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Multiple solutions for Lane–Emden equations with mixed nonlinearities

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This paper is dedicated with esteem to Professor Jean Mawhin
on his 70th birthday anniversary

Abstract - We are concerned with a linear perturbation of the Lane-Emden equation with different growths near the origin and at infinity. By means of a version of the Pucci–Serrin three critical points theorem, we establish the existence of at least two nontrivial solutions in the case of large values of the parameter.

Key words and phrases : Lane–Emden equation, eigenvalue problem, multiple solutions.

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1. Introduction

The Lane–Emden equation describes naturally many physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes (see [8]) as a model for long-range Van der Waals interactions in thin films spreading on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [7]), as well as to describe the flow over an impermeable plate (see [6]). Our main purpose in the present paper is to connect a general class of Lane–Emden equations with the eigenvalue problem for the Laplace operator in order to establish a striking multiplicity result for large values of a certain real parameter. The proof of this existence property for the perturbed equation relies on simple variational tools, namely on a version of the celebrated Pucci–Serrin three critical points theorem (see [14]).

2. Preliminary results

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary. We first consider the Lane–Emden problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where p is a positive number. If $0 < p < 1$ then, by the Brezis-Oswald theorem (see [5]), problem (2.1) has a unique positive solution. If $p = 1$, the existence of a solution depends on the spectrum of the Laplace operator in $H_0^1(\Omega)$. Next, if $1 < p < (N+2)/(N-2)$ (or if $1 < p < \infty$ if $N \in \{1, 2\}$) then problem (2.1) has a solution. This is a direct consequence of the mountain pass theorem of Ambrosetti and Rabinowitz (see [1]).

Existence or non-existence results have been estabished both in the critical and in the supercritical case according to various assumptions on Ω. For instance, if Ω is starshaped, a direct application of the Pohozaev identity shows that problem (2.1) does not have any solution in the supercritical case. Related results have been established for various perturbations of problem (2.1). Indeed, let us consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{2.2}$$

with $1 < p < (N + 2)/(N - 2)$. Let λ_1 denote the first eigenvalue of $(-\Delta)$. Then, by the mountain pass theorem, problem (2.2) has a *positive* solution provided that $\lambda < \lambda_1$. If $\lambda \geq \lambda_1$ then problem (2.2) does not have any positive solution, as we observe easily by multiplying with φ_1 and integrating (as usual, φ_1 denotes the first eigenfunction of the Laplace operator). A refined result, whose proof relies on the dual variational method, establishes that for *all* $\lambda \geq \lambda_1$, problem (2.2) has at least one solution.

The second major issue in this paper is the eigenvalue problem associated to the Laplace operator. As stated in Zworski's paper [15], 'eigenvalues describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life [...]. In most situation however, states do not exist for ever, and a more accurate model is given by a decaying state that oscillates at some rate. Eigenvalues are yet another expression of humanity's narcissist desire for immortality.'

Consider the weighted eigenvalue problem the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda a(x)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

Problems of this type have a long history. If Ω is bounded and $a \equiv 1$, problem (2.3) is related to the Riesz-Fredholm theory of self-adjoint and compact operators (see, Theorem VI.11 in Brezis [3]). The case of a non-constant potential a has been first considered in the pioneering papers of Bocher [2], Hess and Kato [9], Minakshisundaran and Pleijel [12] and Pleijel [13]. For instance, Minakshisundaran and Pleijel (see [12], [13]) studied the case where Ω is bounded, $a \in L^\infty(\Omega)$, $a \geq 0$ in Ω and $a > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$.

3. Main result

This paper is strongly inspired by the celebrated work by Brezis and Oswald [5], which is concerned with the qualitative analysis of solutions of sublinear elliptic equations with Dirichlet boundary condition. By contrast, in the present paper we deal with nonlinearities having a mixed behaviour. More precisely, we study the nonlinear problem

$$\begin{cases} -\Delta u = a(x)u + \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where λ is a positive parameter, $a \in L^\infty(\Omega)$, and $a > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\text{there exists } z > 0 \text{ such that } f \geq 0 \text{ and } f \not\equiv 0 \text{ in } [0, z] \tag{3.2}$$

and, for some $0 < q < 1 < p$,

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{|t|^p} < \infty \qquad \text{and} \qquad \sup_{t\in\mathbb{R}} \frac{|f(t)|}{1 + |t|^q} < \infty\,. \tag{3.3}$$

In particular, the function

$$f(t) = \begin{cases} |t|^{p-1}t & \text{if } |t| \leq 1 \\ |t|^{q-1}t & \text{if } |t| > 1 \end{cases}$$

fulfills assumptions (3.2) and (3.3).

