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Good $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$-gradings on matrix algebras

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Abstract - For any prime number p, we compute the number of isomorphism types of good $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$-gradings on an upper block triangular matrix algebra and, as a special case, on the full matrix algebra.

Key words and phrases : matrix algebra, graded algebra, upper block triangular matrix.

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1. Introduction and preliminaries

In this note we consider good gradings on the upper block triangular matrix algebra

$$A = \begin{pmatrix} M_{m_1}(K) & M_{m_1,m_2}(K) & \dots & M_{m_1,m_r}(K) \\ 0 & M_{m_2}(K) & \dots & M_{m_2,m_r}(K) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{m_r}(K) \end{pmatrix} \subseteq M_n(K)$$

where K is a field, $r \geq 1$, $m_1, ..., m_r \geq 1$, and $n = m_1 + ... + m_r$. Obviously, for $r = 1$ we obtain the full matrix algebra $M_n(K)$.

These upper block triangular matrix algebras are important because they provide interesting examples and counterexamples in ring theory. They are also used in the study of numerical invariants of PI-algebras.

A grading on A by a group G is a decomposition of A as direct sum $A = \oplus_{g \in G} A_g$ of K-subspaces such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. A grading on A is called a good grading if all the matrix units e_{ij} in A are homogenous elements.

The aim of this paper is to find the number of the isomorphism types of good gradings on A by a finite abelian group G of rank three and order p^4, where p is a prime number. Adding this to the results presented in [2], we obtain the combinatorial description of good gradings on A by a finite abelian p-group G of order $\leq p^4$.

Throughout the paper p is a prime number and $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$.

2. Describing the lattice of subgroups of the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$

As we can see in [4], the total number of subgroups of G is $4p^2 + 3p + 5$. An explicit description of all the subgroups of G is given below:

- There is one subgroup of order p^0 of G, namely $\langle(0,0,0)\rangle$.
- There are $p^2 + p + 1$ subgroups of order p in G:

$$\langle(0,0,1)\rangle$$

$$\langle(0,1,0)\rangle$$

$$\langle(0,1,1)\rangle, \langle(0,1,2)\rangle, \ldots, \langle(0,1,p-1)\rangle$$

$$\langle(p,0,0)\rangle, \langle(p,0,1)\rangle, \ldots, \langle(p,0,p-1)\rangle$$

$$\langle(p,1,0)\rangle, \langle(p,1,1)\rangle, \ldots, \langle(p,1,p-1)\rangle$$

$$\vdots$$

$$\langle(p,p-1,0)\rangle, \langle(p,p-1,1)\rangle, \ldots, \langle(p,p-1,p-1)\rangle$$

- There are $2p^2 + p + 1$ subgroups of order p^2 in G. The set of all such subgroups is $\mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{p-1} \cup \mathcal{B} \cup \mathcal{C}_0 \cup \ldots \cup \mathcal{C}_{p-1} \cup \mathcal{D}$, where

$$\mathcal{A}_0 = \{\langle(p,0,0),(p,1,0)\rangle, \langle(p,0,0),(p,1,1)\rangle, \ldots$$
$$\ldots, \langle(p,0,0),(p,1,p-1)\rangle\}$$
$$\mathcal{A}_1 = \{\langle(p,0,1),(p,1,0)\rangle, \langle(p,0,1),(p,1,1)\rangle, \ldots$$
$$\ldots, \langle(p,0,1),(p,1,p-1)\rangle\}$$
$$\vdots$$
$$\mathcal{A}_{p-1} = \{\langle(p,0,p-1),(p,1,0)\rangle, \langle(p,0,p-1),(p,1,1)\rangle, \ldots$$
$$\ldots, \langle(p,0,p-1),(p,1,p-1)\rangle\}$$
$$\mathcal{B} = \{\langle(p,0,0),(p,0,1)\rangle, \langle(p,1,0),(p,1,1)\rangle, \ldots$$
$$\ldots, \langle(p,p-1,0),(p,p-1,1)\rangle\}$$
$$\mathcal{C}_0 = \{\langle(1,0,0)\rangle, \langle(1,0,1)\rangle, \ldots, \langle(1,0,p-1)\rangle\}$$
$$\mathcal{C}_1 = \{\langle(1,1,0)\rangle, \langle(1,1,1)\rangle, \ldots, \langle(1,1,p-1)\rangle\}$$
$$\vdots$$
$$\mathcal{C}_{p-1} = \{\langle(1,p-1,0)\rangle, \langle(1,p-1,1)\rangle, \ldots, \langle(1,p-1,p-1)\rangle\}$$
$$\mathcal{D} = \{\langle(0,0,1),(0,1,0)\rangle\}$$

