Orthogonal polynomials solutions of linear differential equation of order N. What about N?

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Dedicated, with friendship and gratefullness, to Professor Jean Mawhin
for his 70th birthday

Abstract - The aim of this short survey is to emphasize the link between the order N of the differential equation satisfied by a family of orthogonal polynomials and the scalar product generating orthogonality, using many examples. Classical, Semi-classical and Laguerre-Hahn families for which $N = 2$ or $N = 4$ are described. Some typical Sobolev families generating several value of N are also investigated. The examples choosen are presented without proof with elementary mathematical tools.

Key words and phrases : scalar product, Sturm-Liouville, Sobolev orthogonality.

Mathematics Subject Classification (2010) : 33C45, 42C05.

1. Introduction (Basic tools)

A simple, old and seminal way to introduce Orthogonal Polynomials (OP) $p_n(x)$ of degree n is to start with the well known least square approximation problem (see [5]).

Let us consider a real function $f(x)$ continuous in the interval $[a, b]$. One way to approximate this function by polynomials $p_i(x)$, $i = 0, \ldots n$ as $f(x) \simeq \sum_{i=0}^{n} p_i(x)$ is to consider the integral:

$$I(c_i) = \int_a^b \rho(x)[f(x) - \sum_{i=0}^{n} c_i p_i(x)]^2 dx, \qquad (1.1)$$

with $\rho(x)$ integrable and positive in $[a, b]$, and to minimize this integral by the system of the $n + 1$ relations $\dfrac{\partial I(c_i)}{\partial x} = 0$. Of course c_i are easily computed if we choose $p_i(x)$ satisfying $\int_a^b \rho(x) p_j(x)\, p_i(x) dx = 0$ with $i \neq j$

giving immediately

$$c_i = \frac{\int_a^b \rho(x)f(x)p_i(x)dx}{d_i^2} \ , \ d_i^2 = \int_a^b \rho(x)p_i^2(x)dx. \tag{1.2}$$

So the family $\{p_n\}$ of polynomials $p_n(x)$ belongs to a real infinite dimensional vector space $\mathcal{P}$, here Hilbert space, with a scalar product $\langle p_i, p_j \rangle = \int_a^b \rho(x)p_i(x)p_j(x)dx$. By definition this family gives an orthogonal basis in $\mathcal{P}$ and therefore any real polynomial $q_r(x)$ can be expanded uniquely as $q_r(x) = \sum_{i=0}^r c_i p_i(x)$. When $\rho(x)$, called the weight, is equal to 1, the corresponding family is the well known Legendre family.

Two current normalizations are used for the representation of the polynomial $p_n(x) = \sum_{i=0}^n c_i x^i$: the choice $c_n = 1$ (monic OP) and $d_i^2 = 1$ (orthonormal OP).

The fundamental property of each OP $(p_n(x) \equiv p_n)$ is the so-called 3-terms recurrence relation (RR), direct consequence of the orthogonality :

$$xp_n = \alpha_n p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}, \qquad n \geq 1. \tag{1.3}$$

This important relation characterizes the orthogonality from a theorem of Favard (see [7]) proving that for any family of polynomial $q_n(x)$ satisfying (1.3) with $\alpha_n \gamma_n > 0$, there exists a probability measure μ of support I generalizing the weight $\rho(x) = \rho$ $(d\mu \equiv \rho dx)$.

The RR (1.3) is also satisfied by an other family of OP called associate of p_n (see [17]) and defined by :

$$p_{n-1}^{(1)}(x) = \frac{1}{\mu_0} \int_a^b \frac{p_n(x) - p_n(s)}{x - s}\rho(s)ds, \ \ \mu_0 = \int_a^b \rho(s)ds. \tag{1.4}$$

The RR (1.3) for $p_n^{(1)}$ is therefore the same as in (1.3) with a shift of 1 in the coefficients, and, by recurrence, $p_n^{(k)}$ for any integer k, are therefore also OP families from the Favard theorem.

