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Nonlinear elliptic Dirichlet and no-flux boundary value problems

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To Jean–Happy 70th Birthday

Abstract - This paper is devoted to establishing results for semilinear elliptic boundary value problems where the solvability of problems subject to *No Flux* boundary conditions follows from the solvability of related *Dirichlet* boundary value problems. Throughout it is assumed that the nonlinear perturbation terms are gradient dependent. An extension of *No-Flux* problems is discussed, as well.

Key words and phrases : $H^1(\Omega)$ a priori bounds, compactness, Bernstein-Nagumo growth condition, sub-supersolution theorems, Dirichlet and no-flux boundary conditions.

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1. Introduction

Let

$$f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

be a continuous function, such that for every $M > 0$ there exist constants a and b (depending on M) so that

$$|f(t, u, u')| \leq a + b|u'|^2, \ t \in [0,1], \ |u| \leq M,$$

(f satisfies a *Bernstein - Nagumo* condition), then the periodic boundary value problem

$$-u'' = f(t, u, u'), \ u(0) = u(1), \ u'(0) = u'(1), \tag{1.1}$$

has a solution u such that

$$\alpha(t) \leq u(t) \leq \beta(t), \ 0 \leq t \leq 1,$$

whenever α and β are sub - and supersolutions (upper and lower solutions), with

$$\alpha(t) \leq \beta(t), \ 0 \leq t \leq 1,$$

i.e.

$$-\alpha'' \leq f(t,\alpha,\alpha'),\ \alpha(0)=\alpha(1),\ \alpha'(0)\geq\alpha'(1), \tag{1.2}$$

$$-\beta'' \geq f(t,\beta,\beta'),\ \beta(0)=\beta(1),\ \beta'(0)\leq\beta'(1). \tag{1.3}$$

This is an old result and essentially goes back to Knobloch [11]; several alternate proofs (covering more general cases than those in [11]) were given later, e.g., [22], [23], cf., also [7]. Higher dimensional analogues of the periodic boundary value problem, the *no flux* problem introduced later in [2], and several examples (with f independent of gradient terms) were studied in [1], [13], [14], [17], [18], [24]; see also [6].

Our main purpose in this paper is to establish a new version of sub-supersolution theorems when (1.1) is replaced by the following *no-flux* problem

$$\begin{cases} -\mathrm{div}[a(x,u)\nabla u] &= f(x,u,\nabla u) \quad \text{in } \Omega,\\ u &= \text{constant} \quad \text{on } \partial\Omega,\\ \displaystyle\int_{\partial\Omega} a(x,u)\partial_\nu u d\sigma &= 0, \end{cases} \tag{1.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$. It is to be noted that the constant value of the boundary data is not specified and corresponds to the one-dimensional case $u(0) = u(1)$, whereas the requirement in one dimension that $u'(0) = u'(1)$, corresponds to the boundary integral term, in the case that $a \equiv 1$. The approach to prove our sub-supersolution theorem for (1.4) is to solve a family of Dirichlet problems for the same equation and then establish that at least one of these solutions satisfies the boundary condition above. We therefore shall introduce first a sub-supersolution theorem for a Dirichlet problem; this will be done in Section 4.

The property that

$$\int_{\partial\Omega} a(x,u)\partial_\nu u d\sigma \geq \int_{\partial\Omega} a(x,v)\partial_\nu v d\sigma$$

for all $u, v \in H^2(\Omega)$ with $u \leq v$ and $u \equiv v$ on $\partial\Omega$ will play an important role in the existence proof. This motivates us to introduce a generalization of (1.4) by replacing the boundary expression above by a map that shares this property and we shall state a sub-supersolution result for this generalized problem, as well.

We mention that the main points, which make the equation under consideration interesting, are the gradient dependence of the nonlinear term f and the presence of weight $a(x,u)$. We cite the papers of Callegari and Nachman [4, 5] and Fulks and Maybe [9], including some of their references, for providing physical situations from which problems involving the gradient dependence arise, and the paper [15], where degenerate (near the boundary) nonlinear elliptic problems have been studied.

2. General settings

We shall assume, as in Section 1, that a is a smooth function with

$$a(x,s) \geq 1, \tag{2.1}$$

for all $x \in \Omega$ and $s \in \mathbb{R}$, and that

$$a(x,s) \leq a_1(x)|s| + b_1(x), \tag{2.2}$$

for some $a_1 \in L^\infty(\Omega)$ and $b_1 \in L^2(\Omega)$. Under these two conditions, the map

$$\begin{array}{rcl} A : \Omega \times \mathbb{R} \times \mathbb{R}^N & \rightarrow & \mathbb{R}^N \\ (x,s,p) & \mapsto & a(x,s)p \end{array}$$

satisfies the Leray-Lions conditions (see [12]).

We recall here the concept of the class (S_+), which was introduced in [3] (see also [8]).