Let $F(t) := \int_0^t f(s)ds$. Then assumption (3.3) implies that there is $C > 0$ such that for all $u \in \mathbb{R}$,

$$|F(u)| \leq C|u|^{p+1} \qquad \text{and} \qquad |F(u)| \leq C(1 + |u|^{q+1})\,. \tag{3.4}$$

Let $\lambda_1 > 0$ denote the first eigenvalue of the weighted eigenvalue problem (2.3).

Our main result is the following multiplicity property.

Theorem 3.1. *Let Ω be a bounded domain with smooth boundary and assume that f satisfies hypotheses* (3.2) *and* (3.3). *Assume that $\lambda_1 > 1$. The there exists $\lambda^* \geq 0$ such that problem* (3.1) *has at least two solutions, provided that $\lambda > \lambda^*$.*

Proof of Theorem 3.1. The weak solutions of problem (3.1) are the critical points of the associated energy functional $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ defined by $\mathcal{E} = \mathcal{E}_1 + \lambda\mathcal{E}_2$, where

$$\mathcal{E}_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - a(x)u^2)dx \qquad \text{and} \qquad \mathcal{E}_2(u) = -\int_\Omega F(u)dx.$$

We split the proof into several parts.

CLAIM 1. *The mapping $H_0^1(\Omega) \ni u \longmapsto F(u)$ is not constant.*

Let $\varphi_1 > 0$ be the first eigenfunction of problem (2.3). Returning to hypothesis (3.2), we set $\omega := \{x \in \Omega \,|\, \varphi_1(x) > z\}$. Then ω is open and nonempty (this follows after replacing eventually φ_1 with $t\varphi_1$ for t large enough). Set $v := \psi \circ \varphi_1$, where $\psi(t) := \min\{t, z\}$. Then $v \in H_0^1(\Omega)$, $0 \leq v \leq z$ in Ω, and $v \equiv z$ in ω. Therefore $F(v) \geq 0$ in Ω and $F(v(x)) > 0$ for all $x \in \omega$. This concludes the proof of Claim 1.

CLAIM 2. *$\mathcal{E}$ satisfies the Palais-Smale condition.*

We first observe that $\mathcal{E}_1$ is coercive. Indeed, using the variational characterization of λ_1, we deduce that for all $u \in H_0^1(\Omega)$,

$$\mathcal{E}_1(u) = \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int_\Omega |\nabla u|^2 dx = \alpha\, \|u\|^2,$$

where $\alpha > 0$.

Next, using assumption (3.3) and Sobolev embeddings, we obtain

$$\mathcal{E}_2(u) \geq -C\lambda \int_\Omega |u|^{p+1}dx \geq -C\lambda\, \|u\|^{q+1}.$$

Since $q \in (0,1)$ we conclude that $\mathcal{E}$ is coercive. Thus, by Lemma V.4 in Brezis and Nirenberg [4], we conclude that $\mathcal{E}$ satisfies the Palais-Smale condition.

STEP 3. *Proof of Theorem 3.1 concluded.*

The classical three critical points theorem of Pucci and Serrin in [14] (see also [11, Theorem 1.14]) establishes that if X is a Banach space and $J : X \to \mathbb{R}$ is of class C^1 and satisfies the Palais-Smale condition and has two local minima, then J has at least three distinct critical points. Thus, to conclude the proof, it is enough to argue that $\mathcal{E}$ has at least two local minima provided that λ is large enough. For such a purpose we consider the functionals

$$\Phi(t) := \inf_{\mathcal{E}_2(u)<t} \frac{\inf_{\mathcal{E}_2(v)=t}\mathcal{E}_1(v) - \mathcal{E}_1(u)}{\mathcal{E}_2(u) - t}; \qquad t \in \mathbb{R}$$

and

$$\Psi(t) := \sup_{\mathcal{E}_2(u)>t} \frac{\inf_{\mathcal{E}_2(v)=t} \mathcal{E}_1(v) - \mathcal{E}_1(u)}{\mathcal{E}_2(u) - t}\,; \qquad t \in \mathbb{R}\,.$$

A straightforward computation shows that

$$\limsup_{t\to 0^-} \Phi(t) \le \Phi(0) \qquad \text{and} \qquad \liminf_{t\to 0^-} \Psi(t) = +\infty\,.$$

Thus, if $\lambda > \lambda^* := \Phi(0)$, then $\mathcal{E}$ has at least two distinct nontrivial critical points. This concludes the proof. □

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