There exists two general representations of any orthogonal family. The Matrix form, rewritting the RR in a matrix form

$$x\vec{P}_n = [J]\vec{P}_n \text{ with } \vec{P}_n = [p_0, p_1, \ldots, p_n \ldots]^t \tag{1.5}$$

and [J], called Jacobi matrix (see [24]), tridiagonal infinite matrix with β_n in the diagonal and α_n, γ_n in the 2 subdiagonals in different way depending of the normalisation.

The determinental form of size $n+1$ is build from n lines of moment $\mu_k = \int_I \rho(x)x^k d\mu$, the line j being: $\mu_j, \ldots, \mu_{j+n}(j = 0, n)$ and last line being: x^r, $r = 0$ to n (determinant of type Hankel).

Both representations, useful for the theory, are not appropriated to obtain properties of peculiar families. It is therefore interesting to represent p_n, if possible, as solutions of differential equations. Difference equations could also appear when the orthogonality is defined by a discrete scalar product

$$\langle p_n, p_m \rangle = \sum_{k=0}^{K} \rho(k)p_n(k)p_m(k).$$

Generating function as $G(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n$ is an alternative way to introduce the family $p_n(x)$ as done already by Lagrange for the Legendre polynomials $P_n(x)$ (see [3]):

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{1.6}$$

2. Linear differential equations for orthogonal polynomials

i) *Classical OP ($N = 2$)*

Laplace proved already (see [4]) that the polynomials $P_n(x)$ in (1.6) satisfy the second order differential equation (ODE):

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0, \tag{2.1}$$

or in the Sturm-Liouville form:

$$\left[(1-x)^2 P_n'\right]' = \lambda P_n \quad , \quad \lambda = -n(n+1). \tag{2.2}$$

This peculiar hypergeometric equation for orthogonal OP of weight 1 suggests to build the following equation introducing a weight $\rho(x)$

$$\left[\sigma(x)\rho(x)y'\right]' = \lambda\rho(x)y(x),$$

or

$$\sigma\rho y'' + \tau\rho y' = \lambda\rho(x)y(x) \tag{2.3}$$

with

$$(\sigma\rho)' = \tau\rho. \tag{2.4}$$

Equation (2.4), called Pearson equation with $\sigma(x)$ of degree 2, 1 or 0, and $\tau(x)$ of degree 1, generates therefore hypergeometric equation (and also confluent equations) giving families of OP as Jacobi (J), Laguerre (L), Hermite (H) called classical OP with weight:

$$\rho_J = (1-x)^\alpha(1+x)^\beta, \rho_L = e^{-x}x^\alpha, \rho_H = e^{-x^2} \tag{2.5}$$

for $\alpha, \beta > -1$ and σ_J degree 2, σ_L degree 1, σ_H constant and τ of degree 1. The associate family $P_{n-1}^{(1)}(x)$ of these classical OP are solutions [19] of ODE of order 4 ($N = 4$) which can be factorized as ($D = \frac{d}{dx}$):

$$N_n \; L_n^* \left[P_{n-1}^{(1)}(x) \right] = 0 \qquad (2.6)$$

with

$$\begin{aligned} L_n^* &= \sigma D^2 + (2\sigma' - \tau)D + \lambda_n + \sigma'' - \tau' \\ N_n &= \sigma D^2 + (\sigma' + \tau)D + \lambda_n + \tau'. \end{aligned}$$

ii) *Semi-classical OP* ($N = 2$)

From a weight $\rho(x)$ solution of a first order ODE with polynomial coefficients (2.4), it is natural to consider situations in which $\sigma(x)$ and $\tau(x)$ are polynomials of arbitrary degree. These polynomials, more or less rediscovered at the first symposium on OP at Bar-le-Duc in 1984 (see [2]) were already created by Laguerre (see [10]) and explored by Shohat (see [23]). We shortly present the ODE for $y = P_n(x)$, called also Laguerre-Perron using the Laguerre's notations.