Definition 2.1. *We say that $\mathcal{L} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ belongs to the class (S_+) provided that for all sequences $\{u_n\}$ converging weakly to u in $H_0^1(\Omega)$, then u_n converges strongly to u in $H_0^1(\Omega)$, whenever*

$$\limsup_{n\rightarrow\infty} \langle \mathcal{L}u_n, u_n - u\rangle \leq 0. \tag{2.3}$$

The following lemma holds.

Lemma 2.1. *Let $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ be continuous. Assume that $T(H_0^1(\Omega))$ is bounded in $L^\infty(\Omega)$. Then the map $\mathcal{A}_T$ defined by*

$$\langle \mathcal{A}_T u, v\rangle := \int_\Omega a(x,Tu)\nabla u \nabla v dx, \tag{2.4}$$

for all $u, v \in H_0^1(\Omega)$, is continuous and belongs to the class (S_+).

Proof. The continuity of $\mathcal{A}_T$ is obvious because A satisfies the Leray-Lions conditions. Hence, we only provide the proof of the second assertion.

Let $\{u_n\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ converge weakly to u in $H_0^1(\Omega)$ and be uniformly bounded in $L^\infty(\Omega)$. We have

$$\begin{aligned} \|u_n - u\|^2_{H_0^1(\Omega)} &\leq \int_\Omega a(x,Tu_n)\nabla(u_n-u)\nabla(u_n-u)dx \\ &= \int_\Omega a(x,Tu_n)\nabla u_n \nabla(u_n-u)dx \\ &- \int_\Omega a(x,Tu)\nabla u \nabla(u_n-u)dx \\ &+ \int_\Omega (a(x,Tu)-a(x,Tu_n))\nabla u \nabla(u_n-u)dx. \end{aligned} \tag{2.5}$$

Using Hölder's inequality, we see that the third integral in the right hand side of (2.5) converges to 0 as $n \to \infty$. In fact,

$$\left|\int_\Omega (a(x,Tu) - a(x,Tu_n))\nabla u \nabla(u_n - u)dx\right|$$
$$\leq \left(\int_\Omega (a(x,Tu) - a(x,Tu_n))^2|\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega (|\nabla(u_n - u)|^2 dx\right)^{\frac{1}{2}},$$

which tends to 0 because of the boundedness of $\{|\nabla(u_n - u)|\}$ in $L^2(\Omega)$ and that of the set $T(H_0^1(\Omega))$ in $L^\infty(\Omega)$. It follows from the weak convergence of u_n to u in $H_0^1(\Omega)$, that the second integral of the right hand side in (2.5) tends to 0. Now, taking lim sup of both sides of the inequality (2.5) and recalling (2.3), give us the strong convergence of u_n to u in $H_0^1(\Omega)$. □

Throughout this paper, two continuous functions $\underline{u}$ and $\overline{u}$, defined on $\overline{\Omega}$, are said to be well-ordered if $\underline{u}(x) \leq \overline{u}(x)$, for all $x \in \overline{\Omega}$.

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. In this paper, we assume that f satisfies a Bernstein-Nagumo condition on $[\underline{u}, \overline{u}]$ for some well-ordered pair of functions $\underline{u}$ and $\overline{u}$ in $C(\overline{\Omega})$, i.e., there exist $a_2 \in L^2(\Omega)$ and $b_2 \in [0, \infty)$, both of which are allowed to depend on $\underline{u}, \overline{u}$, such that

$$|f(x,s,p)| \leq a_2(x) + b_2|p|^2 \qquad \text{for all } x \in \Omega, s \in [\underline{u}(x), \overline{u}(x)], p \in \mathbb{R}^N. \tag{2.6}$$

With a and f in hand, we establish a sub-supersolution theorem for the equation

$$-\text{div}[a(x,u)\nabla u] = f(x,u,\nabla u) \text{ in } \Omega,$$

subject to Dirichlet boundary conditions and then apply it to obtain a sub-supersolution theorem for the problem containing the same differential equation and the Dirichlet boundary condition replaced by a no-flux one; i.e.,

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu u d\sigma = 0, \tag{2.7}$$

where $d\sigma$ is the surface measure defined on $\partial\Omega$ and ν denotes the outward normal unit vector field to $\partial\Omega$.

3. The Bernstein-Nagumo condition and its consequences

Motivated by [19, 21] and their references, we wish to establish $H_0^1(\Omega)$ *a priori* bounds and the boundedness in $H_0^1(\Omega)$ for the family of functions $\{u\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\left|\int_\Omega a(x,u)\nabla u \nabla v dx\right| \leq \int_\Omega (a_2 + b_2|\nabla u|^2)|v|dx, \tag{3.1}$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Although the results look similar to those in [21], they may not be directly deduced from the results of that paper because of the presence of the weight function. However, the proof in [21] may be used for the case under consideration and we present it in this section to emphasize the beauty of the test functions used (see [25]) and for completeness' sake.