- There are $p^2 + p + 1$ subgroups of order p^3 in G. The set of all such subgroups is $\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_p \cup \mathcal{L}$, where

$$\mathcal{L}_0 = \{\langle (p,0,1),(1,0,0)\rangle, \langle (p,0,1),(1,1,0)\rangle, \ldots$$
$$\ldots, \langle (p,0,1),(1,p-1,0)\rangle\}$$
$$\mathcal{L}_1 = \{\langle (p,1,0),(1,0,0)\rangle, \langle (p,1,0),(1,0,1)\rangle, \ldots$$
$$\ldots, \langle (p,1,0),(1,0,p-1)\rangle\}$$
$$\mathcal{L}_2 = \{\langle (p,1,1),(1,0,0)\rangle, \langle (p,1,1),(1,0,1)\rangle, \ldots$$
$$\ldots, \langle (p,1,1),(1,0,p-1)\rangle\}$$
$$\vdots$$
$$\mathcal{L}_p = \{\langle (p,1,p-1),(1,0,0)\rangle, \langle (p,1,p-1),(1,0,1)\rangle, \ldots$$
$$\ldots, \langle (p,1,p-1),(1,0,p-1)\rangle\}$$
$$\mathcal{L} = \{\langle (p,0,0),(p,0,1),(p,1,0)\rangle\}$$

- There is only one subgroup of order p^4 of G: the group G itself.

Definition 2.1. *Let H and K be two subgroups of G. We say that the subgroup K lies above the subgroup H if we have the strict inclusion $H \subset K$.*

For any $0 \leq i \leq 3$, let $\mathcal{S}_i = \left\{H \leq G : |H| = p^i\right\}$. Given the above description of the subgroups of G, we notice that:

$$\star \ \mathcal{S}_2 = \mathcal{S}_{2,1} \cup \mathcal{S}_{2,2} \cup \mathcal{S}_{2,3}, \text{ where}$$

$$\mathcal{S}_{2,1} = \{\langle (0,0,1),(0,1,0)\rangle\} \cup (\mathcal{B} - \{\langle (p,0,0),(p,0,1)\rangle\}) \cup$$
$$\cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_{p-1}$$
$$\mathcal{S}_{2,2} = \{\langle (p,0,0),(p,0,1)\rangle\} \cup \mathcal{A}_0$$
$$\mathcal{S}_{2,3} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{p-1}$$

$$\star \ \mathcal{S}_1 = \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2} \cup \mathcal{S}_{1,3} \cup \mathcal{S}_{1,4} \cup \mathcal{S}_{1,5} = \mathcal{S}_1^{'} \cup \mathcal{S}_1^{''}, \text{ where}$$

$$\mathcal{S}_{1,1} = \{\langle (0,0,1)\rangle\}$$
$$\mathcal{S}_{1,2} = \{\langle (0,1,0)\rangle, \langle (0,1,1)\rangle, \ldots, \langle (0,1,p-1)\rangle\}$$
$$\mathcal{S}_{1,3} = \{\langle (p,0,0)\rangle\}$$
$$\mathcal{S}_{1,4} = \{\langle (p,0,1)\rangle, \langle (p,0,2)\rangle, \ldots, \langle (p,0,p-1)\rangle\}$$
$$\mathcal{S}_{1,5} = \{\langle (p,\alpha,0)\rangle, \langle (p,\alpha,1)\rangle, \ldots, \langle (p,\alpha,p-1)\rangle :$$
$$\alpha \in \overline{1,p-1}\}$$
$$\mathcal{S}_1^{'} = \mathcal{S}_{1,3} \text{ and } \mathcal{S}_1^{''} = \mathcal{S}_1 - \mathcal{S}_1^{'}.$$

With these notations, the structure of the subgroup lattice of the group G is easely seen to be as described by the next proposition.

Proposition 2.1. 1. *Above any subgroup in* $\mathcal{S}_3$ *lies the subgroup* G.

2. a) *Above any subgroup in* $\mathcal{S}_{2,1}$ *lie one subgroup of order* p^3 *(namely* $\langle (p,0,0),(p,0,1),(p,1,0)\rangle$*) and* G.

b) *Above any subgroup in* $\mathcal{S}_{2,2}$ *lie* $p+1$ *subgroups of order* p^3 *and* G. *More precisely, above* $\langle (p,0,0),(p,0,1)\rangle$ *lie* $\langle (p,0,0),(p,0,1),(p,1,0)\rangle$ *and all the subgroups from* $\mathcal{L}_0$ *and above any subgroup* $\langle (p,0,0),(p,1,\alpha)\rangle$ *lie* $\langle (p,0,0),(p,0,1),(p,1,0)\rangle$ *and all the* p *subgroups from* $\mathcal{L}_{\alpha+1}$ *where* $0 \le \alpha \le p-1$.

c) *Above any subgroup in* $\mathcal{S}_{2,3}$ *lie* $p+1$ *subgroups of order* p^3 *(one subgroup from each of the sets* $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_p$*) and* G.

