-\kappa\epsilon_T}.$$

**Proof.** From Lemma 2.3, we know $\bar{T}^M$ uniquely exists, and then $z = \bar{T}^M \bar{b}$ is a solution of the equation $\bar{T}z = \bar{b}$. Now from our proof of Theorem 4.1, we see $x = T^M b + (I_X - T^M T)\bar{T}^M \bar{b}$. Noting that $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$, we have

$$\begin{aligned} x &= T^M b + (I_X - T^M T)\bar{T}^M \bar{b} \\ &= T^M b + \pi_{\mathcal{N}(T)} \bar{T}^M \bar{b} = T^M b + \pi_{\mathcal{N}(\bar{T})} \bar{T}^M \bar{b} \\ &= T^M b. \end{aligned}$$

So by using Theorem 4.1, we can obtain the result. □

We now consider the problem:

$$\min \|x\| \quad \text{subject to} \quad \|Tx - b\| = \inf_{y \in X} \|Ty - b\|. \quad (4.7)$$

Suppose that the problem (4.7) is perturbed to the following problem:

$$\min \|z\| \quad \text{subsect to} \quad \|\bar{T}z - \bar{b}\| = \inf_{y \in X} \|\bar{T}y - \bar{b}\|, \quad (4.8)$$

where $b$, $\bar{b} = b + \delta b \in Y$ and $\bar{T} = T + \delta T$. It follows from [28, Proposition 2.3.7] that the equation (4.7) (resp. (4.8)) has solutions when $X$ and $Y$ are reflexive and $\mathcal{R}(T)$ (resp. $\mathcal{R}(\bar{T})$) is closed. Moreover, if the Moore–Penrose metric generalized inverse $T^M$ (resp. $\bar{T}^M$) exists, then from Definition 2.5 (or cf. [27, Theorem 3.2]), we see that the vector $x = T^M b$ (resp. $z = \bar{T}^M \bar{b}$) is the solution of (4.7) (resp. (4.8)). In order to use Theorem 3.1, from now on, we always assume that $X, Y$ are smooth, strictly convex and reflexive Banach spaces.

**Theorem 4.2.** *Let $T$, $\delta T \in B(X,Y)$ such that $\|T^M\|\|\delta T\| < 1$ and $T^M$ is quasi–additive on $\mathcal{R}(\delta T)$. Assume that $\mathcal{N}(\bar{T}) = \mathcal{N}(T)$ and $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$. Then for the problems* (4.7) *and* (4.8), *we have*

$$\frac{\|\bar{T}^M \bar{b} - T^M b\|}{\|T^M b\|} \le \frac{1}{1-\kappa\epsilon_T}\left(\kappa\epsilon_T + \frac{\|T\|\|T^M(b+\delta b) - T^M b\|}{\|\pi_{\mathcal{R}(T)} b\|}\right).$$

**Proof.** From our assumption and by using Theorem 3.1, we see $\bar{T}^M$ exists and

$$\bar{T}^M = T^M(I_Y + \delta T T^M)^{-1} = (I_X + T^M \delta T)^{-1} T^M.$$

Since $(I_X + T^M \delta T)^{-1} \in B(X, X)$, we have

$$\bar{T}^M \bar{b} - T^M b = (I_X + T^M \delta T)^{-1}[T^M(b + \delta b) - T^M b] + \big[(I_X + T^M \delta T)^{-1} - I_X\big] T^M b$$

and consequently,

$$\begin{aligned}
\|\bar{T}^M \bar{b} - T^M b\| &\le \frac{\|T^M(b + \delta b) - T^M b\|}{1 - \|T^M\|\|\delta T\|} + \frac{\|T^M\|\|\delta T\|}{1 - \|T^M\|\|\delta T\|}\|T^M b\|, \\
\frac{\|\bar{T}^M \bar{b} - T^M b\|}{\|T^M b\|} &\le \frac{\kappa \epsilon_T}{1 - \kappa \epsilon_T} + \frac{\|T^M(b + \delta b) - T^M b\|}{1 - \kappa \epsilon_T} \frac{1}{\|T^M b\|}. \qquad (4.9)
\end{aligned}$$

Note that $\pi_{\mathcal{R}(T)} b = T T^M b$ and $\|\pi_{\mathcal{R}(T)} b\| \le \|T\| \|T^M b\|$. So the assertion follows from (4.9). □

**Remark 4.1.** Under the assumption of Theorem 4.2, if we also have $\delta b \in \mathcal{R}(\delta T)$, then it follows from Theorem 4.2 that

$$\frac{\|\bar{T}^M \bar{b} - T^M b\|}{\|T^M b\|} \le \frac{\kappa}{1 - \kappa \epsilon_T} \left( \epsilon_T + \frac{\|\delta b\|}{\|\pi_{\mathcal{R}(T)} b\|} \right).$$

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## References

[1] X. BAI, Y. WANG, G. LIU and J. XIA, Definition and criterion of homogeneous generalized inverse, *Acta Math. Sinica (Chin. Ser.)*, **52** (2009), 353-360.

[2] V. BARBU and TH. PRECUPANU, *Convexity and Optimization in Banach Spaces*, Editura Academiei, Bucureşti, 1978.