Assume that there are two well-ordered continuous functions $\underline{u} \leq \overline{u}$. Let u satisfy (3.1) with $u \in [\underline{u}, \overline{u}]$. Fix $t > 0$. Using the test function $v_t = e^{tu^2}u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ gives

$$\begin{aligned} \int_\Omega e^{tu^2}(2tu^2+1)|\nabla u|^2 dx &\leq \int_\Omega e^{tu^2}(2tu^2+1)a(x,u)|\nabla u|^2 dx \\ &\leq \int_\Omega (a_2 + b_2|\nabla u|^2)e^{tu^2}|u| dx. \end{aligned}$$

It follows that

$$\int_\Omega e^{tu^2}(2tu^2+1-b_2|u|)|\nabla u|^2 dx \leq Me^{tM^2}\|a_2\|_{L^1(\Omega)} = C(M),$$

where

$$M = \max\{\|\underline{u}\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\Omega)}\}.$$

We have written $C(M)$, instead of $C(M, \|a_2\|_{L^1(\Omega)})$, because a_2 may itself depend on M. Noting that $e^{tu^2} \geq 1$ and choosing t large, we have the following theorem.

Theorem 3.1. *Let $\underline{u}$ and $\overline{u}$ be a well-ordered pair of continuous functions. Then there exists $C > 0$, depending on $\underline{u}$ and $\overline{u}$, such that for all $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ which solve* (3.1) *with $u \in [\underline{u}, \overline{u}]$,*

$$\|u\|_{H_0^1(\Omega)} \leq C. \tag{3.2}$$

Theorem 3.2. *Let $\underline{u}, \overline{u}$ be as in Theorem* 3.1. *The set $\{u\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ of solutions to* (3.1) *with $u \in [\underline{u}, \overline{u}]$ is compact in $H_0^1(\Omega)$.*

Proof. Let $\{u_n\}$ be an arbitrary sequence in the set of solutions to (3.1) of the theorem. Applying Theorem 3.1, we obtain the boundedness in $H_0^1(\Omega)$ of $\{u_n\}$. Assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n &\to u \quad \text{in } L^2(\Omega), \\ u_n &\to u \quad \text{a.e. in } \Omega, \end{aligned}$$

for some $u \in H_0^1(\Omega)$. It is obvious that $u \in [\underline{u}, \overline{u}]$.

Using $v_t = e^{t(u_n-u)^2}(u_n - u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as a test function for (3.1), we have

$$\begin{aligned}
&\int_\Omega e^{t(u_n-u)^2}(2t(u_n-u)^2+1)|\nabla(u_n-u)|^2 dx \\
&\leq \int_\Omega e^{t(u_n-u)^2}(2t(u_n-u)^2+1)a(x,u_n)|\nabla(u_n-u)|^2 dx \\
&\leq \int_\Omega e^{t(u_n-u)^2}(a_2 + b_2|\nabla(u_n-u)|^2)|u_n-u|dx.
\end{aligned}$$

Letting $M = \|\overline{u} - \underline{u}\|_{L^\infty(\Omega)}$ gives

$$\begin{aligned}
&\int_\Omega e^{t(u_n-u)^2}(2t(u_n-u)^2+1-b_2|u_n-u|)|\nabla(u_n-u)|^2 dx \\
&\leq e^{tM^2}\int_\Omega a_2|u_n-u|dx.
\end{aligned}$$

The right hand side of the inequality tends to 0 for all $t > 0$ by an application of Hölder's inequality. Choosing t large, we obtain the strong convergence of $\{u_n\}$ to u. It is not hard to verify that u is a solution of (3.1). □

The remark bellow explains how to link the Bernstein-Nagumo condition to the equation under consideration and inequality (3.1).

Remark 3.1. Let $\underline{u}$ and $\overline{u}$, with $\underline{u} \leq \overline{u}$, be two given continuous functions and let f satisfy a Bernstein-Nagumo condition on $[\underline{u}, \overline{u}]$. If $u \in [\underline{u}, \overline{u}]$ is a solution of

$$|-\mathrm{div}[a(x,u)\nabla u]| \leq |f(x,u,\nabla u)|$$

in the classical sense, then (3.1) is obviously true; therefore, (3.2) holds and the set of such functions $\{u\}$ is compact in $H_0^1(\Omega)$.

4. A sub-supersolution theorem for Dirichlet boundary problems

During the last several years we have studied sub-supersolution theorems for boundary value problems (and other types of boundary conditions, like Neumann or Robin); see [15, 16, 18, 20, 19]. In these papers, we paid attention to the case that the principal part does not depend on u. Hence, the presence of the weight $a(x,u)$ makes the results in this paper somewhat more general, although the arguments used to verify them are not significantly more complicated.