3. a) *Above the subgroup in* $\mathcal{S}_{1,1}$ *lie* $p+1$ *subgroups of order* p^2 *(namely* $\langle (0,0,1),(0,1,0)\rangle$ *and all the* p *subgroups from* $\mathcal{B}$*),* $p+1$ *subgroups of order* p^3 *(*$\langle (p,0,0),(p,0,1),(p,1,0)\rangle$ *and all the* p *subgroups from* $\mathcal{L}_0$*) and* G.

b) *Above any subgroup in* $\mathcal{S}_{1,2}$ *lie* $p+1$ *subgroups of order* p^2 *(namely* $\langle (0,0,1),(0,1,0)\rangle$ *and one subgroup from each set* $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{p-1}$*),* $p+1$ *subgroups of order* p^3 *(*$\langle (p,0,0),(p,0,1),(p,1,0)\rangle$ *and all the* p *subgroups from a* $\mathcal{L}_j, j \ne 0$ *i.e. above the subgroup* $\langle (0,1,\alpha)\rangle$ *we find the* p *subgroups from* $\mathcal{L}_{\alpha+1}$ *for any* $0 \le \alpha \le p-1$*) and* G.

c) *Above the subgroup in* $\mathcal{S}_{1,3}$ *lie* p^2+p+1 *subgroups of order* p^2 *(namely* $\langle (p,0,0),(p,0,1)\rangle$*, all the* p *subgroups from* $\mathcal{A}_0$ *and all the* p^2 *subgroups from* $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{p-1}$*), all the* p^2+p+1 *subgroups of order* p^3 *and* G.

d) *Above any subgroup in* $\mathcal{S}_{1,4}$ *lie* $p+1$ *subgroups of order* p^2 *(namely* $\langle (p,0,0),(p,0,1)\rangle$ *and the* p *subgroups from an* $\mathcal{A}_j, j \ne 0$ *i.e. above the subgroup* $\langle (p,0,\alpha)\rangle$ *we find all the subgroups from* $\mathcal{A}_\alpha$*, for any* $1 \le \alpha \le p-1$*),* $p+1$ *subgroups of order* p^3 *(*$\langle (p,0,0),(p,0,1),(p,1,0)\rangle$ *and all the* p *subgroups from* $\mathcal{L}_0$*) and* G.

e) *Above any subgroup in* $\mathcal{S}_{1,5}$ *lie* $p+1$ *subgroups of order* p^2 *(namely one subgroup from the set* $\mathcal{B} - \{\langle (p,0,0),(p,0,1)\rangle\}$ *and one subgroup from each of the sets* $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{p-1}$*),* $p+1$ *subgroups of order* p^3 *(namely* $\langle (p,0,0),(p,0,1),(p,1,0)\rangle$ *and all the* p *subgroups from a* $\mathcal{L}_j$ *which are lying above the subgroup from* $\mathcal{A}_0$*) and* G.

We conclude that above the subgroup in $\mathcal{S}_1^{'}$ *we find* p^2+p+1 *subgroups of order* p^2*,* p^2+p+1 *subgroups of order* p^3 *and one subgroup of order* p^4 *and above any subgroup from* $\mathcal{S}_1^{''}$ *we find* $p+1$ *subgroups of order* p^2*,* $p+1$ *subgroups of order* p^3 *and one subgroup of order* p^4.

4. *Above the subgroup in* $\mathcal{S}_0$ *lie* p^2+p+1 *subgroups of order* p *(*p^2+p *subgroups in* $\mathcal{S}_1^{''}$ *and one subgroup in* $\mathcal{S}_1^{'}$*),* $2p^2+p+1$ *subgroups of order* p^2 *(*p^2 *subgroups in* $\mathcal{S}_{2,1}$ *and* p^2+p+1 *subgroups in* $\mathcal{S}_{2,2} \cup \mathcal{S}_{2,3}$*), all the* p^2+p+1 *subgroups of order* p^3 *and* G.