[3] A. BEN-ISRAEL and T.N.E. GREVILLE, *Generalized Inverses: Theory and Applications*, first ed., Wiley, New York, 1974, second ed., Springer-Verlag, NewYork, 2003.

[4] J. CAO and Y. XUE, *Perturbation analysis of bounded homogeneous generalized inverses on Banach spaces*, arXiv preprint, http://arxiv.org/abs/1302.3965.

[5] J. CAO and Y. XUE, Perturbation analysis of the Moore–Penrose metric generalized inverses in $L^p(\Omega, \mu)$, *Numer. Funct. Anal. Optim.*, submitted.

[6] G. CHEN and Y. XUE, Perturbation analysis for the operator equation $Tx = b$ in Banach spaces, *J. Math. Anal. Appl.*, **212** (1997), 107-125.

[7] G. CHEN, Y. WEI and Y. XUE, Perturbation analysis of the least square solution in Hilbert spaces, *Linear Algebra Appl.*, **244** (1996), 69-80.

[8] G. CHEN, Y. WEI and Y. XUE, The generalized condition numbers of bounded linear operators in Banach spaces, *J. Aust. Math. Soc.*, **76** (2004), 281-290.

[9] T. CHEN, H. HUDZIK, W. KOWALEWSKI et al., Approximative compactness and continuity of metric projector in Banach spaces and applications, *Sci. China Ser. A*, **51** (2008), 293-303.

[10] J. DIESTEL, *Geometry of Banach Spaces-Selected Topics*, Lecture Notes in Mathematics, Springer-Verlag, New York, 1975.

[11] J. DING, On the expression of generalized inverses of perturbed bounded linear operators, *Missouri J. Math. Sci.*, **15**, *1* (2003), 40-47.

[12] J. DING, Lower and upper bounds in the perturbation of general linear algebraic equations, *Appl. Math. Lett.*, **17** (2004), 55-58.

[13] J. DING and L. HUANG, On the perturbation of the least squares solutions in Hilbert spaces, *Linear Algebra Appl.*, **212** (1994), 487-500.

[14] J. DING and L. HUANG, A generalization of a classic theorem in the perturbation of linear operators, *J. Math. Anal. Appl.*, **239** (1999), 118-123.

[15] H. HUDZIK, Y. WANG and W. ZHENG, Criteria for the metric generalized inverse and its selection in Banach spaces, *Set-Valued Anal.*, **16** (2008), 51-65.

[16] P. LIU and Y. WANG, The best generalized inverse of the linear operator in normed linear space, *Linear Algebra Appl.*, **420** (2007), 9-19.

[17] H. MA, *Construction of some generalized inverses of operators between Banach spaces and their selections, perturbations and applications*, PhD Thesis, 2012.

[18] H. MA, H. HUDZIK and Y. WANG, Continuous homogeneous selections of set-valued metric generalized inverses of linear operators in Banach spaces, *Acta Math. Sin. (Engl. Ser.)*, **28** (2012), 45-56.

[19] M.Z. NASHED and G.F. VOTRUBA, A unified approach to generalized inverses of linear operators. II. Extremal and proximal properties, *Bull. Amer. Math. Soc.*, **80** (1974), 831-835.

[20] M.Z. NASHED and G.F. VOTRUBA, *Generalized Inverses and Applications*, Academic Press, New York, 1976.

[21] R. NI, Moore–Penrose metric generalized inverses of linear operators in arbitrary Banach spaces (Chinese), *Acta Math. Sinica (Chin. Ser.)*, **49**, *6* (2006), 1247-1252.

[22] I. SINGER, *The Theory of Best Approximation and Functional Analysis*, Springer-Verlag, New York, 1970.

[23] D. STĂNICĂ, The stability of some operator equations in Hilbert spaces, *Ann. Univ. Buchar. Math. Ser.*, **1 (LIX)** (2010), 155-164.

[24] Y. WANG, *Generalized Inverse of Operator in Banach Spaces and Applications*, Science Press, Beijing, 2005.

[25] Y. WANG and J. LIU, Metric generalized inverse for linear manifolds and extremal solutions of linear inclusion in Banach spaces, *J. Math. Anal. Appl.*, **302** (2005), 360-371.

[26] Y. WANG and SH. PAN, An approximation problem of the finite rank operator in Banach spaces, *Sci. China Ser. A*, **46**, *2* (2003), 245-250.

[27] H. WANG and Y. WANG, Metric generalized inverse of linear operator in Banach space, *Chinese Ann. Math. Ser. B*, **24**, *4* (2003), 509-520.

[28] Y. XUE, *Stable Perturbations of Operators and Related Topics*, World Scientific, 2012.

[29] Y. XUE and G. CHEN, Some equivalent conditions of stable perturbation of operators in Hilbert spaces, *Appl. Math. Comput.*, **147** (2004), 765-772.

[30] Y. ZHAO and Y. WANG, The structure of the set of extremal solutions of ill–posed operator equation $Tx = y$ with codim$R(T) = 1$, *Numer. Funct. Anal. Optim.*, **30**, *7-8* (2009), 870-880.

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