Let us recall the concepts of weak subsolution, supersolution and solution to the problem

$$\begin{cases} -\mathrm{div}[a(x,u)\nabla u] = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Definition 4.1. *The function $u \in H^1(\Omega)$ is called a weak subsolution (supersolution) of* (4.1) *if, and only if:*

i. $u|_{\partial\Omega} \leq (\geq) 0$,

ii. for all nonnegative functions $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,u)\nabla u \nabla v dx \leq (\geq) \int_\Omega f(x,u,\nabla u)dx.$$

Definition 4.2. *The function $u \in H_0^1(\Omega)$ is a weak solution if, and only if,*

$$\int_\Omega a(x,u)\nabla u \nabla v dx = \int_\Omega f(x,u,\nabla u)v dx$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We have the theorem.

Theorem 4.1. *Assume that* (4.1) *has a subsolution $\underline{u}$ and a supersolution $\overline{u}$, both of which are in $C^1(\overline{\Omega})$. Assume further that*

i. $\underline{u} \leq \overline{u}$ *in* Ω,

ii. f *satisfies a Bernstein-Nagumo condition on* $[\underline{u}, \overline{u}]$.

Then, (4.1) *has a solution* $u \in C^1(\overline{\Omega})$.

The proof of this theorem is a combination of arguments used in [15, 16, 18], where the dependence of f on the gradient term ∇u was not assumed and those in [19], where f may depend upon ∇u.

Proof of Theorem 4.1. Define

$$h_n(p) := \begin{cases} p & |p| \leq n \\ \frac{np}{|p|} & |p| > n \end{cases} \qquad \text{for all } n \geq 1, p \in \mathbb{R}^N,$$

and

$$Tu := \max\{\min\{u, \overline{u}\}, \underline{u}\} \qquad \text{for all } u \in H_0^1(\Omega).$$

It is obvious that $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous and $T(H_0^1(\Omega))$ is bounded in $L^\infty(\Omega)$. Hence, the map $\mathcal{A}_T$, defined in (2.4), is of class (S_+) by Lemma 2.1.

Consider the map $\mathcal{L}_n : H_0^1(\Omega) \to H^{-1}(\Omega)$, defined by

$$\langle \mathcal{L}_n u, v\rangle := \int_\Omega a(x, Tu)\nabla u \nabla v dx - \int_\Omega f(x, Tu, h_n(\nabla Tu))v dx.$$

It follows from the continuity of T and $\mathcal{A}_T$ and the growth condition of f in (2.6) that $\mathcal{L}_n$ is demicontinuous; i.e. if $u_m \to u$ in $H_0^1(\Omega)$ as $m \to \infty$, then

$$\lim_{m\to\infty} \langle \mathcal{L}_n u_m, v\rangle = \langle \mathcal{L}_n u, v\rangle.$$

Moreover, since $\mathcal{A}_T$ is of class (S_+), so is $\mathcal{L}_n$ because of (2.6). Moreover, since the weight a is such that $a \geq 1$, $\mathcal{L}_n$ is coercive. Employing the topological degree defined by Browder [3] and arguing as in [19], we can find a zero of $\mathcal{L}_n$ in $H_0^1(\Omega)$, called u_n. In other words, u_n is a solution of

$$\begin{cases} -\mathrm{div}[a(x,Tu_n)\nabla u_n] = f(x,Tu_n,h_n(\nabla Tu_n)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We next prove that $u_n \in [\underline{u}, \overline{u}]$, for n sufficiently large. In fact, using the test function $v = (u_n - \overline{u})^+ \in H_0^1(\Omega)$ in (4.2) gives

$$\begin{aligned} \int_\Omega a(x,\overline{u})\nabla u_n \nabla (u_n - \overline{u})^+ dx &= \int_\Omega a(x,Tu_n)\nabla u_n \nabla (u_n - \overline{u})^+ dx \\ &= \int_\Omega f(x,Tu_n,h_n(\nabla Tu_n))(u_n - u)^+ dx \\ &= \int_\Omega f(x,\overline{u},h_n(\nabla \overline{u}))(u_n - u)^+ dx. \end{aligned}$$

We now consider n so large such that

$$n \geq \max\{\|\nabla \underline{u}\|_{L^\infty(\Omega)}, \|\nabla \overline{u}\|_{L^\infty(\Omega)}\} \tag{4.3}$$

and therefore

$$h_n(\nabla \overline{u}) = \nabla \overline{u}.$$

This implies

$$\begin{aligned} \int_\Omega a(x,\overline{u})\nabla u_n \nabla (u_n - \overline{u})^+ dx &= \int_\Omega f(x,\overline{u},\nabla \overline{u})(u_n - u)^+ dx \\ &\leq \int_\Omega a(x,\overline{u})\nabla \overline{u} \nabla (u_n - \overline{u})^+ dx \end{aligned}$$

and

$$\int_\Omega a(x,\overline{u})\nabla |(u_n - u)^+|^2 dx \leq 0.$$

We have obtained $u_n \leq \overline{u}$ a.e. in Ω. Similarly, with n satisfying (4.3), $u_n \geq \underline{u}$.