3. Combinatorial description of good $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$-gradings on A

Denote by $\mathcal{S}_m$ the symmetric group of the set $\{1, \ldots, m\}$. Then the group $\mathcal{S}_{m_1} \times \ldots \times \mathcal{S}_{m_r}$ acts from the left on the set G^n by

$$(f_1, f_2, \ldots, f_r) \cdot (g_1, \ldots, g_n) = (g_{f_1(1)}, \ldots, g_{f_1(m_1)}, g_{m_1+f_2(1)}, \ldots, g_{m_1+f_2(m_2)},$$

$$\ldots, g_{m_1+\ldots+m_{r-1}+f_r(1)}, \ldots, g_{m_1+\ldots+m_{r-1}+f_r(m_r)})$$

for any $f_1 \in \mathcal{S}_{m_1}, \ldots, f_r \in \mathcal{S}_{m_r}$ and $(g_1, \ldots, g_n) \in G^n$, while G acts on G^n from the right by translations: $(g_1, \ldots, g_n) \cdot \sigma = (g_1\sigma, \ldots, g_n\sigma)$ for any $(g_1, \ldots, g_n) \in G^n$ and $\sigma \in G$. Thus we obtain a left $\mathcal{S}_{m_1} \times \ldots \times \mathcal{S}_{m_r}$, right G-biaction on G^n.

In [1] it was proved that good G-gradings on A are classified by the orbits of this $(S_{m_1} \times ... \times S_{m_r}, G)$-biaction on the set G^n, which means that counting these orbits we obtain the number of isomorphism types of good G-gradings on A.

For any additive group G and any positive integer m let

$$\mathcal{Y}(m, G) = \left\{ (a_{g,m})_{g \in G} : a_{g,m} \in \mathbb{Z}, a_{g,m} \geq 0 \text{ and } \sum_{g \in G} a_{g,m} = m \right\}$$

There is a right action β of G on the set $\mathcal{Y}(m_1, G) \times ... \times \mathcal{Y}(m_r, G)$ given by

$$((a_{g,m_1})_{g \in G}, ..., (a_{g,m_r})_{g \in G}) \cdot h = ((a_{g+h,m_1})_{g \in G}, ..., (a_{g+h,m_r})_{g \in G}).$$

As we can see in [2], there is a bijection between the orbits of the biaction $(S_{m_1} \times ... \times S_{m_r}, G)$ on the set G^n and the orbits of the right action β of G on the set $\mathcal{Y}(m_1, G) \times ... \times \mathcal{Y}(m_r, G)$ given by

$$\varphi : G^n \to \mathcal{Y}(m_1, G) \times ... \times \mathcal{Y}(m_r, G) \text{ defined by } \varphi(z) = (a_1(z), ..., a_r(z))$$

where $z = (z_1, ..., z_r)$ with $z_1 \in G^{m_1}, ..., z_r \in G^{m_r}$ and $a_i(z) \in \mathcal{Y}(m_i, G)$ is given by $a_i(z) = (a_i(z)_g)_{g \in G}$ with $a_i(z)_g =$ the number of appearances of g in z_i, for any $1 \leq i \leq r$.

Since we have this bijection described above, the isomorphism types of good G-gradings on A are in bijection with the orbits of the action β, so all we have to do is to determine the number of orbits of the action β. Thus we have to compute the following sum:

$$\sum_{0 \leq t \leq 4} h_t = \sum_{0 \leq t \leq 4} \frac{1}{p^t} e_t$$

where h_t is the number of orbits of length p^t of the action β and e_t is the number of elements having the orbit of length p^t.

For any subgroup H of G and for any $1 \leq i \leq r$, let $\mathcal{Y}(m_i, G)_H = \{z \in \mathcal{Y}(m_i, G) : \mathrm{Stab}_G(z) = H\}$. Also let $\gamma_{t,i} = |\mathcal{Y}(m_i, G)_H|$. Now we have to be sure that all the $\gamma_{t,i}$ we work with are well defined (i.e. they don't depend on the choice of the subgroup H).

As shown in [3], if the subgroup H has the order p^{4-t} (for $0 \leq t \leq 4$), then:

$$|\mathcal{Y}(m_i, G)_H| = N - \sum_{H<K\leq G} |\mathcal{Y}(m_i, G)_K|$$

where $N = \begin{cases} \binom{\frac{m_i}{p^{4-t}} + p^t - 1}{p^t - 1} & , \ p^{4-t} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases}$.