Writing the Pearson's equation (2.4) for the weight $\rho(x)$ as:

$$\frac{\rho'}{\rho} = \frac{2V(x)}{W(x)} \qquad (2.7)$$

or generating an equation for the Stieltjes function $S(x)$:

$$S(x) = \int_I \frac{\rho(t)dt}{x - t} = \sum_n \mu_n x^n, \qquad (2.8)$$

$$W(x)S'(x) = 2V(x)S(x) + U(x) \quad (U(x) \text{ Polynomial}), \qquad (2.9)$$

$y(x)$ satisfies:

$$W(x)\theta_n y'' + \left[(2V + W')\theta_n - W\theta_n' \right] y' + k_n y_n = 0, \qquad (2.10)$$

with $\theta_n = \theta_n(x)$ and $k_n = k_n(x)$ polynomials of degree independent of n.

The semi-classical class of weights is very large and as extension of the classical class contains for example weights $\bar{\rho} = \pi\rho$, $\pi = \pi(x)$ rational functions (see [18]), $\rho^* = \rho + \sum_{k=1}^{K} \lambda_k \delta(x - x_k)$ with $\delta(x - x_k)$ Dirac distribution ($\lambda_k \geq 0$), called 'classical type OP' (see [21]), $\check{\rho} = \rho(x)H(x - c)$ (see [22]), $H(x - c)$, Heaviside 'cutting' functions $H(x - c) = 0$ $x < c$, $H(x - c) = 1$, $x \geq c$ etc.

Of course the recurrence coefficients in the RR (1.3) and the polynomials θ_n and k_n in (2.10) are relatively difficult to compute explicitly even if $\rho(x)$

is classical.

iii) *Laguerre-Hahn families* ($N = 4$)

The semi-classical OP also generates first and r^{th} associate polynomials as in (1.4) and also satisfy one ODE of order 4 for $P_n^{(r)}$ with

$$P_{n-r}^{(r)}(x) = \text{associate of } P_{n-r+1}^{(r-1)}(x) \quad r = 1, 2, \ldots$$

The genial hint, proposed by Magnus (see [13]), uses the continued fraction of the Stieltjes function as:

$$S(x) = \gamma_0 \Big/_{x-\beta_0} -\gamma_1 \Big/_{x-\beta_1} - \ldots \tag{2.11}$$

linking $S^1(x)$, Stieltjes function of $P_{n-1}^{(1)}(x)$ by:

$$S(x) = \frac{\gamma_0}{x - \beta_0 - S^1(x)}, \tag{2.12}$$

from the expansion of (2.11).

Now, $S(x)$ being solution of a first order linear ODE, $S^1(x)$, and any $S^r(x)$, are solutions of a Riccati equation of type

$$AS' = BS^2 + CS + D, \quad S = S^1, \ldots, S^r, \tag{2.13}$$

with $A(x), B(x), C(x), D(x)$, polynomials. Magnus proved in [13] the existence of a 4th order ODE for the rth associate families, and J. Dzoumba (see [6]), in a thesis in Paris VI with P. Maroni, not published, obtained the 4th order ODE for OP built from any Stieltjes function solution of a Riccati equation. These results are in accordance with a theorem of Hahn in [8] saying that the order of any OP families solution of an ODE of any order can be reduced to an ODE of order 2 or 4.

iv) *Sturm-Liouville equations (N even)*

ODE for OP can be also Sturm-Liouville equations generating orthogonal solutions, not always polynomials, and of type:

$$L[y(x)] = \sum_{i=1}^{2r} \sum_{j=0}^{i} (\ell_{ij} x^j) y^{(i)}(x) = \lambda_n y(x) \tag{2.14}$$

in which the degree n of polynomial solution $y = P_n(x)$ does not appear in the operator L. This property is already lost for the ODE of associate classical as in (2.6) and also for semi-classical OP as shown in the equation (2.10). H.L. Krall and A.M. Krall investigate in great details the operators L in [9] generating families of OP and give some exemples involving operators

of order 4 and 6. H.L. Krall also showed that the order of L is always even $(N = 2r)$.

If Sturm-Liouville equations are easier to manipulate than equations of Laguerre-Perron with polynomials θ_n and K_n often unknown, the weight of the OP $P_n(x)$ are difficult to obtain because the moments μ_n satisfy recurrence relations involving the coefficients ℓ_{ij}.