Since $u_n \in [\underline{u}, \overline{u}]$, $Tu_n = u_n$ when n is large. For such n, u_n solves

$$\begin{cases} -\mathrm{div}[a(x,u_n)\nabla u_n] = f(x,u_n,h_n(\nabla u_n)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Noting that $|h_n(p)| \leq |p|$ for all $p \in \mathbb{R}^N$, we see that u_n satisfies (3.1). Hence, by Theorem 3.1 and Theorem 3.2, $u_n \to u$ in $H_0^1(\Omega)$ for some function u, which is obviously a solution to (4.1). Moreover, the zero boundary value of u and the uniform boundedness of u imply that $u \in C^1(\overline{\Omega})$ (see [12]). □

Remark 4.1. The C^1 requirements for the pair of sub-supersolution in Theorem 4.1 is necessary because of (4.3). In the case that f does not depend on ∇u, this smoothness condition can be relaxed.

By a simple substitution, say $v = u - c$ for any constant c, we have the following theorem.

Theorem 4.2. *Assume that*

$$\begin{cases} -div[a(x,u)\nabla u] &= f(x,u,\nabla u) \quad in\ \Omega, \\ u &= c \qquad on\ \partial\Omega \end{cases} \tag{4.5}$$

has a subsolution $\underline{u}$ *and a supersolution* $\overline{u}$*, both of which are in* $C^1(\overline{\Omega})$*. Assume further that*

i. $\underline{u} \leq \overline{u}$ *in* Ω*,*

ii. f *satisfies a Bernstein-Nagumo condition on* $[\underline{u}, \overline{u}]$*.*

Then, (4.5) *has a solution* $u \in C^1(\overline{\Omega})$*.*

In the theorem above, $\underline{u} \in H^1(\Omega)$ is a subsolution of (4.5) if and only if $\underline{v} = \underline{u} - c$ is a subsolution of

$$\begin{cases} -\mathrm{div}[a(x,v+c)\nabla v] &= f(x,v+c,\nabla v) \quad \text{in } \Omega, \\ v &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

The concepts of supersolution and solution to (4.5) are defined in the same manner.

5. A sub-supersolution theorem for no-flux problems

In this section, we are concerned with

$$\begin{cases} -\mathrm{div}[a(x,u)\nabla u] &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ u &= \text{constant} \qquad \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} a(\xi,u)\partial_\nu d\sigma(\xi) &= 0, \end{cases} \tag{5.1}$$

where the constant value of u on $\partial\Omega$ is not specified. This suggests to use the functional space

$$V = \{v \in H^1(\Omega) \,|\, v|_{\partial\Omega} = \text{constant}\} \tag{5.2}$$

to study (5.1).

If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution to

$$\begin{cases} -\mathrm{div}[a(x,u)\nabla u] &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ u &= c \qquad \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where c is a constant, then

$$\int_{\partial\Omega} a(\xi, c)\partial_\nu u d\sigma = -\int_\Omega f(x, u, \nabla u)dx \tag{5.4}$$

by the divergence theorem. We have the following lemma.

Lemma 5.1. *If* $u \in H^1(\Omega)\cap L^\infty(\Omega)$ *solves equation* (5.3) *in the weak sense, the identity* (5.4) *is still true.*

Proof. Since $u \in L^\infty(\Omega)$ and $u|_{\partial\Omega}$, u belongs to $C^1(\overline{\Omega})$ (see [12]). This explains the well-definedness of the boundary expression in the left hand side of (5.4).

For $n \geq 1$, define the function

$$\alpha_n(s) = \begin{cases} s & 0 \leq s < \frac{1}{n} \\ 1/n & s \geq \frac{1}{n}, \end{cases}$$

and for each $x \in \Omega$, let $\delta(x)$ denote the Euclidean distance from x to $\partial\Omega$. It follows from the smoothness of $\partial\Omega$, that δ is smooth on a neighborhood of $\partial\Omega$ (see [10]). Using

$$v_n = \alpha_n \circ \delta \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

as a test function in the variational formulation of (5.3) gives

$$n\int_{\Omega_{\frac{1}{n}}} a(x, u)\nabla u \nabla v_n dx = n\int_\Omega f(x, u, \nabla u)v_n dx \tag{5.5}$$

for all $n \geq 1$, where

$$\Omega_{\frac{1}{n}} = \left\{x \in \Omega \,|\, \delta(x) < \frac{1}{n}\right\}.$$

It is obvious that

$$\lim_{n\to\infty} n\int_\Omega f(x, u, \nabla u)v_n dx = \int_\Omega f(x, u, \nabla u)dx. \tag{5.6}$$