Let $1 \leq i \leq r$. Looking at our group $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ lattice description, we remark that $\gamma_{t,i}$ is well defined for $t \in \{0, 1, 4\}$, but for $t \in \{2, 3\}$ we have to split $\gamma_{t,i}$ as sum between $\gamma'_{t,i}$ and $\gamma''_{t,i}$, any term of the sum being well defined. Namely, using the above formula and $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$'s lattice description we can compute $\gamma_{t,i}$ as follows:

$$\gamma_{0,i} = |\mathcal{Y}(m_i, G)_G| = \begin{cases} 1 & , \ \text{if } p^{4-0} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases}$$

$$\gamma_{1,i} = |\mathcal{Y}(m_i, G)_{H\in\mathcal{S}_3}| = \begin{cases} \binom{\frac{m_i}{p^{4-1}} + p - 1}{p - 1} & , \ p^{4-1} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases} - \gamma_{0,i}$$

$$\gamma'_{2,i} = |\mathcal{Y}(m_i, G)_{H\in\mathcal{S}_{2,1}}| = \begin{cases} \binom{\frac{m_i}{p^{4-2}} + p^2 - 1}{p^2 - 1} & , \ p^{4-2} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases} - \gamma_{1,i} - \gamma_{0,i}$$

$$\gamma''_{2,i} = |\mathcal{Y}(m_i, G)_{H\in\mathcal{S}_{2,2}\cup\mathcal{S}_{2,3}}| = \begin{cases} \binom{\frac{m_i}{p^{4-2}} + p^2 - 1}{p^2 - 1} & , \ p^{4-2} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases} - \\ -(p+1)\gamma_{1,i} - \gamma_{0,i}$$

$$\gamma'_{3,i} = |\mathcal{Y}(m_i, G)_{H\in\mathcal{S}'_1}| = \begin{cases} \binom{\frac{m_i}{p^{4-3}} + p^3 - 1}{p^3 - 1} & , \ p^{4-3} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases} - \\ -(p^2 + p + 1)\gamma''_{2,i} - (p^2 + p + 1)\gamma_{1,i} - \gamma_{0,i}$$

$$\gamma''_{3,i} = |\mathcal{Y}(m_i, G)_{H\in\mathcal{S}''_1}| = \begin{cases} \binom{\frac{m_i}{p^{4-3}} + p^3 - 1}{p^3 - 1} & , \ p^{4-3} \,|\, m_i \\ 0 & , \ \text{otherwise} \end{cases} - p\gamma'_{2,i} - \\ -\gamma''_{2,i} - (p+1)\gamma_{1,i} - \gamma_{0,i}$$

$$\gamma_{4,i} = |\mathcal{Y}(m_i, G)_{\langle(0,0,0)\rangle}| = \binom{m_i + p^4 - 1}{p^4 - 1} - \gamma'_{3,i} - (p^2 + p)\gamma''_{3,i} - p^2\gamma'_{2,i} - \\ -(p^2 + p + 1)\gamma''_{2,i} - (p^2 + p + 1)\gamma_{1,i} - \gamma_{0,i}.$$

For any $t\in\{0,1,4\}$ let $\gamma_t=\prod_{i\in\overline{1,r}}\gamma_{t,i}$. For any $t\in\{2,3\}$ let $\gamma'_t=\prod_{i\in\overline{1,r}}\gamma'_{t,i}$ and $\gamma''_t=\prod_{i\in\overline{1,r}}\gamma''_{t,i}$.

Also, let $s'_{4,4-2}=|\mathcal{S}_{2,1}|=p^2$, $s''_{4,4-2}=|\mathcal{S}_{2,2}\cup\mathcal{S}_{2,3}|=p^2+p+1$, $s'_{4,4-3}=|\mathcal{S}'_1|=1$ and $s''_{4,4-3}=|\mathcal{S}''_1|=p^2+p$ and let $s_{4,4-t}$ be the number of subgroups of G having the order p^{4-t} for $0\le t\le 4$.

For any $t\in\{0,1,4\}$ we obtain that there are $\gamma_t s_{4,4-t}$ elements having orbits of length p^t. Also for any $t\in\{2,3\}$ there are $\gamma'_t s'_{4,4-t}+\gamma''_t s''_{4,4-t}$ elements having the orbit of length p^t.

In view of the introduction of this section we obtain:

Theorem 3.1. *The number of isomorphism types of good $\mathbb{Z}_{p^2}\times\mathbb{Z}_p\times\mathbb{Z}_p$-gradings on the algebra A is:*

$$\sum_{t\in\{0,1,4\}}\frac{1}{p^t}\gamma_t s_{4,4-t}+\sum_{t\in\{2,3\}}\frac{1}{p^t}[\gamma'_t s'_{4,4-t}+\gamma''_t s''_{4,4-t}]$$

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