L.L. Littlejohn and A.M. Krall presented in [12] the following equation $(N = 4)$:

$$L_4[y(x)] = x^2 y'''' - (2x^2 - 4x)y''' + (x^2 - [2R+6]x)y''$$
$$+[2R+2]x - 2Ry' = \lambda_n y. \tag{2.15}$$

with $\lambda_n = n[2R + n - 1]$. The polynomial solution $P_n(x)$ are OP of weight $\rho(x)$

$$P(x) = \frac{1}{R}\delta(x) + e^{-x}H(x). \tag{2.16}$$

The recurrence relation for the moments can be solved in this case and gives:

$$\mu_0 = \frac{R+1}{R} \quad , \quad \mu_m = m! \qquad m = 1, 2, \ldots$$

On the other side, these polynomials $P_n(x)$ are also solutions of the Laguerre-Perron equation $(N = 2)$

$$q_2(x, n)y'' + q_1(x, n)y' + q_0(x, n)y = 0 \tag{2.17}$$

$$q_2 = (R^2 + R + \lambda_n)x^2 - Rx,$$
$$q_1 = -(R^2 + B + \lambda_n)x^2 + (R^2 + 2R + \lambda_n)x - 2R$$
$$q_0 = (2R\lambda_n + 22\lambda_n - \chi_n)x - \lambda_n,$$

with $\chi_n = (3R^2 + 45R + 42)n + 18n(n-1) - n(n-1)(n-2)$.

3. Sobolev orthogonality. Very short survey $(N =?)$

i) *Introduction*

The first appearance of Orthogonal Polynomials with Sobolev inner product is generally credited to Lewis (see [19]), who considered the generalization of a least square approximation problem.

In order to approximate by polynomial $q_n(x)$ of degree n a function $f(x)$, taking into account all derivatives of $f(x)$ up to the order r, the following L^2 minimization problem appears in a natural way:

$$I(c_i) = \int_{\mathbb{R}} dx \sum_{k=0}^{r} \rho_k(x) \left[\sum_{i=0}^{n} c_i q_i^{(k)}(x) - f^{(k)}(x)\right]^2 \tag{3.1}$$

where $\rho_k(x)$ are non negative weight functions. The condition $\frac{\partial I}{\partial c_i} = 0$, $i = 0 \ldots n$, privileges polynomials $q_i(x)$ which are orthogonal with respect to the Sobolev (diagonal) inner product:

$$\langle q_i, q_j \rangle_s = \int_{\mathbb{R}} dx \sum_{k=0}^{r} \rho_k(s) q_i^{(k)}(x) q_j^{(k)}(x) \qquad (3.2)$$

The presence of derivative in the scalar product changes drastically the standard properties of polynomials orthogonal with respect to a positive measure: the 3 terms recurrence relation (1.3) is lost, the zeroes are not always inside the supports of the measures, differential equations satisfied by these polynomials (when existing) are not always of even order.

ii) *Brenner exemple* $(N = 4)$

In 1972, Brenner (see [1]) investiged polynomials $R_m(x)$, orthogonal with respect to the Sobolev scalar product:

$$\langle R_n, R_m \rangle_s = \int_0^{\infty} e^{-x} \left[R_n(x) R_m(x) + \gamma R_n'(x) R_m'(x) \right] \, dx \ (\gamma > 0) \qquad (3.3)$$

and showed that the polynomials $R_n(x)$ are linked to the Laguerre polynomials $L_n(x)$, [weight e^{-x}, support $(0, \infty)$] by a relation as:

$$\gamma[R_n'' - R_n'] - R_n = A_n L_{n+1}' + B_n L_n' \qquad (3.4)$$

giving recurrence relations for A_n and B_n. He seemed not interested by the ODE of $R_n(x)$, but it is easy to build it, eliminating L_n' and L_{n+1}', using the killing operator $\mathcal{L}(D) = xD^2 + (2 - x)D + nI$ with $D = \frac{d}{dx}$.

iii) *Sobolev type OP* $(N = 2)$ *and general Sobolev*

The 3 term RR (1.3) for 'standard' orthogonality (not Sobolev) is a consequence of the trivial property:

$$\langle xp_i, p_j \rangle = \langle p_i, xp_j \rangle. \qquad (3.5)$$

For Sobolev OP $q_n(x)$, in order to expand a polynomial $F_r(x) q_n(x)$ in a limited sum of q_j from $n + r$ to $n - r$:

$$F_r(x) q_n(x) = \sum_{n-r}^{n+r} c_{m,n} q_m(x), \qquad (3.6)$$

the scalar product must satisfy the symmetry property:

$$\langle F_r q_i, q_j \rangle_s = \langle q_i, F_r q_j \rangle_s. \qquad (3.7)$$

F_r is called 'symmetric function' and is essential in order to create an ODE for the q_n family.