We next evaluate the limit of the left hand side of (5.5) as $n \to \infty$ by the method of substitution. For each n large, define the map P that sends $x \in \Omega_{\frac{1}{n}}$ to (ξ, ρ), where ξ is the projection of x on $\partial\Omega$ and $\rho = \delta(x)$. The map $P^{-1}(\xi, \rho)$ is given by

$$(\xi, \rho) \mapsto \xi - \rho\nu(\xi).$$

Thus, if we let T_ξ be the tangent space to $\partial\Omega$ at ξ and $B(\xi)$ be the orthonormal basis of $\mathbb{R}^N$, defined by an orthonormal basis of T_ξ and $\nu(\xi)$, then

$$mat_{B(\xi)}(DP^{-1}(\xi, \rho)) = \begin{bmatrix} Id + \rho D\nu(\xi) & 0 \\ * & 1 \end{bmatrix}.$$

Hence,

$$\det DP^{-1}(\xi,\rho) = 1 + \rho \mathrm{div} D\nu(\xi) + O(\rho^2) = 1 + O(\rho) = 1 + O\left(\frac{1}{n}\right),$$

because $\mathrm{div} D\nu(\xi)$ does not depend on n. We now write the left hand side of (5.5) as

$$\begin{aligned} & n\int_{\Omega_{\frac{1}{n}}} a(x,u)\nabla u \nabla v_n dx \\ &= n\int_{\partial\Omega}\int_0^{\frac{1}{n}} a(\xi+\rho\nu(\xi),u)\nabla u(\xi+\rho\nu(\xi))\alpha_n'(\rho)\nabla\delta(\xi+\rho\nu(\xi)) \\ &\qquad\qquad \times \det DP^{-1}(\xi+\rho\nu(\xi))d\rho d\sigma \\ &= n\int_{\partial\Omega}\int_0^{\frac{1}{n}} a(\xi+\rho\nu(\xi),u)\nabla u(\xi+\rho\nu(\xi))\alpha_n'(\rho)\nabla\delta(\xi+\rho\nu(\xi)) \\ &\qquad\qquad \times(1+O\left(\frac{1}{n}\right) d\rho d\sigma. \end{aligned}$$

We observe that when x is in $\Omega_{\frac{1}{n}}$, $\alpha_n'(\delta(x)) = 1$ and $\nabla(\delta(x)) = -\nu(\xi)$, hence we may let $n \to \infty$ to obtain

$$\lim_{n\to\infty} n\int_{\Omega_{\frac{1}{n}}} a(x,u)\nabla u\nabla v_n dx = -\int_{\partial\Omega} a(\xi,u)\nabla u \nu d\sigma.$$

This, together with (5.5) and (5.6), shows (5.4). □

We next discuss the concept of subsolution and supersolution for (5.1).

Definition 5.1. *The function $u \in V \cap C^1(\overline{\Omega})$ is called a subsolution (supersolution) to* (5.1) *if, and only if,*

i. for all nonnegative functions $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,u)\nabla u\nabla v dx \le (\ge) \int_\Omega f(x,u,\nabla u) v dx,$$

ii.

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu u d\sigma \le (\ge) 0.$$

Definition 5.2. *The function $u \in V \cap C^1(\overline{\Omega})$ is called a solution to* (5.1) *if, and only if,*

i. for all functions $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,u)\nabla u\nabla v dx = \int_\Omega f(x,u,\nabla u) v dx,$$

ii.

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu u d\sigma = 0.$$

The following is our main result in this section.

Theorem 5.1. *Assume that* (5.1) *has a subsolution* $\underline{u}$ *and a supersolution* $\overline{u}$. *Assume further that*

i. $\underline{u} \le \overline{u}$ *in* Ω,

ii. f *satisfies a Bernstein-Nagumo condition on* $[\underline{u}, \overline{u}]$.

Then, (5.1) *has a solution* $u \in C^1(\overline{\Omega})$.

Proof. Let $\alpha = \underline{u}|_{\partial\Omega}$ and $\beta = \overline{u}|_{\partial\Omega}$. For each $t \in [0,1]$, define

$$c_t = t\beta + (1-t)\alpha.$$

Applying Theorem 4.2, we can find $u_t \in C^1(\overline{\Omega})$ solving

$$\begin{cases} -\operatorname{div}[a(x,u_t)\nabla u_t] &= f(x,u_t,\nabla u_t) \quad \text{in } \Omega, \\ u_t &= c_t \quad \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Let U_t be the set of such solutions u_t and $U = \cup_{t\in[0,1]} U_t$. Let U^1 (respectively U^2) denote the set of solution $u_t \in V$ of (5.7) so that

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu d\sigma < (\text{respectively} >) 0.$$

Suppose that (5.1) has no solution staying between $\underline{u}$ and $\overline{u}$. Then

$$U = U^1 \cup U^2.$$

Theorem 3.2 implies that U is compact in $H^1(\Omega)$. This, together with Lemma 5.1, shows that both U^1 and U^2 are also compact in $H^1(\Omega)$.

If $u \in U_0$ then $u = \underline{u}$ on $\partial\Omega$ and, since $u \ge \underline{u}$,

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu u d\sigma \le \int_{\partial\Omega} a(\xi,\underline{u})\partial_\nu \underline{u} d\sigma \le 0,$$

and hence, because of our assumption,

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu u d\sigma < 0.$$

Similarly, if $u \in U_1$, then

$$\int_{\partial\Omega} a(\xi,u)\partial_\nu u d\sigma > 0.$$

Let
$$t_* = \sup\{t \in [0,1] \,|\, u_t \in U^1\}.$$
The compactness of U^1 shows that there exists a solution $u_{t_*} \in U^1 \cap U_{t_*}$. Considering u_{t_*} and $\overline{u}$ as a pair of subsolutions and supersolutions to (5.7) with $t \in (t_*, 1)$ gives us a decreasing sequence of solution $u^{(n)}$ to (5.7) with $u^{(n)}|_{\partial\Omega} \searrow c_{t_*}$ with $u^{(n)} \geq u_{t_*}$. Denote the limit of this sequence by v_{t_*}. The compactness of U^2 implies that $v_{t_*} \in U^2 \cap U_{t_*}$, and we now get the contradiction
$$0 < \int_{\partial\Omega} a(x, v_{t_*})\partial_\nu v_{t_*} d\sigma \leq \int_{\partial\Omega} a(x, u_{t_*})\partial_\nu u_{t_*} d\sigma < 0,$$
which completes the proof. □

6. A generalization of the no-flux problem

We shall next derive a result similar to Theorem 5.1 for the more general boundary value problem
$$\begin{cases} -\mathrm{div}[a(x,u)\nabla u] &= f(x,u,\nabla u), \quad \text{in } \Omega, \\ u &= c \qquad\qquad\quad \text{on } \partial\Omega \\ \Phi(u) &= 0, \end{cases} \tag{6.1}$$
where
$$\Phi : H^2(\Omega) \to \mathbb{R}$$
is a functional satisfying assumptions spelled out below in Assumption 6.1.

Remark 6.1. As usual, a weak solution u of (6.1) belongs to $H^1(\Omega)$. Assume that $c = u|_{\partial\Omega}$. Denoting by b the map $x \mapsto a(x, u(x))$, we can apply the standard rules in differentiation to verify that u satisfies the problem (in the unknown v)
$$\begin{cases} -\Delta v &= \dfrac{f(x,u,\nabla u) + \nabla b \nabla u}{b} \quad \text{in } \Omega, \\ v &= c \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases}$$
Since any solution of the problem above is in $H^2(\Omega)$ (see [10]), so is u. This explains how to define the term $\Phi(u)$ in (6.1) when the domain of Φ is $H^2(\Omega)$.

The following lemma will be useful.

Lemma 6.1. *Assume that Φ is continuous. Let $\underline{u}$ and $\overline{u}$ be a well-ordered pair of continuous functions on $\overline{\Omega}$. Let U^- (resp. U^+) be the set of all solutions in $[\underline{u}, \overline{u}]$ to*
$$\begin{cases} -div[a(x,u)\nabla u] &= f(x,u,\nabla u) \quad in\ \Omega, \\ u &= constant \qquad\ \ on\ \Omega, \end{cases} \tag{6.2}$$

with $\Phi(u) \leq$ *(resp.* $\geq$*)*0*. If* f *satisfies a Bernstein-Nagumo condition on* $[\underline{u}, \overline{u}]$*, then* U^- *and* U^+ *are both compact in* $H^1(\Omega)$*.*

Proof. Let $\{u_n\}$ be a sequence in U^-. Theorem 3.1 and 3.2 help us find a solution u of (6.2) with $u_n \to u$ in $H^1(\Omega)$. Repeating the arguments in Remark 6.1, we see that $u \in H^2(\Omega)$ and therefore $\Phi(u)$ is well-defined. Hence, proving $\Phi(u) \leq 0$ is sufficient to the compactness of U^-.

For all $n \geq 1$, it is not hard to see that u_n solves

$$\begin{cases} -\Delta u_n &= \dfrac{f(x, u_n, \nabla u_n) + \nabla b_n \nabla u_n}{b_n} \quad \text{in } \Omega, \\ \quad u_n &= c_n \quad \text{on } \partial\Omega. \end{cases}$$

where c_n is a constant and $b_n(x) = a(x, u_n(x))$. Letting $v_{n,m} = u_n - u_m$, we have

$$\begin{cases} -\Delta v_{n,m} &= g_{n,m} \quad \text{in } \Omega, \\ \quad u_{n,m} &= c_n - c_m \quad \text{on } \partial\Omega. \end{cases}$$

where

$$g_{n,m}(x) = \frac{f(x, u_n, \nabla u_n) + \nabla b_n \nabla u_n}{b_n} - \frac{f(x, u_m, \nabla u_m) + \nabla b_m \nabla u_m}{b_m}.$$

It follows from the continuity of a, the Bernstein-Nagumo requirement on f and the Cauchy property of $\{u_n\}$ in $H^1(\Omega)$ that $\|g_{m,n}\|_{L^2(\Omega)} \to 0$. Applying the H^2 regularity results in [10], we have $\|v_{m,n}\|_{H^2(\Omega)} \to 0$, which shows $\{u_n\}$ is Cauchy in $H^2(\Omega)$. Its limit must be u. By the continuity of Φ, $\Phi(u) \leq 0$.