An instructive situation appears with the Sobolev-type scalar product (see [14])

$$\langle q_n, q_m \rangle_s = \int_I dx \rho(x) q_n(x) q_m(x) + \lambda^{-1} q_n^{(r)}(c) q_m^{(r)}(c) \quad (\lambda > 0), \qquad (3.8)$$

where the family $P_n(x)$ is orthogonal with respect to the (standard) scalar product

$$\langle p_n, p_m \rangle = \int_I dx \rho(x) p_n(x) p_m(x) \qquad (3.9)$$

and the hypothesis $\rho(x)$ semi-classical weight.

For all semi-classical family as $p_n(x)$ there exists a differential relation called 'structure relation' (see [16]) between p_n' and p_j:

$$\sigma(x) p_n'(x) = \sum_{i=n-1-t}^{i=n-1+s} \alpha_{i,n} p_i(x) \qquad (3.10)$$

$$\text{with } s = \text{degree of } \sigma(x),$$

$$\text{and } t = \max(s-2, \text{degree of } \tau - 1)$$

The expansions of monic polynomials $p_n(x)$ and $q_n(x)$

$$q_n(x) = p_n(x) + \sum_{j=0}^{n-1} \beta_{n,j} p_j(x) \qquad (3.11)$$

$$(x-c)^{r+1} q_n(x) = \sum_{j=n-r-1}^{n+r+1} \alpha_{n,j} p_j(x), \quad \text{from (3.7) with } F_r = (x-c)^{r+1}$$

$$(3.12)$$

and the structure relation allows, after many transformations to link p_n to q_n in order to obtain a second order ODE for $q_n(x)$. It is interesting to point that N is 2, whatever the degree r of the derivative in the scalar product, because the next exemple involving also r derivatives produces an ODE of order $N = 2r + 2$. This non Sobolev type situation was treated in [15] using the Sobolev inner product (with $\lambda_j > 0$)

$$\langle q_n, q_m \rangle_s = \int_I dx \rho(x) q_n(x) q_m(x) + \int_I \sum_{j=1}^{r} \lambda_j \rho(x) q_n^{(j)}(x) q_m^{(j)}(x) dx. \qquad (3.13)$$

In this case, in order to limit expansions of q_n in q_j ($j = 0$ to n) as in the Sobolev type in iii), the symetric function must be replaced by a symmetric differential operator $\mathcal{F}$ of order $2r$ such that:

$$\langle \mathcal{F} q_n, q_m \rangle_s = \langle q_n, \mathcal{F} q_m \rangle_s \qquad (3.14)$$

One ODE of order $N = 2r + 2$ is build, generalising the Brenner case with $r = 1$.

More general situations involving several weights $\rho_i(x)$ (only one weight in (3.13)) were already investigated for a scalar product, in a matrix form as:

$$(f, g) = \int_{\mathbb{R}} \vec{F}^t [A] \vec{G} \ , \ [A] = [A]^t$$

with $\vec{F}^t = [f(x), f(x), \ldots f^{(r)}(x)] \quad , \quad \vec{G} = [g(x), g'(x) \ldots g^{(r)}(x)]$

and $[A] \equiv A_{ij} \rho_{ij}(x) \quad (i, j \leq r)$

where $\rho_{ij}(x)$ are arbitrary positive weights of support I_{ij}.

Let us conclude by saying that many more other situations, described in hundred of publications, can be investigated with for instance scalar product as [20]

$$\langle f, g \rangle = \int_I \rho(x) dx \mathcal{L} f \cdot \mathcal{L} g \tag{3.15}$$

with $\mathcal{L}$ appropriate linear differential operator which can generate for the corresponding polynomial one ODE of order odd...

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