The compactness of U^+ can be proved in the same manner. □

We again need the notion of sub- and supersolution for (6.1).

Definition 6.1. *A function* $u \in V$ *is called a subsolution (resp. supersolution) of* (6.1) *if, and only if:*

i. for all nonnegative functions $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$*,*

$$\int_\Omega a(x, u)\nabla u \nabla v dx \leq (\geq) \int_\Omega f(x, u, \nabla u) v dx,$$

ii. $\Phi(u) \leq (\geq)\ 0$.

In the definition above, we employ again the functional space V in the previous section.

We shall impose the following assumption on the functional Φ.

Assumption 6.1. The functional Φ is continuous and satisfies: If $u, v \in H^2(\Omega)$ are such that $u \leq v$ *in* Ω and $u \equiv v$ *on* $\partial\Omega$, then $\Phi(u) \geq \Phi(v)$.

We next establish a theorem similar to the result about the no-flux problem (assuming conditions as before on f)

Theorem 6.1. *Assume there exist functions $\underline{u}, \overline{u} \in V \cap C^1(\overline{\Omega})$ which are, respectively, sub- and supersolutions of* (6.1) *and satisfy*

$$\underline{u}(x) \leq \overline{u}(x),\ x \in \Omega.$$

Let the functional Φ satisfy Assumption 6.1 *and assume that $|f|$ satisfies a Bernstein-Nagumo condition on $[\underline{u}, \overline{u}]$. Then there exists a solution u of* (6.1) *such that*

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x),\ x \in \Omega.$$

Proof. Let

$$u_\lambda(x) := (1-\lambda)\underline{u}(x) + \lambda\overline{u}(x),\ x \in \Omega$$

and for any $\lambda \in [0,1]$ consider the Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}[a(x,u)\nabla u] &=& f(x,u,\nabla u), & \text{in } \Omega, \\ u &=& u_\lambda, & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Since $\underline{u}$ and $\overline{u}$ are, respectively sub- and supersolutions of (6.1), then for any such λ they are, respectively, sub- and supersolutions of (6.3), as follows from the definitions. We may therefore conclude from Theorem 4.2 that each problem (6.3) has a solution $u \in V$ with

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x),\ x \in \Omega.$$

Let us denote, for each such λ, by U_λ the set of all such solutions. It follows from Theorem 3.2 that

$$U := \cup_{0\leq\lambda\leq 1} U_\lambda$$

is a compact family in $H^1(\Omega)$. By Remark 6.1, we have that $U \subset H^2(\Omega)$. We claim that there exists $\lambda \in [0,1]$ and a solution $u \in U_\lambda$ of (6.3) such that

$$\Phi(u) = 0$$

and hence that u is a solution of (6.1). This we argue indirectly. As in Lemma 6.1, let

$$U^- = \{u \in U \,|\, 0 < \Phi(u)\}$$

and

$$U^+ = \{u \in U \,|\, \Phi(u) > 0\},$$

These two sets are nonemppty because Assumption 6.1 implies $u_0 \in U^-$ and $u_1 \in U^+$. Further, if we let

$$\bar{\lambda} = \sup\{\lambda \in [0,1] \,|\, u_\lambda \in U^-\},$$

then, using the compactness of the families U^- and U^+, Theorem 4.2 and our assumption, we conclude that for this value $\bar{\lambda}$ there must exist solutions $u, v \in U_{\bar{\lambda}}$ with $u \in U^-$ and $v \in U^+$ such that

$$u(x) \leq v(x),\ x \in \Omega.$$

This, however, will imply the impossible statement

$$0 < \Phi(u) \leq \Phi(v) < 0.$$

The contradiction, arrived at, concludes the proof. □

Remark 6.2. Let $\underline{u}$, $\overline{u}$ and f be as in Theorem 6.1 with the condition that both $\underline{u}$ and $\overline{u}$ take constant values on $\partial\Omega$ can be generalized to the case that $\underline{u}|_{\partial\Omega}$ and $\overline{u}|_{\partial\Omega}$ are the traces of two $H^2(\Omega)$ functions on $\partial\Omega$. By the same arguments above, we can find a solution in $H^2(\Omega)$ to

$$\begin{cases} -\text{div}[a(x,u)\nabla u] &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ \Phi(u) &= 0 \end{cases}$$

with $u|_{\partial\Omega}$ being a convex combination of $\underline{u}|_{\partial\Omega}$ and $\overline{u}|_{\partial\Omega}$.

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